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Shortcutting Planar Digraphs

by

Mikkel Thorup[1,2,3]
Programming Research Group
of Oxford University
8–11 Keble Road
Oxford OX1 3QD
United Kingdom

[1]DIMACS visitor 92/93.
[2]Supported by the Danish Technical Research Council and the Danish Research Academy.
[3]Moving to (10/1/93): University of Copenhagen, Department of Computer Science, Universitetsparken 1, 2100 København Ø, Denmark.

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ABSTRACT

It is shown that given a planar digraph, by at most doubling the number of arcs but without changing the transitive closure, we can obtain a digraph such that whenever there is a dipath from one vertex to another there is such a dipath of length $O(\alpha(p,p)(\log p)^2)$, where p is the number of vertices and α is a functional inverse of Ackermann's function.

1 Introduction

The *diameter* of a digraph is defined to be the minimum number d such that whenever we have a dipath from one vertex to another, there is such a dipath of length at most d. We say we *shortcut* a digraph G if we add arcs to G without changing its transitive closure. Thus, if in G there is a dipath from a vertex v to a vertex w, then we can shortcut with (v, w). A shortcutting of a digraph is said to be *matched* if it at most doubles the number of arcs. In [15] the following conjecture was made:

Conjecture 1 *For all digraphs there is a matched shortcutting achieving a diameter which is poly-logarithmic in the number of vertices.*

The result of this paper is to support the conjecture by establishing it for all planar digraphs.

Theorem 2[1] *Denote by $\delta_2(p)$ the least number such that for any planar digraph with p vertices, there is a matched shortcutting achieving a diameter of at most $\delta_2(p)$. Then $\delta_2(p) = O(\alpha(p,p)(\log p)^2)$.*

Here α denotes the functional inverse of Ackermann's function defined in [12]. Theorem 2 is at present the strongest evidence for Conjecture 1. For the sake of completeness, it should be mentioned that if one is willing to add $O(p \log p)$ shortcuts, using the separator theorem from [10], it is straightforward to achieve a diameter of $O(\log p)$ for all planar digraphs [8]. Both Yao and Chazelle have been working with similar problems but in a quite different framework based on undirected graphs. Translating their results, first in 1982 Yao showed that for linear chains optimal matched shortcutting achieves a diameter in $\Theta(\alpha(p,p))$ [16]. Later in 1987 Chazelle proved that this diameter can be achieved for all directed trees [4]. In a loose way Chazelle asks for possible generalizations, for example to planar graphs, so Theorem 2 can be taken as an answer to Chazelle's call for generalizations. The exact result and problem of Chazelle will be stated in Section 3 together with the translation to the directed case.

The results of Yao and Chazelle have already found applications within certain static databases [4, 11, 16] and sparse matrix multiplication [2]. Also they have been used theoretically for speedups in proofs of bounded arithmetic [1]. These early applications indicate that our more general result for planar digraphs could be useful, and in particular indicate the importance of settling Conjecture 1 in general for all digraphs. A concrete application of the latter is within parallel computing. We have poly-logarithmically fast parallel algorithms for deciding if there is a dipath from one vertex to another, but they all use matrix multiplication, computing the full transitive closure [7]. Hence for sparse digraphs they require at least squared space and processor number which is unacceptable in connection with, say, large databases. As observed in [15], if d is the diameter of a digraph, it is easy to find all vertices reachable from a certain vertex in time d using linear space and processor number. Thus

[1] Notice that the theorem contains the definition of a bounding function with the same index. Most of the following results will be presented in this form, the idea being to make it easier to trace the derivation of the various bounds.

Conjecture 1 states that given any digraph, by shortcutting we can preprocess the digraph, at most doubling its size, such that on-line queries about reachability can be answered in poly-logarithmic time, with a linear number of processors, and in linear space. In a broader perspective, it is hoped that the study of shortcutting will deepen our understanding of digraphs, allowing us to solve (not just preprocess for) reachability and other fundamental problems in parallel computing. Notice that we already know efficient parallel algorithms for reachability in planar digraphs [6], so in this connection Theorem 2 is primarily interesting as evidence for Conjecture 1.

The paper, which is basically a proof of Theorem 2, deals with more and more complicated classes of digraphs ending up with the class of planar digraphs. After some preliminaries in Section 2, in Section 3 we consider Chazelle's result for trees. Section 4 contains the most complicated step in the proof. It generalizes the result for trees to the class of acyclic planar digraphs where all vertices are reachable from a single "root" vertex. Then, in Section 5, we take the last step, generalizing to the general planar case. Finally, Section 6 contains a few concluding remarks. Appendix A contains a brief summary of some of the essential new definitions in the paper and Appendix B contains all the figures.

2 Preliminaries

For a definition of the standard terminology of digraphs, the reader is referred to [3]. Our *digraphs* may have no vertices, and in this case the diameter is defined to be 0. They have no parallel arcs, so an arc is uniquely represented as an ordered pair of vertices. Thus, formally, a digraph is a pair $G = (V, A)$ where $V = \mathrm{V}(G)$ is the set of vertices and the set of arcs, $A = \mathrm{A}(G)$, is a subset of V^2. When it causes no confusion we will use standard set notation as if our digraphs consisted of one set of both vertices and arcs. For example, if $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ are digraphs, their union $G_1 \cup G_2$ is $(V_1 \cup V_2, A_1 \cup A_2)$. Also, if $G = (V, A)$ is a digraph and $B \subseteq V^2$, we have $G \cup B = (V, A \cup B)$. If A is an arbitrary set of ordered pairs and V is precisely the set of elements occurring in these pairs, we write $\mathrm{G}(A) = (V, A)$, and call $\mathrm{G}(A)$ the digraph *consisting* of the arcs in A. If $G = (V, A)$ is a digraph and $W \subseteq V$, by $G|W$ we denote the sub-digraph of G induced by W. Moreover, we denote by $G \setminus W$ the sub-digraph $G|(V \setminus W)$. The last notation is also used in the case where W contains elements outside V. A *dipath* P of length n is represented as a sequence $(v_0, \ldots, v_n)$ of $n+1$ distinct vertices from G where the pairs (v_i, v_{i+1}) are arcs in G. If $n = 0$ we say that P is *trivial*. The dipath P is often identified with the sub-digraph of G consisting of the arcs (v_i, v_{i+1})—if $n = 0$, it is just the singular vertex $v_0 = v_n$. If $Q = (w_0, \ldots, w_m)$ is another dipath and the only common vertex of P and Q is $v_n = w_0$, we denote the dipath $(v_0, \ldots, v_n = w_0, \ldots, w_m)$ by PQ. Given a digraph, the terms *connected* and *component* refer to the structure of the underlying undirected graph whilst we have the terms *strongly connected* and *strong component* that take into account the directions in the digraph. When we say that a digraph is *acyclic* we mean that it contains no directed cycle. In a digraph a vertex w is *reachable* from a vertex v if and only if there is a directed path from v to w. A *rooted tree* is a directed tree where all arcs are directed away from some vertex, called the *root*. Generally we will say a digraph is *rooted* if there is a vertex, called the *root*, from which we can reach all other vertices. For any digraph G, by $\mathrm{diam}(G)$ we denote the diameter as defined in the introduction. A shortcutting S of G is often identified with the set of arcs it adds to G. Thus S is matched if $|S| \leq |\mathrm{A}(G)|$, and the achieved diameter is $\mathrm{diam}(G \cup S)$. A matched shortcutting is called *optimal* if it achieves the minimal diameter possible.

3 Directed trees

We will now state the result of Chazelle for undirected trees, and then show how it can be translated to a matched shortcutting result for directed trees.

Theorem 3 (Chazelle [4]) *For $m \geq n$, denote by $\delta_3(m,n)$ the least number d with the following property: for any n-edge tree T with a weight function w mapping the edges into the elements of some semigroup $(\mathcal{S},+)$, there is a set W of at most m elements from $\mathcal{S}$ such that for any non-trivial path P in T the path weight $w(P) = \sum_{e \in P} w(e)$ can be expressed as the sum of at most d values from W. Then $\delta_3(m,n) = \Theta(\alpha(m,n) + n/(m-n+1))$.*

Lemma 4 *Let T be a tree with n edges. Moreover, let m and d be natural numbers with $m \geq n$. For the following statements we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv):*

(i) Let w be any weight function mapping every edge e from T into an element $w(e)$ of some semigroup $(\mathcal{S},+)$. Then there exists a set W with at most m semigroup values such that for any non-trivial path P in T, the path weight $w(P)$ can be expressed as the sum of at most d values from W.

(ii) Let D be any directed version of T. Then there exists a shortcutting S of D that, using at most $(m-n)/2$ arcs, achieves a diameter of at most d.

(iii) There is a rooted version R of T with a shortcutting S which, using at most $(m-n)/2$ arcs, achieves a diameter of at most d.

(iv) Let w' be any weight function mapping every edge e from T into an element $w'(e)$ of some semigroup $(\mathcal{S}',+)$. Then there exists a set W' with at most m semigroup values such that for any non-trivial path P in T, the path weight $w(P)$ can be expressed as the sum of at most $2d$ values from W'.

Proof: First notice that (iii) follows trivially from (ii).

(i) $\Rightarrow$ (ii): Assume that (i) holds true for any w and $(\mathcal{S},+)$. Define $\mathcal{S}$ as the set of all edge sequences of non-trivial paths in T together with some distinguished element $\perp$. Moreover, define $+$ such that for any $s_1, s_2 \in \mathcal{S}$,

$$s_1 + s_2 = \begin{cases} s_1 s_2 & \text{if } s_1, s_2 \neq \perp \text{ and } s_1 s_2 \text{ is an edge sequence of } T, \\ \perp & \text{otherwise.} \end{cases}$$

For any edge e, let $w(e)$ be the edge sequence (e). Thus for any non-trivial path P, the path weight $w(P) = \sum_{e \in P} w(e)$ is the edge sequence of P. Let W be the subset of $\mathcal{S}$ described in (i).

In order to prove (ii), consider an arbitrary directed version D of T. Let S be the smaller of the following two sets; if they are of equal size, take S_1 say:

$$
\begin{aligned}
S_1 = \{(u,v) | \text{ if } & P \text{ is the path in } T \text{ from } u \text{ to } v \text{ then} \\
& P \text{ has length at least } 2, \\
& P \text{ is a dipath in } D, \text{ and} \\
& w(P) \text{ is in } W\}
\end{aligned}
$$

$$
\begin{aligned}
S_2 = \{(u,v) | \text{ if } & P \text{ is the path in } T \text{ from } u \text{ to } v \text{ then} \\
& P \text{ has length at least } 2, \\
& P \text{ is a dipath in } D, \text{ and} \\
& w(P^{-1}) \text{ is in } W\}
\end{aligned}
$$

We shall see that S is a satisfactory shortcutting of D. First we will prove that S_1 achieves a diameter $\leq d$. Let u and v be arbitrary vertices with a dipath P from u to v. By construction of W we may choose $k \leq d$ non-trivial dipaths $P_1, \ldots, P_k$ such that $P_1 \cdots P_k = P$ and $w(P_1), \ldots, w(P_k) \in W$. For $i = 1, \ldots, k$, denote by u_i and v_i the first and the last vertex of P_i. If P_i is of length 1 the arc (u_i, v_i) is in D; otherwise (u_i, v_i) is in S_1. Thus we have a dipath in $D \cup S_1$ from u to v of length $k \leq d$, as desired. Next consider the shortcutting S_2. Let u, v, and P be unchanged. As above, by construction of W we can find $l \leq d$ non-trivial dipaths $Q_1, \ldots, Q_l$ such that $Q_l^{-1} \cdots Q_1^{-1} = P^{-1}$ and $w(Q_1^{-1}), \ldots, w(Q_l^{-1}) \in W$. The former implies that $Q_1 \cdots Q_l = P$, so as in the case of S_1 we conclude that there is a dipath in $D \cup S_2$ of length $l \leq d$. Thus both S_1 and S_2 achieve the desired diameter.

Now we will bound the cardinality of S. Let E be the set of edge sequences of length 1. Clearly all such sequences are in W. No two dipaths of length ≥ 2 can have the same edge sequences, so the shortcuts in S_1 and S_2 all correspond to different elements in $W \setminus E$. Moreover, the subsets W_1 and W_2 of W corresponding to S_1 and S_2 must be disjoint since for no path P in T both P and P^{-1} can be dipaths in D. Thus $|S_1| + |S_2| = |W_1| + |W_2| \leq |W \setminus E| \leq m - n$, and hence $|S| \leq (m-n)/2$, as desired.

(iii) $\Rightarrow$ (iv): Basically, the proof is a simple reversal of the last construction. Assume that (iii) holds with R the rooted version of T and S the shortcutting of R. Any path P in T divides into a first part P_1 going towards the vertex which is the root of R, and a second part P_2 going away from this root vertex. More precisely we have $P_1 P_2 = P$ where both P_1^{-1} and P_2 are dipaths in R. Now W' is constructed simply as the set of weights $w'(P)$ where either P is an edge of T, or P is a path in T from a vertex u to a vertex v where $\{(u,v),(v,u)\} \cap S \neq \emptyset$. Then any path weight can be expressed as the sum of at most $2d$ values from W', and, moreover, $|W'| = n + 2|S| \leq n + 2(m-n)/2 = m$, as desired. ■

Theorem 5 *Denote by $\delta_5(p)$ the least number such that for any directed tree D with p vertices, there is a matched shortcutting of D achieving a diameter of at most $\delta_5(p)$. Then $\delta_5(p) = \Theta(\alpha(p,p))$.*

Proof: When translating Theorem 3 with Lemma 4 we have $n = p - 1$ since D is a tree. Moreover, since we want the shortcutting to be matched, we take $(m - n)/2 = n$, so $m = 3(p - 1)$. Hence, from (i) $\Rightarrow$ (ii), we get $\delta_5(p) \leq \delta_3(3(p-1), p-1) = O(\alpha(3(p-1), p-1) + (p-1)/(3(p-1) - (p-1) + 1)) = O(\alpha(p,p))$, and from (ii) $\Rightarrow$ (iv), we get $\delta_5(p) \geq \frac{1}{2}\delta_3(3(p-1), p-1) = \frac{1}{2}\Omega(\alpha(3(p-1), p-1) + (p-1)/(3(p-1)-(p-1)+1)) = \Omega(\alpha(p,p))$. Thus, $\delta_5(p) = \Theta(\alpha(p,p))$. ■

The equivalence of the weight problem and the shortcutting problem is completely dependent on the fact that between any two vertices in a tree there is a unique path. In [4], as a comment to Theorem 3, Chazelle writes

> One might wonder what happens if arbitrary graphs are substituted for free trees. Difficulties arise because of the nonuniqueness of a path between two vertices. Restricting oneself to shortest paths or planar graphs, or perhaps representing paths as regular expressions, are possible approaches to the general problem, which we believe deserve further study.

Shortcutting of an acyclic digraph corresponds to looking for the weight of an unspecified dipath between any given vertices. In this way Theorem 2 can be taken as an answer to Chazelle's call for generalizations of Theorem 3.

4 Rooted acyclic planar digraphs

This section is devoted to proving the following theorem, taking a major step forward towards proving Theorem 2, establishing Conjecture 1 for all planar digraphs.

Theorem 6 *Denote by $\delta_6(p)$ the least number such that for any rooted acyclic planar digraph with p vertices, there is a matched shortcutting achieving a diameter of at most $\delta_6(p)$. Then $\delta_6(p) = O(\alpha(p,p)\log p)$.*

The proof of the theorem is based on an intricate induction over something called areas which, unfortunately, requires quite a few new definitions. Besides giving and discussing these definitions below, in order to support the reader's memory, a brief summary of the definitions has been included in Appendix A.

For the rest of this section, let $\mathcal{G}$ be a fixed rooted acyclic planar graph embedded into some specific sphere $\mathcal{X}$. Often we will identify the vertices and arcs of $\mathcal{G}$ with the points and directed curves into which they are embedded in $\mathcal{X}$. No other graph components than those of $\mathcal{G}$ will be embedded into $\mathcal{X}$. In particular, all added shortcuts will be connected to $\mathcal{X}$ only via the end-vertices. Thus, for any subset Z of $\mathcal{X}$, by *the vertices and arcs in Z* we understand the vertices and arcs from $\mathcal{G}$ whose embeddings into $\mathcal{X}$ are completely contained in Z—for arcs this includes the end-vertices. The sets of these vertices and these arcs are denoted by $\mathcal{V}(Z)$ and $\mathcal{A}(Z)$, respectively, and the digraph $(\mathcal{V}(Z), \mathcal{A}(Z))$ is denoted by $\mathcal{G}(Z)$.

We call a subset X of $\mathcal{X}$ an *area* if it is a region of some non-empty connected sub-digraph of $\mathcal{G}$ (a region of a non-empty spheric graph is a maximal connected subset of the sphere that does not intersect the graph). Thus, equivalently, an area is a non-empty open simply connected subset of $\mathcal{X}$ whose boundary is a sub-digraph of $\mathcal{G}$. As an extreme case, we have areas of the form $\mathcal{X} \setminus \{v\}$ where v is a vertex of $\mathcal{G}$. As already mentioned, the proof of Theorem 6 will be by induction over areas. Unfortunately, before we can state the induction, we need to define several area related concepts. First some simple notation: if X is an area, by ∂X we denote the boundary of X, and by $\overline{X}$ we denote the closure of X—so $\overline{X} = \partial X \cup X$. In Figure 1 is an example of an area X_{EX}. Notice that $\overline{X}_{\mathrm{EX}}$ is not simply connected.

Consider some area X. A dipath P in $\overline{X}$—and hence in $\mathcal{G}(\overline{X})$—is called an X*-dipath* if it is contained in X with the possible exception of its end-vertices. With v and w denoting the first and the last vertex in P we say that w is X*-reachable* from v. If, moreover, all dipaths in $\overline{X}$ from v to w are X-dipaths, we say that w is *inherently* X-reachable from v. Let v and w be two different vertices in ∂X. If w is inherently X-reachable from v, we say that the arc (v, w) is a *crossover* for X. A crossover may, or may not be an arc in $\mathcal{G}$. By $\mathrm{Cross}(X)$ we denote the set of all crossovers for X. Since $\mathcal{G}$ is acyclic, $\mathrm{Cross}(X)$ is exactly the least set of arcs such that for all $v, w \in \partial X$, if there is a dipath in $\overline{X}$ from v to w, there is a dipath in $\partial X \cup \mathrm{Cross}(X)$ from v to w. In our example $\mathrm{Cross}(X_{\mathrm{EX}}) = \{(v_6, v_9)\}$. Notice that v_{12} is X_{EX}-reachable from v_{10}, but not inherently X_{EX}-reachable.

For any area X, let $\mathcal{G}^+(X)$ denote $\mathcal{G}(X \cup \mathcal{V}(\partial X))$. Thus $\mathcal{G}^+(X)$ equals $\mathcal{G}(\overline{X})$ minus the arcs in ∂X. Also $\mathcal{G}^+(X)$ is the union of all X-dipaths. However, a dipath in $\mathcal{G}^+(X)$ is only an X-dipath if all its interior vertices are in X.

By a *division* of X we mean a sub-digraph D of $\mathcal{G}^+(X)$ such that $\partial X \cup D$ is connected. Notice that $\mathcal{G}^+(X)$ is itself a division since $\mathcal{G}$ is connected. Also notice that the condition that $\partial X \cup D$ be connected is equivalent to requiring that all the regions of $\partial X \cup D$ which are contained in X are areas. We say that D *divides* X into these sub-areas of X, and the set of them is denoted by X/D.

By $\mathrm{CrossExt}(D, X)$ we denote the extension of D with the crossovers for all areas in X/D, i.e. $\mathrm{CrossExt}(D, X) = D \cup \mathrm{G}(\bigcup_{X_i \in X/D} \mathrm{Cross}(X_i))$. Notice that there might be vertices from ∂X in $\mathrm{G}(\bigcup_{X_i \in X/D} \mathrm{Cross}(X_i))$ which are not in D, but that all such vertices are in ∂X. From our previous remarks concerning crossovers it follows that if v and w are vertices in $\partial X \cup D$ and w is inherently X-reachable from v, there is a dipath from v to w in $\mathrm{CrossExt}(D, X)$. In Figure 1 we have the division D_{EX} dividing X_{EX} into two areas, and $\mathrm{CrossExt}(D_{\mathrm{EX}}, X_{\mathrm{EX}}) = D_{\mathrm{EX}} \cup \{(v_{10}, v_9)\}$. Notice that for every area X, we have $\mathrm{CrossExt}(\mathcal{G}^+(X), X) = \mathcal{G}^+(X)$ because none of the areas in $X/\mathcal{G}^+(X)$ contain any parts of $\mathcal{G}$, and hence they cannot have any crossovers. Also, notice that for the empty division $(\emptyset, \emptyset)$ of X, we have $\mathrm{CrossExt}((\emptyset, \emptyset), X) = \mathrm{G}(\mathrm{Cross}(X))$ since $X/(\emptyset, \emptyset) = \{X\}$.

Let D be a division of an area X, and let S be a shortcutting of $\mathrm{CrossExt}(D, X)$. By ii-diam(D, X, S) we denote the minimum number $d \geq 0$ such that if v and w are vertices in $D|\mathcal{V}(X)$ and w is inherently X-reachable from v, there is a dipath in $\mathrm{CrossExt}(D, X) \cup S$ from v to w of length $\leq d$. Moreover, by ib-diam(D, X, S) we denote the minimum number $d \geq 0$ such that if v and w are vertices with one in $D|\mathcal{V}(X)$ and one in ∂X, and if w is inherently X-reachable from v, then there is a dipath in $\mathrm{CrossExt}(D, X) \cup S$ from v to w of length $\leq d$. The difference between the two measures is that with ii-diam we quantify over variables both inside X, whilst with ib-diam we quantify over variables with one inside X and one in the boundary of X. In our example we have ii-diam$(D_{\mathrm{EX}}, X_{\mathrm{EX}}, \emptyset) = 2$ and ib-diam$(D_{\mathrm{EX}}, X_{\mathrm{EX}}, \emptyset) = 3$. With $S_{\mathrm{EX}} = \{(v_{10}, v_{13})\}$, we get ii-diam$(D_{\mathrm{EX}}, X_{\mathrm{EX}}, S_{\mathrm{EX}}) = 1$ and ib-diam$(D_{\mathrm{EX}}, X_{\mathrm{EX}}, S_{\mathrm{EX}}) = 2$.

If D is a division of an area X, a shortcutting S of $\mathrm{CrossExt}(D, X)$ is called a *division shortcutting of* (D, X) if the following conditions are satisfied:

- $\mathrm{Cross}(X) \subseteq \mathrm{CrossExt}(D, X) \cup S$,
- $|\mathrm{Cross}(X)| + |S| \leq |\mathrm{A}(D)| + \sum_{X_i \in X/D} |\mathrm{Cross}(X_i)|$.

Notice that, since $|\mathrm{A}(\mathrm{CrossExt}(D, X))| \leq |A(D)| + \sum_{X_i \in X/D} |\mathrm{Cross}(X_i)|$, the second condition is satisfied if $|\mathrm{Cross}(X)| + |S| \leq |\mathrm{A}(\mathrm{CrossExt}(D, X))|$. The inequality $|\mathrm{A}(\mathrm{CrossExt}(D, X))| \leq |A(D)| + \sum_{X_i \in X/D} |\mathrm{Cross}(X_i)|$ is not sharp, for it is possible both that $\mathrm{Cross}(X_i) \cap \mathrm{Cross}(X_j) \neq \emptyset$ for some $i \neq j$, and that some $\mathrm{Cross}(X_i) \cap \mathrm{A}(D) \neq \emptyset$. The following lemma describes the essential properties of the definitions:

Lemma 7 *Let X be an area, let D be a division of X, and S be a division shortcutting of (D, X). Moreover for each of the areas X_i in X/D let D_i be a division of X_i, and let S_i be a division shortcutting of (D_i, X_i). Then*

(i) $D' = D \cup \bigcup_i D_i$ is a division of X,

(ii) $S' = S \cup \bigcup_i S_i$ is a division shortcutting of (D', X),

(iii) $$\text{ii-diam}(D', S') \leq \max\{\ \text{ii-diam}(D, X, S) + 2\max_i\{\text{ib-diam}(D_i, X_i, S_i)\},\ \max_i\{\text{ii-diam}(D_i, X_i, S_i)\}\},$$

(vi) $\text{ib-diam}(D', S') \leq \text{ib-diam}(D, X, S) + \max_i\{\text{ib-diam}(D_i, X_i, S_i)\}$.

Proof: For all i we have that $D_i \cup \partial X_i$ is connected, and that $\partial X_i \subseteq \partial X \cup D$. Moreover $D \cup \partial X$ is connected, so $D' \cup \partial X$ must be connected. Besides D' cannot contain any arc from ∂X, so D' is a division of X as stated in (i). We will now settle (ii), that S' is a division shortcutting of (D', X). First observe that

$$\begin{aligned}\mathrm{Cross}(X) &\subseteq \mathrm{CrossExt}(D, X) \cup S \;=\; D \cup \textstyle\bigcup_i \mathrm{Cross}(X_i) \cup S \\ &\subseteq D \cup \textstyle\bigcup_i (\mathrm{CrossExt}(D_i, X_i) \cup S_i) \cup S \;=\; \mathrm{CrossExt}(D', X) \cup S',\end{aligned}$$

Hence (ii) follows if we can show that $|\mathrm{Cross}(X)| + |S'| \leq |\mathrm{A}(D')| + \sum_{Y_j \in X/D'} |\mathrm{Cross}(Y_j)|$. By assumption S and the S_is are all division shortcuttings, so $|\mathrm{Cross}(X)| + |S| \leq |\mathrm{A}(D)| + \sum_i |\mathrm{Cross}(X_i)|$, and $|\mathrm{Cross}(X_i)| + |S_i| \leq |\mathrm{A}(D_i)| + \sum_{Y_j \in X_i/D_i} |\mathrm{Cross}(Y_j)|$ for all i. Summing these inequalities we get

$$\begin{aligned}&|\mathrm{Cross}(X)| + \sum_i |\mathrm{Cross}(X_i)| + |S| + \sum_i |S_i| \leq \\ &\quad |\mathrm{A}(D)| + \sum_i |\mathrm{A}(D_i)| + \sum_i |\mathrm{Cross}(X_i)| + \sum_{i, Y_j \in X_i/D_i} |\mathrm{Cross}(Y_j)|.\end{aligned}$$

Notice that $|S'| \leq |S| + \sum_i |S_i|$, that $|\mathrm{A}(D')| = |\mathrm{A}(D)| + \sum_i |\mathrm{A}(D_i)|$ since D and the D_is are arc- disjoint, and that $\sum_{i,Y_j \in X_i/D_i} |\mathrm{Cross}(Y_j)| = \sum_{Y_j \in X/D'} |\mathrm{Cross}(Y_j)|$ since $\bigcup_i (X_i/D_i) = X/D'$. Hence the summed inequality can be reduced to $|\mathrm{Cross}(X)| + |S'| \leq |\mathrm{A}(D')| + \sum_{Y_j \in X/D'} |\mathrm{Cross}(Y_j)|$, as required for settling (ii).

In order to settle (iii), consider any two vertices v and w in $D'|\mathcal{V}(X)$ such that w is inherently X-reachable from v. We want to find a dipath from v to w in $\mathrm{CrossExt}(D', X) \cup S'$ of length at most

$$d = \max\{\text{ii-diam}(D, X, S) + 2\max_i\{\text{ib-diam}(D_i, X_i, S_i)\}, \max_i\{\text{ii-diam}(D_i, X_i, S_i)\}\}.$$

First suppose that for some X_i we have $v, w \in X_i$ and w inherently X_i-reachable from v. Then $v, w \in D_i|\mathcal{V}(X)$, so by definition we have a dipath in $\mathrm{CrossExt}(D_i, X_i) \cup S_i \subseteq \mathrm{CrossExt}(D', X) \cup S'$ from v to w of length at most $\text{ii-diam}(D_i, X_i, S_i) \leq d$, as desired. Now suppose the opposite that there is no such X_i. Then there must be an X_i and X_j with vertices v' and w' satisfying the following conditions:

- $v \in \overline{X_i}$, $v' \in \partial X_i$, and v' is inherently X_i-reachable from v.
- $w \in \overline{X_j}$, $w' \in \partial X_j$, and w is inherently X_j-reachable from w'.
- $v', w' \in D|\mathcal{V}(X)$, and w' is inherently X-reachable from v'.

Notice that $v = v'$ if $v \in \partial X_i$; otherwise $v \in D_i|\mathcal{V}(X)$. The same holds true for w relative to w', so by definition we can find a dipath in $\mathrm{CrossExt}(D', X) \cup S'$ from v to v' to w' to w of length at most $\text{ib-diam}(D_i, X_i, S_i) + \text{ii-diam}(D, X, S) + \text{ib-diam}(D_j, S_j) \leq d$, as desired.

Settling (iv) is done similarly. Let v and w be vertices with one in $D|\mathcal{V}(X)$ and one in ∂X and such that w is inherently X-reachable from v. By symmetry we may assume that it is v which is in $D|\mathcal{V}(X)$. There is then an X_i and a vertex v' such that $v \in \overline{X_i}$ and $v' \in \partial X_i$, such that v' is inherently X_i-reachable from v, and such that w is inherently X-reachable from v'. Hence there must be a dipath in $\mathrm{CrossExt}(D', X) \cup S'$ from v to v' to w of length at most $\text{ib-diam}(D_i, X_i, S_i) + \text{ib-diam}(D, X, S)$ which is within the bound in (iv). This completes the proof of the lemma. ∎

Given an area X, by a *left side* of X we understand a maximal dipath P in ∂X such that P is a part of the closed curve clock-wise around X. A *right side* of X is defined similarly except that P has to be part of the closed curve counter-clock-wise around X. Suppose that ∂X is not a single vertex. Since $\mathcal{G}$ is acyclic, there must exist non-trivial dipaths $L_0, \ldots, L_n, R_1, \ldots, R_n$ such that $R_1^{-1} L_1 \cdots R_n^{-1} L_n$ is the closed curve directed clock-wise around the boundary of X. Then $\{L_1, \ldots, L_n\}$ is the unique set of left sides for X, and $\{R_1, \ldots, R_n\}$ the unique set of right sides. We say an area is *m-sided* if m is the number of left sides plus the number of right sides, so in the above case X is $(2n)$-sided. If ∂X is a single vertex, X is 0-sided. In our example $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)(v_1, v_3, v_4, v_9, v_8, v_7)^{-1}$ is the closed curve clock-wise around X_{EX}, so X_{EX} is 2-sided. In $X_{\mathrm{EX}}/D_{\mathrm{EX}}$ we have one 2-sided and one 6-sided area. A *corner* is an end-vertex of a side; if the corner is the first vertex of a side, the corner is a *source corner* and if it is a last vertex, it is a *terminal* corner. Thus v_1 is the source corner of X_{EX} whilst v_7 is the terminal corner.

We will need an "inner" view of areas X such that if a non-trivial X-dipath, say, starts in ∂X but not in a corner, the side it starts in is unique. A seemingly pathological case could be if ∂X was a single dipath, for then X would be a 2-sided area with this dipath constituting both the sides. We say a non-trivial X-dipath P *starts (finishes) in a particular left side* L if the first (last) vertex of P is in L and, walking along the closed curve clock-wise around X, we are in L when the dipath P leaves towards (enters from) the right. Similarly we say P *starts (finishes) in a right side* R if the first (last) vertex of P is in R and, walking along the closed curve counter-clock-wise around X, we are in R when P leaves towards (enters from) the left. For example, although the first vertex of the X_{EX}-dipath (v_3, v_{14}, v_5) is in both the right side and the left side of X_{EX}, we say that (v_3, v_{14}, v_5) leaves the right side but not the left side.

We have now gone through all area related concepts necessary for talking properly about areas. The aim of the rest of this section is to prove the following result that together with Lemma 7 implies Theorem 6:

Key Lemma 8 *Denote by $\delta_8(n)$ the least number with the following property: given a 2-sided area X containing n vertices, all reachable from ∂X, there is a division D of X such that*

(i) All areas in X/D containing vertices are 2-sided and contain at most $n/2$ vertices.

(ii) For some division shortcutting S of (D, X), we have that both ii-diam(D, X, S) and ib-diam(D, X, S) are bounded by $\delta_8(n)$.

Then $\delta_8(n) = O(\alpha(n, n))$.

Notice that (i) is trivially satisfied in the extreme case where $D = \mathcal{G}^+(X)$. Also notice that since $\mathcal{G}$ is rooted and acyclic, the condition that all vertices in X are reachable from ∂X is satisfied if and only if X does not contain the root of $\mathcal{G}$. In fact the condition that $\mathcal{G}$ is rooted is not needed for Key Lemma 8. Finally notice that the property that all vertices are reachable from the boundary is inherited by sub-areas.

Proof that Key Lemma 8 implies Theorem 6: Denote by p the number of vertices in $\mathcal{G}$. We want to show that $\delta_6(p) = O(\delta_8(p) \log p)$. The result is trivial if $p < 2$, so assume $p \geq 2$. Fix $d = \delta_8(p)$. Then $d \geq 1$.

Define $\log' : \mathcal{N}_0 \to \mathcal{N}_0$ by the recursion:

$$\begin{aligned}&\log' 0 = 0 \text{ and}\\&\log' n = \log' \lfloor n/2 \rfloor + 1 \text{ if } n > 0.\end{aligned}$$

Then $\log' n = \lfloor \log_2 n \rfloor + 1$ for $n > 0$, but in contrast to $\log_2$, we have that $\log'$ is defined for $n = 0$.

Claim 1 *Let X be any 2-sided area containing $n \le p$ vertices, all reachable from ∂X. Then there is a division shortcutting S of $(\mathcal{G}^+(X), X)$ such that*

$$\begin{aligned} &\textit{ii-diam}(\mathcal{G}^+(X), X, S) \le \max\{2\, d \log' n - d, 0\}, \\ &\textit{ib-diam}(\mathcal{G}^+(X), X, S) \le d \log' n. \end{aligned}$$

Proof: The claim is settled by induction over n. If $n = 0$, all crossovers for X are arcs in $\mathcal{G}^+(X)$, so $\emptyset$ is a division shortcutting of $(\mathcal{G}^+(X), X)$, and since there are no interior vertices, we have $\text{ii-diam}(\mathcal{G}^+(X), X, \emptyset) = \text{ib-diam}(\mathcal{G}^+(X), X, \emptyset) = 0$, as required. Suppose now that $n > 0$ and that the claim holds for all smaller values. By Key Lemma 8 we have a division D of X such that all areas in X/D containing vertices are 2-sided containing at most $n/2$ vertices, and such that there is a division shortcutting S of (D, X) achieving $\text{ii-diam}(D, X, S), \text{ib-diam}(D, X, S) \le \delta_8(n) \le d$. Moreover, for each $X_i \in X/D$, by the induction hypothesis we have a division shortcutting S_i of $(\mathcal{G}^+(X_i), X_i)$ such that $\text{ii-diam}(\mathcal{G}^+(X_i), X_i, S_i) \le \max\{2d \log' \lfloor n/2 \rfloor - d, 0\}$ and $\text{ib-diam}(\mathcal{G}^+(X_i), X_i, S_i) \le d \log' \lfloor n/2 \rfloor$. Since $\mathcal{G}^+(X) = D \cup \bigcup_i \mathcal{G}^+(X_i)$, by Lemma 7 we have that $S' = S \cup \bigcup_i S_i$ is a division shortcutting of $(\mathcal{G}^+(X), X)$ achieving

$$\begin{aligned} \text{ii-diam}(\mathcal{G}^+(X), X, S') &\le \max\{\ \text{ii-diam}(D, X, S) + 2 \max_i\{\text{ib-diam}(\mathcal{G}^+(X_i), X_i, S_i)\}, \\ &\qquad\qquad \max_i\{\text{ii-diam}(\mathcal{G}^+(X_i), X_i, S_i)\}\} \\ &\le \max\{\ d + 2d \log' \lfloor n/2 \rfloor, \\ &\qquad\qquad \max\{2d \log' \lfloor n/2 \rfloor - d, 0\}\} \\ &= 2d \log' n - d, \end{aligned}$$

$$\begin{aligned} \text{ib-diam}(\mathcal{G}^+(X), X, S') &\le \text{ib-diam}(D, X, S) + \max_i\{\text{ib-diam}(\mathcal{G}^+(X_i), X_i, S_i)\} \\ &\le d + d \log' \lfloor n/2 \rfloor \\ &= d \log' n, \end{aligned}$$

as desired. □

Let α be the root of $\mathcal{G}$, and let β be any vertex such that (α, β) is an arc in $\mathcal{G}$. Such a vertex β exists since $\mathcal{G}$ is rooted and $p \ge 2$. Set $\mathcal{Y} = \mathcal{X} \setminus (\alpha, \beta)$; then $\mathcal{G}^+(\mathcal{Y}) = \mathcal{G} \setminus \{(\alpha, \beta)\}$. Observe that $\mathcal{Y}$ is a 2-sided area with the dipath (α, β) constituting both the left and the right side. Moreover, all the $p - 2 \le p$ vertices in $\mathcal{Y}$ are reachable from $\partial \mathcal{Y}$ since α is outside $\mathcal{Y}$. Thus by Claim 1 there is a division shortcutting $\mathcal{S}$ of $(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y})$ such that $\text{ii-diam}(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y}, \mathcal{S}) \le 2d \log' p - d$ and $\text{ib-diam}(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y}, \mathcal{S}) \le d \log' p$. Since $\mathcal{S}$ is a division shortcutting of $\mathcal{G}^+(\mathcal{Y})$ we have $|\text{Cross}(\mathcal{Y})| + |\mathcal{S}| \le |\text{A}(\mathcal{G}^+(\mathcal{Y}))| + \sum_{X_i \in \mathcal{Y}/\mathcal{G}^+(\mathcal{Y})} |\text{Cross}(X_i)|$. Clearly, none of the areas in $\mathcal{Y}/\mathcal{G}^+(\mathcal{Y})$ have any crossovers, so $\sum_{X_i \in \mathcal{Y}/\mathcal{G}^+(\mathcal{Y})} |\text{Cross}(X_i)| = 0$, implying $|\mathcal{S}| \le |\text{A}(\mathcal{G}^+(\mathcal{Y}))|$. Thus $\mathcal{S}$ is a matched shortcutting of $\mathcal{G}^+(\mathcal{Y})$, hence of $\mathcal{G}$.

Consider in $\mathcal{G}$ any two different vertices v and w such that there is a dipath from v to w; then $w \ne \alpha$. We will find a dipath in $\mathcal{G} \cup \mathcal{S}$ from v to w of length at most $2d \log' p$. Suppose first that $v, w \in \mathcal{Y}$ and that w is inherently $\mathcal{Y}$-reachable from v; then by definition we have a dipath from v to w of length $\le \text{ii-diam}(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y}, \mathcal{S}) \le 2d \log' p - d < 2d \log' p$. Otherwise, either $v = \alpha$ and then we must have a dipath of length $\le 1 + \text{ib-diam}(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y}, \mathcal{S}) \le$

$1 + d\log' p \leq 2d\log' p$, or β is inherently $\mathcal{Y}$-reachable from v and w from β giving us a dipath from v to β to w of length at most $2\,\text{ib-diam}(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y}, \mathcal{S}) \leq 2d\log' p$. Thus we may conclude that the matched shortcutting $\mathcal{S}$ of $\mathcal{G}$ achieves a diameter of at most $2d\log' p = O(d\log p)$, as desired. By Key Lemma 8, we have $d = \delta_8(p) = O(\alpha(p,p))$, so, since $\mathcal{G}$ was an arbitrarily chosen rooted acyclic planar digraph, this completes the proof of Theorem 6 under the assumption that Key Lemma 8 holds true. ■

For brevity we introduce iib-diam := max{ii-diam, ib-diam}. Our proof of Key Lemma 8 starts with two technical lemmas simplifying the problem of finding adequate division shortcuttings.

Lemma 9 *Given an area X with a division D, if S is a shortcutting of* CrossExt(D,X) *we have*

$$\textit{iib-diam}(D,X,S) - 1 \leq \textit{ii-diam}(D,X,S) \leq \textit{diam}((\mathrm{CrossExt}(D,X) \cup S)|\mathcal{V}(X))$$

Proof: The last inequality follows trivially from the definitions. We will now prove the first inequality by proving $\text{ii-diam}(D,X,S) \geq \text{ib-diam}(D,X,S) - 1$. Let v and w be vertices with one in ∂X and one in $D|\mathcal{V}(X)$, such that w is inherently X-reachable from v, and such that a shortest dipath P in $\mathrm{CrossExt}(D,X) \cup S$ from v to w is of length $\text{ib-diam}(D,X,S)$. By symmetry we may assume that it is v which is in ∂X. Denote by u the second vertex in P, and by Q the part of P from u to w, i.e. $P = (v,u)Q$. Then Q is a shortest dipath in $\mathrm{CrossExt}(D,X) \cup S$ from u to w. Moreover, since w is inherently X-reachable from v, we have u in X and that w is inherently X-reachable from u. Hence by definition $\text{ii-diam}(D,X,S)$ is at least the length of Q, so $\text{ii-diam}(D,X,S) \geq \text{ib-diam}(D,X,S) - 1$, as desired. ■

Lemma 10 *Let X be a 2-sided area with a division D. Suppose S is a matched shortcutting of* $\mathrm{CrossExt}(D,X)|\mathcal{V}(X)$. *Then S can be extended to a division shortcutting S' of (D,X).*

Proof: Denote by C_1 the set of arcs in $\mathrm{Cross}(X)$ not occurring in $\mathrm{CrossExt}(D,X)$, and set $S' = S \cup C_1$. We want to show that S' is a division shortcutting (D,X), which follows if

$$|\mathrm{Cross}(X)| + |C_1 \cup S| \leq |\mathrm{CrossExt}(D,X)|.$$

The arcs (v,w) counted on the right hand side are divided into three sets: A_0 contains those where both v and w are in X; A_1 contains those where exactly one of v and w belongs to X, and A_2 contains those where both v and w are in ∂X. With C_2 denoting the complement of C_1 in $\mathrm{Cross}(X)$, we have

$$\begin{array}{ll} |C_2| \leq |A_2| & \text{since } C_2 \subseteq \mathrm{CrossExt}(D,X), \text{ and,} \\ |S| \leq |A_0| & \text{since } S \text{ match } \mathrm{CrossExt}(D,X)|\mathcal{V}(X). \end{array}$$

On the left hand side, the arcs from C_1 are exactly the arcs that are counted twice, so the lemma follows if we can establish the following:

Claim 1 $2|C_1| \leq |A_1|$.

Proof: With each arc (v, w) in C_1 associate a dipath in CrossExt(D, X) from v to w. By definition of C_1 this dipath is of length at least 2. We will show that no two of these dipaths have a common end-arc. This will give us the desired result because both the first and the last arc are in A_1. Let L and R denote the left side and the right side of X. By symmetry, to prove the uniqueness of the end-arcs, it is sufficient to consider the case of a first arc starting in L. Thus let the crossover (v, w) be such that v is in L, and let (v, v_1) be the first arc in the associated dipath P. Since P is of length at least 2 we have v_1 in X, so (v, v_1) cannot be the last arc of any of the associated dipaths. Suppose that (v, v_1) is the first arc of one of the other associated dipaths P', and let w' be the last vertex in P'; so P' is the dipath associated with the crossover (v, w'). Since $\mathcal{G}$ is acyclic, neither w nor w' can be in L, for otherwise there would be a dipath from v to w or w' in $L \subseteq \partial X$ contradicting the definition of crossovers. Thus both w and w' must be in R, but then we have dipath between them in $R \subseteq \partial X$, so they cannot both be inherently X-reachable from v, so this contradicts that both (v, w) and (v, w') are crossovers. Hence we have established the desired uniqueness of the end-arcs of the associated dipaths. □

Summing up, we have shown that

$$|\mathrm{Cross}(X)| + |S'| \leq |C_1| + |C_2| + |S| + |C_1| \leq |A_0| + |A_1| + |A_2| = |\mathrm{CrossExt}(D, X)|,$$

and, moreover, by choice of C_1, we have $\mathrm{Cross}(X) \subseteq \mathrm{CrossExt}(X, D) \cup C_1 \subseteq \mathrm{CrossExt}(X, D) \cup S'$, so indeed S' is a division shortcutting. ■

The first application of the above two technical lemmas is yet another technical lemma.

Lemma 11 *Denote by $\delta_{11}(n)$ the least number such that if D is a division of an area X with n vertices, and, moreover, $D = (P \cup Q) \cap \mathcal{G}^+(X)$ where P and Q are dipaths in $\overline{X}$, then there is a division shortcutting S of (D, X) such that iib-diam$(D, X, S) \leq \delta_{11}(n)$. Then $\delta_{11}(n) = O(\alpha(n, n))$.*

Proof: Let $\boldsymbol{P}$ consist of the components of $P \setminus \mathcal{V}(\partial X)$ and let $\boldsymbol{Q}$ consist of the components of $Q \setminus \mathcal{V}(\partial X \cup P)$. Clearly, the elements of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are disjoint dipaths in $D|\mathcal{V}(X)$, and, moreover, any vertex from $D|\mathcal{V}(X)$ is in one of these dipaths. Let S be the union of optimal matched shortcuttings for each of the dipaths in $\boldsymbol{P} \cup \boldsymbol{Q}$. Then S is a matched shortcutting of $D|\mathcal{V}(X)$, and by Theorem 5, we have that S achieves a local diameter of at most $\delta_5(n) = O(\alpha(n, n))$ for each of the dipaths in $\boldsymbol{P} \cup \boldsymbol{Q}$.

Claim 1 *ii-diam$(D, X, S) = 3\delta_5(n) + 2$.*

Proof: Let v and w be vertices in $D|\mathcal{V}(X)$ such that w is inherently X-reachable from v. In particular, v and w are in $P \cup Q$. Suppose that neither v nor w are in P but that any X-dipath from v to w contains at least one vertex from P. Since P and Q are both dipaths and $\mathcal{G}$ is acyclic, we can find dipaths P_1, Q_1, Q_2 such that P_1 is a sub-dipath of P, Q_1 and Q_2 are sub-dipaths of Q, and $Q_1(v_1, v_2)P_1(v_3, v_4)Q_2$ is a dipath in CrossExt(D, X) from v to w. Here (v_1, v_2) and (v_3, v_4) are either crossovers from the areas in X/D, or arcs from Q with one end-vertex in P. None of P_1, Q_1, Q_2 can contain a vertex from ∂X, for that would imply the existence of a dipath in $\overline{X}$ from v to w intersecting ∂X, and this would contradict that w is *inherently* X-reachable from v. Hence P_1, Q_1, Q_2 are all segments of dipaths in $\boldsymbol{P} \cup \boldsymbol{Q}$, so we have a dipath in $\mathrm{CrossExt}(D, X)$ from v to w of length at most $3\delta_5(n) + 2$. It is straightforward to verify that our assumptions about the positions of v and w constitute a worst case, and therefore the claim follows. □

Now, S is a matched shortcutting of $D|\mathcal{V}(X) \subseteq \mathrm{CrossExt}(D, X)|\mathcal{V}(X)$, so by Lemma 10 we may extend S to a division shortcutting S' of (D, X). Clearly $\text{ii-diam}(D, X, S') \leq \text{ii-diam}(D, X, S)$, so by Lemma 9 we have $\text{iib-diam}(D, X, S') \leq 3\delta_5(n) + 3 = O(\alpha(n, n))$. ■

Notice that our proof of Lemma 11 can be generalized to work for any constant number of dipaths. This, however, is not relevant for this presentation. We are now ready for the first reduction of Key Lemma 8. The aim is to get down the number of vertices in the sub-areas.

Lemma 12 *Denote by $\delta_{12}(n)$ the least number such that for any 2-sided area X with n vertices, all reachable from ∂X, there is a division D with a division shortcutting S satisfying the following conditions*

(i) Any area Y in X/D is either 2-sided or 4-sided; if Y is 2-sided then any 2-sided strict sub-area of Y contains at most $n/2$ vertices, and if Y is 4-sided then Y contains at most $n/2$ vertices.

(ii) iib-diam$(D, X, S) \leq \delta_{12}(n)$.

Then $\delta_{12}(n) = O(\alpha(n, n))$.

Proof: Let X' be any 2-sided sub-area of X such that X' contains at least $n/2$ vertices and such that any 2-sided strict sub-area of X' contains at most $n/2$ vertices. Notice that possibly $X' = X$. Denote by L, R, s, and t the left side, the right side, the source corner, and the terminal corner of X, and denote by L', R', s', and t' the left side, the right side, the source corner, and the terminal corner of X'. Suppose that s' is not in ∂X. Since all vertices in X are reachable from ∂X, there is a dipath P from ∂X to s'. Choose the dipath P such that it intersects ∂X only at its starting point. Notice that since $\mathcal{G}$ is acyclic, s' is the only intersection between P and $\partial X'$. If $s' \in \partial X$, we set $P = (s')$. The division D is chosen as $(P \cup \partial X') \cap \mathcal{G}^+(X)$. Notice that $P \cup \partial X' = PL' \cup R'$ and that both PL' and R' are dipaths in $\overline{X'}$. Hence the result follows by Lemma 11 if we can show that D satisfies (i).

Before settling (i) we fix the meaning of a few more symbols. Denote by t_0 the second common vertex of L' and R'. Let L_0 and R_0 be the initial segments of L' and R' finishing in t_0, and let L_1 and R_1 be the final segments of L' and R' starting in t_0. Then $L' = L_0 L_1$ and $R' = R_0, R_1$. Notice that since $\mathcal{G}$ is acyclic, t_0 is the only intersection point between $L_0 \cup R_0$ and $L_1 \cup R_1$. The assignments of the variables is illustrated in Figure 2. Observe that $L_0 \cup R_0$ contains exactly the points of $\partial X'$ that can be reached by a curve in $\mathcal{X}$ starting in ∂X and only intersecting $\partial X' \cup \partial X$ in its end-points.

Now, by our choice of X', we have (i) satisfied for $Y = X'$. Moreover, since X' contains at least $n/2$ vertices, no other area in X/D can contain more than $n/2$ vertices. Hence (i) follows if all areas in $X/D \setminus \{X'\}$ are 2-sided or 4-sided.