$\mathbf{D_4}$, $\mathbf{E_8}$, Leech and Certain Other Lattices are Symplectic

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August 9, 1993

ABSTRACT

A recent paper of Buser and Sarnak has raised new interest in an old question of Schottky and others: which lattices arise from Jacobi varieties of compact Riemann surfaces? A necessary condition is that the lattice be symplectic. In this note it is shown that the root lattices D_4, E_8, K_{12}, Λ_{16}, the Leech lattice Λ_{24}, and certain others are symplectic.

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A recent paper of Buser and Sarnak [B-S] has raised new interest in an old question of Schottky and others: which lattices can arise from Jacobi varieties of compact Riemann surfaces (via the period matrix)? A necessary condition is that the lattice be symplectic. In this note it is shown that the root lattices D_4, E_8, the Leech lattice Λ_{24}, and certain others are symplectic. (This will appear as an appendix to [B-S].)

A $2n$-dimensional lattice Λ is symplectic if it has a Gram matrix P of the form

$$P = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & Y^{-1} \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} ,$$

with X, Y symmetric, $Y > 0$ (this is equivalent to the version used in the Introduction). Any two-dimensional lattice may be scaled to be symplectic. We show that suitably scaled versions of the lattices (named as in [C–S]) D_4, D_{4m}^{+} $(m \geq 1)$, $A_6^{(2)}$, E_8, Λ_{24} (the Leech lattice) and certain others, are symplectic.

Theorem. A lattice Λ is symplectic if and only if there is an orthogonal transformation ι on the space such that $\iota^2 = -1$ and $\iota(\Lambda)$ is the dual lattice Λ^*.

Proof. The following argument is rather more complicated than necessary but also yields a basis. (Only if:) We regard the matrix $\begin{bmatrix} Y & 0 \\ 0 & Y^{-1} \end{bmatrix}$ as the Gram matrix of the direct sum of a lattice L in a space V and a copy $\overline{L}^*$ of the dual lattice L^* in a space $\overline{V}$. Let $e_1, \ldots, e_n$ be an integral basis for L, $e_1^*, \ldots, e_n^*$ the dual basis for L^*, and $\overline{e}_1, \ldots, \overline{e}_n$ and $\overline{e}_1^*, \ldots, \overline{e}_n^*$ the corresponding bases for $\overline{L}$ and $\overline{L}^*$. The matrix $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ represents a base change under which

$\overline{e}_i^*$ is replaced by

$$-\sum_{j=1}^{n} x_{ij} e_j + \overline{e}_i^* \qquad (i = 1, \ldots, n) \ .$$

Then $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & Y^{-1} \end{bmatrix} \begin{bmatrix} I & -{}^tX \\ 0 & I \end{bmatrix}$ is the Gram matrix of a lattice whose basis vectors are

$$e_i, \quad -\sum_{j=1}^{n} x_{ij} e_j + \overline{e}_i^* \qquad (i = 1, \ldots, n) \ . \tag{A2.1}$$

The dual basis is

$$e_i^* + \sum_{j=1}^{n} x_{ji} \overline{e}_j, \quad \overline{e}_i \qquad (i = 1, \ldots, n) \ . \tag{A2.2}$$

The symmetry condition $x_{ij} = x_{ji}$ (all i, j) holds precisely when the transformation

$$\iota = \sum_{i=1}^{n} (a_i e_i + b_i \overline{e}_i) \mapsto \sum_{i=1}^{n} (b_i e_i - a_i \overline{e}_i)$$

maps Λ to Λ^*. (If:) Let ι be as stated in the theorem. We first select vectors $e_1', \ldots, e_n'$ which are linear combinations of vectors in Λ, choosing e_k' $(k = 1, \ldots, n)$ to be orthogonal to $\iota(e_1'), \ldots, \iota(e_{k-1}')$. e_k' is automatically orthogonal to $\iota(e_k')$. Let $V = \mathbb{R}\langle e_1', \ldots, e_n' \rangle$, $\overline{V} = \mathbb{R}\langle \iota(e_1'), \ldots, \iota(e_n') \rangle$. Let $e_1, \ldots, e_n$ be a basis for $\Lambda \cap V$, let $\overline{e}_1, \ldots, \overline{e}_n$ be their ι-images in $\overline{V}$,and let $\{e_i^*\} \subseteq V$, $\{\overline{e}_i^*\} \subseteq \overline{V}$ be the dual bases. Any $u \in \Lambda$ can be written $u = v + w$, $v \in V$, $w \in \overline{V}$. Since ι maps Λ to Λ^*, there are u's whose projections w include all the $\overline{e}_i^*$. Thus we can find a basis for Λ of the form (A2.1), whose dual basis is (A2.2). But applying ι to (A2.1) we obtain a second basis for Λ^*, namely

$$-e_i^* - \sum_{j=1}^{n} x_{ij} \overline{e}_j, \quad \overline{e}_i \qquad (i = 1, \ldots, n) \ . \tag{A2.3}$$

Comparing (A2.2), (A2.3) we see that x_{ij} and x_{ji} differ by integers, and so, by adding multiples of the $\overline{e}_j$'s, X may be taken to be symmetric.

Corollary. Let Λ be a classically integral lattice. Then Λ is symplectic if and only if it is unimodular and has an automorphism ι satisfying $\iota^2 = -1$.

The theorem shows that D_{4m}^+ $(m \geq 1)$, $A_6^{(2)}$, E_8 and Λ_{24} are symplectic (an ι for Λ_{24} is given in [C–S], p. 178). Some explicit matrices X, Y are as follows:

$$D_4: \qquad X = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \qquad Y = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$A_6^{(2)}: \qquad X = -\frac{1}{12}\begin{bmatrix} 4 & -2 & 4 \\ -2 & -1 & 6 \\ 4 & 6 & 8 \end{bmatrix}, \qquad Y = \frac{1}{\sqrt{7}}\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix},$$

$$E_8: \qquad X = -\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix},$$

$$\Lambda_{24}: \qquad X = -\frac{1}{2}\begin{bmatrix} 14 & 3j_{11} \\ 3\,{}^t j_{11} & C \end{bmatrix}, \qquad Y = \begin{bmatrix} 24 & -10 j_{11} \\ -10\,{}^t j_{11} & 2I_{11} + 4J_{11} \end{bmatrix},$$

where j_{11} is a vector of 11 1's, J_{11} is an 11×11 matrix of 1's, and C is a retro-circulant with first row $(1,1,0,1,1,1,0,0,0,1,0)$. An integral basis for Λ_{24} corresponding to this pair X, Y consists of the rows of the matrix

$$\begin{bmatrix} -4 & 4j_{11} & 0 & 0 \\ 2\,{}^t j_{11} & 4I_{11} - 2J_{11} & 0 & 0 \\ 5 & j_{11} & 9 & j_{11} \\ 0 & 2C & 2\,{}^t j_{11} & 2I_{11} \end{bmatrix}.$$

For E_8 the subgroup of the automorphism group that commutes with ι has structure 4.2^4S_6 and order 46080. For Λ_{24} the corresponding group is $(4 \times G_2(4)) : 2$ of order 2012774400.