Expressing $(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2)^3$ as a Sum of 23 Sixth Powers

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ABSTRACT

It is shown that $(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3$ can be written as a sum of 23 sixth powers of linear forms. This is one less than is required in Kempner's 1912 identity. There is a corresponding set of 23 points in the four-dimensional unit ball which provide an exact quadrature rule for homogeneous polynomials of degree 6 on S^3. It appears that this result is best possible, i.e. that no 22-term identity exists.

Mathematics Subject Classification: Primary 11P05 (Waring's problem); secondary 65D32 (quadrature formulas).

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1. Introduction

In connection with Waring's problem of expressing integers as a sum of sixth powers, Lucas published in 1876 the incorrect identity

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3 = \frac{1}{10} \sum_{i<j}^{(12)} (x_i \pm x_j)^6 \ ,$$

and in the following year replaced it by

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3 = \frac{2}{5} \sum^{(4)} x_i^6 + \frac{1}{10} \sum_{i<j}^{(12)} (x_i \pm x_j)^6 \ ,$$

also incorrect (cf. Dickson [Dic], p. 718). A correct version,

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3 = \frac{8}{15} \sum^{(4)} x_i^6 + \frac{1}{15} \sum_{i<j}^{(12)} (x_i \pm x_j)^6 + \frac{1}{120} \sum^{(8)} (x_1 \pm x_2 \pm x_3 \pm x_4)^6 \quad (1)$$

was apparently given for the first time by Kempner [Kem] in 1912.

More generally, for any even number q, we look for N points $P^{(i)} = [P_1^{(i)}, \ldots, P_d^{(i)}] \in \mathbb{R}^d$ and positive coefficients $c_i \in \mathbb{R}$, $1 \le i \le N$, such that there is an identity

$$(x_1^2 + \cdots + x_d^2)^{q/2} = \sum_{i=1}^{N} c_i \left(\sum_{j=1}^{d} P_j^{(i)} x_j \right)^q \ . \quad (2)$$

The existence of such identities, for all d and even q, with c_i and $P_j^{(i)}$ rational, was a key step in Hilbert's (1909) general solution of Waring's problem (Ellison [Ell]). Chapter XXV of Dickson [Dic] surveys the classical results concerning such identities. They have recently resurfaced in works by Lyubich and Vaserstein [LyV], Reznick [Rez] and Seidel [Sei], where many new results are given as well as connections with numerical quadrature, designs, and isometric embeddings of one space in another.

Let $N(d, q)$ denote the smallest value of N for which an identity (2) is possible. Equation (1) shows that $N(4, 6) \le 24$, and Proposition 9.2 of [Rez] shows that $N(4, 6) \ge 21$. The purpose of this note is to present a new identity

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3 = \sum_{i=1}^{23} c_i \left(\sum_{j=1}^{4} P_j^{(i)} x_j \right)^6 , \tag{3}$$

which establishes $N(4,6) \leq 23$. The points $P^{(i)}$ and coefficients c_i are given in Table 1. (However, since some of the c_i and $P^{(i)}$ are irrational, (3) does not contribute to Waring's problem.)

It follows from Proposition 3.14 of [LyV] or Corollary 8.27 of [Rez] that

$$\int_{S^3} f(x) dx \approx \frac{5\pi^2}{32} \sum_{i=1}^{23} c_i f(P^{(i)}) \tag{4}$$

is then a quadrature formula for S^3 which is exact for all homogeneous polynomials f of degree 6.

Equation (3) was discovered in the following way. Let us rewrite it as $\left(\sum_{i=1}^{4} x_i^2 \right)^3 = \sum_{i=1}^{23} \left(\sum_{j=1}^{4} Q_j^{(i)} x_j \right)^6$, where $Q_j^{(i)} = c_i^{1/6} P_j^{(i)}$. We used a computer to find sets of $Q_j^{(i)}$'s which satisfy this identity to within a small tolerance. The algorithm was a modification of that used in the GOSSET program for searching for experimental designs [HS1], [HS2], [HS3]. These sets of 23 points were then analyzed by hand. There appear to be many inequivalent solutions, and it was not easy to find a set of points with enough symmetry to identify their coordinates $Q_j^{(i)}$ as algebraic numbers. Having identified the $Q_j^{(i)}$, there is still the problem of finding a convenient way to factorize them as $c_i^{1/6} P_j^{(i)}$. There is no unique way to do this, and our aim was to make the final answer as simple as possible. (Equation (4) is exact for homogeneous polynomials of degree 6 for any such factorization.) The computer took a couple of hours to find each approximate solution, but "beautification" of the points by hand took a couple of weeks. We will not attempt to describe this fairly mysterious process, but just give our best answer, which emerged after a series of miraculous simplifications, in Table 1.

Our computer investigations have failed to produce a 22-term identity, and we conjecture that $N(4,6) = 23$.

We have also found sets of points which suggest that

$$N(3,10) \leq 24, \quad N(3,12) \leq 32, \quad N(3,14) \leq 41, \quad N(3,16) \leq 52, \quad N(3,18) \leq 66,$$
$$N(4,8) \leq 43, \quad N(6,6) \leq 63, \quad N(7,6) \leq 91, \quad N(8,4) \leq 45, \quad N(9,4) \leq 59 .$$

We are currently in the process of formally establishing these identities. Details will be published elsewhere.

2. The points $\mathbf{P^{(i)}}$ and the coefficients $\mathbf{c_i}$

The $P^{(i)}$ and c_i are listed in Table 1, both exactly (as algebraic numbers) and approximately (as decimal numbers, rounded to six significant figures). The geometric structure of the points is as follows. $P^{(1)}$ is at the north pole, and $P^{(2)}$ to $P^{(13)}$ form a regular icosahedron (with x_1 coordinate $\sqrt{3\tau}\,5^{-1/4}$, where $\tau = (1+\sqrt{5})/2$). The remaining ten points lie on the equatorial hyperplane $x_1 = 0$ and consist of a singleton and three equilateral triangles. The vertices of those equilateral triangles lie on the edges (possibly produced) of the icosahedron (with x_1 coordinate omitted). The only symmetries of the configuration are the cyclic shifts of the last three components. However, if the negatives or antipodes of the points are included, the resulting 46-point configuration has symmetry group of structure $C_2 \times C_2 \times D_6$, where D_6 is a dihedral group of order 6. This group contains the cyclic shift just mentioned and the reflection in $[0, 1, 1/\tau, -\tau]$, which together generate a three-dimensional triangular antiprismatic group (denoted by $[2^+, 6]$ in the notation of [CoM], by $2{*}3$ in the Conway-Thurston orbifold notation, and with structure $C_2 \times D_6$). The full group is generated by this subgroup and negation of the first coordinate.

Table 1(a). The points $P^{(1)}, \ldots, P^{(23)}$ in the identity (3). Double parentheses indicate that all cyclic shifts of the last three components are to be included.

[1		0	0	0]	=	[1		0	0	0]
[α	((	0	± 1	$\pm\tau$))]	=	[1.473370	((	0	± 1	± 1.618034))]
[0		1	1	1]	=	[0		1	1	1]
[0	((	1	$\tau - 1$	$-\tau$))]	=	[0	((	1	.618034	-1.618034))]
[0	((	$-\tau$	β	$\beta\tau + 1$))]	=	[0	((	-1.618034	1.868517	4.023324))]
[0	((	$-\tau$	γ	$\gamma\tau + 1$))]	=	[0	((	-1.618034	$-.535184$	.134054))]

where $\alpha = \sqrt{3\tau}\,5^{-1/4}$, $\beta = (2+\sqrt{13})/3$, $\gamma = (2-\sqrt{13})/3$.

Table 1(b). The coefficients $c_1, \ldots, c_{23}$ in the identity (3).

$$\frac{16}{25} = .64 \quad \text{(once)} ,$$

$$\frac{1}{36\tau^3\sqrt{5}} = .00293258 \quad \text{(twelve times)} ,$$

$$\frac{16}{735} = .0217687 \quad \text{(once)} ,$$

$$\frac{1}{120} = .00833333 \quad \text{(three times)} ,$$

$$-\frac{1460+\sqrt{5}-\sqrt{13}(577-76\sqrt{5})}{2^4\, 3^2\, 5\, 7^2\sqrt{13}} = .0000427301 \quad \text{(three times)}$$

$$\frac{1460+\sqrt{5}+\sqrt{13}(577-76\sqrt{5})}{2^4\, 3^2\, 5\, 7^2\sqrt{13}} = .0230332 \quad \text{(three times)}$$

The points $P^{(i)}$ lie on six different spheres, one for each of the six rows of Table 1(a). (So in the notation of [Rez], equation (3) is a sixth-caliber representation of $(x_1^2 + \cdots + x_4^2)^3$.)

As to the proof that (3) is an identity: the verification could be carried out by hand, but would be rather tedious. The c_i and $P^{(i)}$ are now simple enough, however, that a computer algebra system such as Maple [Map] is able to verify the identity exactly in a few seconds. Additional confirmation is provided by the fact that floating point evaluations of the coefficients on each side of (3) agree very closely.

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