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ON EDGE BIJECTIONS OF GRAPHS

by

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ABSTRACT

Let G_1 and G_2 be undirected graphs, and $\mathcal{F}_1(G_1)$ and $\mathcal{F}_2(G_2)$ be families of edge sets of G_1 and G_2, respectively. *An* $(\mathcal{F}_1, \mathcal{F}_2)$*-semi-isomorphism of* G_1 *onto* G_2 is an edge bijection between G_1 and G_2 that induces an injection from $\mathcal{F}_1(G_1)$ to $\mathcal{F}_2(G_2)$. This concept generalizes a well known concept of a circuit isomorphism of graphs due to H. Whitney. If $\mathcal{F}_1(G_1)$ has a "dual nature" with respect to $\mathcal{F}_2(G_2)$ then the concept of $(\mathcal{F}_1, \mathcal{F}_2)$-semi-isomorphism of graphs turns into a concept of *a* $(\mathcal{F}_1, \mathcal{F}_2)$*-semi-duality of graphs.* This gives a natural generalization of the circuit duality of graphs due to H.Whitney.

In this paper we investigate $(\mathcal{F}_1, \mathcal{F}_2)$-semi-isomorphisms and $(\mathcal{F}_1, \mathcal{F}_2)$-semi-dualities of graphs for verious families $\mathcal{F}_1(G_1)$ and $\mathcal{F}_2(G_2)$. In particular, we consider families of circuits and cocircuits of graphs from this point of view, and obtain some strengthenings of Whitney's 2-isomorphism theorem and Whitney's planarity criterion for 3-connected graphs.

Keywords graph, matroid, circuit, cocircuit, non-separating circuit, non-separating cocircuit, planarity, semi-isomorphism, semi-duality.

Contents

1 Introduction

In 1932–1933 H. Whitney introduced the concepts of a *circuit isomorphism* and a *circuit duality* (or *combinatorial duality*) of undirected graphs [21–24]. A *circuit isomorphism* of a graph G onto a graph G' is a bijection of the edge set $E(G)$ of G onto the edge set $E(G')$ of G' which induces a bijection of the set of circuits of G onto the set of circuits of G'. Similarly a *circuit duality* of G onto G' is a bijection of $E(G)$ onto $E(G')$ which induces a bijection of the set of circuits of G onto the set of cocircuits of G'. Whitney's circuit isomorphism (or 2-isomorphism) theorem claims that G and G' are circuit isomorphic if and only if one of them can be obtained from the other by a sequence of some simple operations [23]. Whitney's circuit duality theorem (or Whitney's planarity criterion of graphs) claims that G is planar if and only if it has a circuit dual graph G' (and G and G' are geometrically dual). Various strengthenings, refinements and generalizations of these interesting results have been found since than [4–12, 14–16, 18, 19]. In this paper we introduce some concepts which generalize the concepts of a circuit isomorphism and a circuit duality of graphs. Investigation of these concepts gives some new generalizations or strengthenings of the above classical results due to H. Whitney. In particular, we concider the concepts of a *circuit semi-isomorphism* and a *circuit semi-duality* of graphs. We prove that for 3-connected graphs Whitney's circuit isomorphism theorem remains true if we replace the concept "circuit isomorphism of graphs" by the concept "circuit semi-isomorphism of graphs", and so we get the corresponding strengthening of Whitney's circuit isomorphism theorem for 3-connected graphs. We also prove that for 3-connected graphs Whitney's planarity criterion remains true if the concept "circuit isomorphism of graphs" is replaced by the concept "circuit semi-isomorphism of graphs". This gives the corresponding strengthening of Whitney's planarity criterion for 3-connected graphs. The results of this paper were presented to the Moscow Discrete Mathematics Seminar in April, 1987, to the

22-nd Southeastern International Conference on Combinatorics, Graph Theory and Computing, Baton Rouge, Louisiana, February, 1991 (see also [17]).

2 Some notions and notation

All the concepts about graphs and matroids used but not defined here can be found in [2, 3, 20].

Undirected graphs are considered which may have *loops* and *parallel edges*. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. Given a vertex x of G let $E(x, G)$ or simply $E(x)$ denote the set of edges of G incident to x; we call $E(x, G)$ a *vertex star* of x in G. Put $|E(x, G)| = d(x, G)$. The number $d(x, G)$ or simply $d(x)$ is called the *degree* of the vertex x in G. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given $Z \in V(G) \cup E(G)$ let $G[Z]$ denote the *subgraph of G induced by Z*. In particular, if $Z \in V(G)$ then $G[Z] = G \setminus (V(G) \setminus Z)$, and if $Z \in E(G)$ then $E(G[Z]) = Z$, and $V(G[Z])$ is the set of vertices of G incident to at least one edge in Z.

Given two disjoint vertex subsets (or subgraphs) X and Y of G, let $[X, Y]$ denote the set of edges of G connecting X and Y.

An edge subset A of G is called a *cut* of G if $G \setminus A$ has more components than G.

A *path* P of G is a connected subgraph of G such that exactly two distinct vertices of P are of degree 1 and all the remaining vertices of P are of degree 2 in P. The vertices of degree 1 of a path are called the *end-vertices* of the path. A path P will sometimes be denoted by xPy to identify x and y as the *end-vertices* of P.

A graph F is *k-regular* if every vertex of F is of degree k.

A *circuit* C is a connected 2-regular graph. Let $\mathcal{C}(G)$ denote the set of circuit-subgraphs of G.

Let $\mathcal{C}[M]$ and $\mathcal{C}^*[M]$ denote the set of circuits and the set of cocircuits of a matroid M, respectively.

Let $M(G)$ denote the *matroid of a graph* G, i.e. the set $\mathcal{C}[M(G)]$ of circuits of $M(G)$ is defined by

$$\mathcal{C}[M(G)] = \{A \subseteq e(G) : A = E(C), C \in \mathcal{C}(G)\}.$$

We write $\mathcal{C}[G]$ and $\mathcal{C}^*[G]$ instead of $\mathcal{C}[M(G)]$ and $\mathcal{C}^*[M(G)]$, respectively.

It is well-known and it is easy to prove that a cocircuit of $M(G)$ has the following properties.

Proposition 2.1 *An edge set A of a graph G is a cocircuit of $M(G)$ if and only if A is a minimal (under inclusion) cut of G.*

Proposition 2.2 *An edge set A of a connected graph G is a cocircuit of $M(G)$ if and only if there exists a partition $\{X_1, X_2\}$ of $V(G)$ such that $A = [X_1, X_2]$ and $G[X_i]$ is connected for $i = 1, 2$.*

Given $e_1, e_2 \in E(G)$, we write $e_1 \sim^c e_2$ if either $e_1 = e_2$ or $e_1 \neq e_2$ and e_1, e_2 belong to a common circuit. It is easy to see that $\sim^c$ is an equivalence relation, and so $E(G)$ is the union of the disjoint $\sim^c$-equivalence classes in G. The subgraph of G induced by an $\sim^c$-equivalence class in G is called a *block of* G. A graph G is called *cyclically connected* if G has at list two edges, and any two edges of G belong to a common circuit (i.e. G has one block).

A graph G is called *2-connected*, if

(1) G has no loops and at least two vertices,

(2) for $|VG| = 2$, G has at least two parallel edges, and

(3) for $|VG| \geq 3$, the graph obtained from G by deleting any vertex is connected.

It is easy to see that

Proposition 2.3 *A graph G is cyclically connected if and only if the graph obtained from G by deleting the isolated vertices is 2-connected.*

A graph G is called *3-connected*, if

(1) G has neither loops nor parallel edges and at least four vertices, and

(2) the graph obtained from G by deleting any vertex or any two vertices is connected.

Let Λ be an algorithm which for any finite set E of elements and for any set $\mathcal{F} \subseteq 2^E$ of subsets of E gives a set of subsets of E: $\Lambda(E, \mathcal{F}) \subseteq 2^E$. Given a graph G, put $\Lambda(G) = \Lambda(E(G), \mathcal{C}[G])$ and $\Lambda^*(G) = \Lambda(E(G), \mathcal{C}^*[G])$.

Let us consider two graphs G and G' and two algorithms Λ and Λ' described above. A bijection $e : E(G) \to E(G')$ is called a *(Λ, Λ')–isomorphism* (a *(Λ, Λ')–semi-isomorphism*) *of G onto G'* if e induces a bijection (respectively, an injection) from $\Lambda(G)$ to $\Lambda'(G')$, i.e. if $A \in \Lambda(G) \Leftrightarrow e(A) \in \Lambda'(G')$ (respectively, $A \in \Lambda(G) \Rightarrow e(A) \in \Lambda'(G')$). Here $e(A) = \sum\{e(a) : a \in A\}$.

For a subgraph H of G we sometimes write $e(H)$ instead of $e(E(H))$.

We say Λ–*isomorphism* and Λ–*semi-isomorphism* instead of (Λ, Λ)-isomorphism and (Λ, Λ)-semi-isomorphism, respectively. We say Λ–*duality* and Λ–*semi-duality* instead of (Λ, Λ^*)–isomorphism and (Λ, Λ^*)–semi-isomorphism, respectively.

To contract an edge u of a graph G means to identify the end-vertices of G and to delete u from G. Given an edge subset A of a graph G, let $G \backslash A$ and G/A denote the graph obtained from G by deleting and by contracting the edges in A, respectively.

Given a cyclically connected (i.e. a 2-connected up to isolated vertices) graph G, a circuit A (a cocircuit A^*) of $M(G)$ is called *non-separating* if G/A (respectively, $G \backslash A^*$) is also cyclically connected, and *separating* otherwise. C is a *non-separating circuit of* G if $E(C)$ is a non-separating circuit of $M(G)$.

Let $\mathcal{N}[G]$ and $\mathcal{N}^*[G]$ denote the sets of non-separating circuits and non-separating cocircuits of $M(G)$, respectively.

The following statements on non-separating cocircuits of a graph are known [8] and can be easily proved.

Proposition 2.4 *Let G be a 2-connected graph. If A is a non-separating cocircuit of $M(G)$ then A is a vertex star of G, i.e. there exists a vertex x of G such that $A = E(x, G)$.*

Proposition 2.5 *Let G be a 2-connected graph. Every vertex star of G is a non-separating cocircuit of $M(G)$ if and only if G is either a 3-connected graph or can be obtained from a 3-connected graph by replacing edges by parallel edges.*

If $\Lambda(G) = \mathcal{C}[G]$ then we have the following concepts:

$\mathcal{C}$–isomorphism (or a *circuit isomorphism*),

$\mathcal{C}$–semi-isomorphism (or a *circuit semi-isomorphism*),

$\mathcal{C}$–duality (or a *circuit duality* or *combinatorial duality*),

$\mathcal{C}$–semi-duality (or a *circuit semi-duality*).

It is known and easy to see that $\mathcal{C}^*[G]$ is uniquely defined by $\mathcal{C}[G]$. So if Λ is an algorithm which gives $\mathcal{C}^*[G]$ from $(E, \mathcal{C}[G])$ then $\Lambda(G) = \mathcal{C}^*[G]$. Thus we have the following concepts:

$\mathcal{C}^*$–isomorphism (or a *cocircuit isomorphism*),

$\mathcal{C}^*$–semi-isomorphism (or a *cocircuit semi-isomorphism*),

$\mathcal{C}^*$–duality (or a *cocircuit duality*), and

$\mathcal{C}^*$–semi-duality (or a *cocircuit semi-duality*).

If $\Lambda(G)$ is either $\mathcal{N}$ or $\mathcal{N}^*$ (it is easy to see that the corresponding algorithms exist) then we have the concepts of $\mathcal{N}$– or $\mathcal{N}^*$–*isomorphism* and *semi-isomorphism*, as well as the concepts of $\mathcal{N}$– and $\mathcal{N}^*$–*duality* and *semi-duality*.

Obviously a circuit isomorphism is a cocircuit isomorphism and a circuit duality is a cocircuit duality. But for circuit and cocircuit semi-isomorphisms as well as for circuit and cocircuit semi-dualities the situation turns out to be different.

3 Preliminaries

The following Whitney circuit isomorphism theorem is classical [21, 23].

Theorem 3.1 (on a circuit isomorphism of 2-connected graphs).

Suppose that G is a 2-connected graph and $e : EG \to EG'$ is a circuit isomorphism of G onto G'. Then e is induced by a 2-isomorphism of G onto G'.

Corollary 3.1 (on a circuit isomorphism of 3-connected graphs).

Suppose that G is a 3-connected graph and $e : EG \to EG'$ is a circuit isomorphism of G onto G'. Then e is induced by an isomorphism of G onto G'.

The following circuit duality theorem is well-known as Whitney's planarity criterion for graphs [22, 24].

Theorem 3.2 (on a circuit duality of graphs).

A graph G is planar if and only if there exists a graph G^* which is circuit dual to G.

Many papers were devoted to refinements, strengthenings and generalizations of Whitney's circuit isomorphism theorem (see, for example, [4–12, 14–16, 18, 19]). Much fewer papers were devoted to development of Whitney's circuit duality theorem (see, for example, [16]).

One of the natural directions in which the above mentioned Whitney results can be developed is to consider semi-isomorphisms and semi-dualities of some types.

Here is one of these results. Let $\mathcal{F}(G)$ denote the set of edge subsets A of G such that $G[A]$ is a 2-regular subgraph of G. Obviously every component of a 2-regular graph is a circuit. We proved the following theorem (reported to the Moscow Discrete Mathematics Seminar in February 1976, published in [11]).

Theorem 3.3 (on a $(\mathcal{C}, \mathcal{F})$–semi-isomorphism of graphs).

Suppose that G is a 3-connected graph, G' is a connected graph, and $e : EG \to EG'$ is a $(\mathcal{C}, \mathcal{F})$–semi-isomorphism of G onto G'. Then e is induced by an isomorphism of G onto G'.

Corollary 3.2 (a circuit semi-isomorphism theorem).

Suppose that G is a 3-connected graph and $e : EG \to EG'$ is a circuit-semi-isomorphism of G onto G'. Then e is induced by an isomorphism of G onto G'.

This circuit semi-isomorphism theorem was proved in another way by Y.H. Sanders and D. Sanders [18].

In this paper we proceed these results by studying a cocircuit semi-isomorphism as well as a circuit and cocircuit semi-duality (and also some other semi-isomorphisms and semi-dualities) of graphs.

4 On $\mathcal{N}$-semi-duality of graphs

Theorem 4.1 (on $\mathcal{N}$-semi-duality of graphs).

Suppose that

(h1) G is a subdivision of a 3-connected graph, and

(h2) $e : EG \to EG^*$ is a bijection such that if A is a non-separating circuit of $M(G)$ then $e(A)$ is a non-separating cocircuit of $M(G^*)$ (i.e. e is an $\mathcal{N}$-semi-duality of G onto G^*).

Then

(a1) G and G^* are planar graphs, and

(a2) A is a circuit of $M(G)$ if and only if $e(A)$ is a cocircuit of $M(G^*)$ (i.e. e is a circuit duality of G onto G^*).

Proof.

By Proposition 2.4, a non-separating cocircuit of a graph is a vertex star. Therefore each edge of G^* belongs to at most two non-separating cocircuits of $M(G^*)$. By $(h2)$, each edge of G belongs to at most two non-separating circuits of $M(G)$. Since by $(h1)$, G is a subdivision of a 3-connected graph, it follows from Theorem 6.2 in [8] that each edge of G belongs to at least two non-separating circuits of $M(G)$. Hence each edge of G belongs to exactly two non-separating circuits of $M(G)$, and so by $(a2)$, each edge of G^* belongs to exactly two non-separating circuits of $M(G^*)$. It follows that each vertex star in G^* is a non-separating cocircuit of $M(G^*)$. Therefore G^* is a 3-connected graph up to parallel edges, and C is a non-separating circuit of $M(G)$ if and only if $e(C)$ is a non-separating cocircuit of $M(G^*)$. Since by $(h1)$, G is a subdivision of a 3-connected graph, it follows from Theorem 6.7 (the graph planarity criterion in terms of non-separating circuits) [8] that G is planar and for any embedding G_{emb} of G into the plane the set of circuit-faces of G_{emb} is exactly the set of non-separating circuits of G. Then from the above consideration it follows that G^* is also planar, and C is a circuit of $M(G)$ if and only if $e(C)$ is a cocircuit of $M(G)$, i.e. the matroid of G is dual to the matroid of G^*: $M(G) = M^*(G^*)$. □

5 On a circuit semi-duality of graphs

Lemma 5.1 (on 2-colored paths in a graph).

Suppose that

(h1) G is a 2-connected graph,

(h2) C is a non-separating circuit of G, i.e. $C \in \mathcal{N}(G)$,

(h3) there exists a path $x_1C'x_2$ of C such that each vertex of C' is of degree at least 3 in G and each vertex of $C \setminus C'$ is of degree 2 in G, and

(h4) the edge set $E(G)\setminus E(C)$ is colored arbitrarily in two colors: $E(G)\setminus E(C) = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, $E_1 \neq \emptyset$, and $E_2 \neq \emptyset$ (we say that an edge from E_i is of color i).

Then there exists a two-colored path xPy of G such that $C \cup P = \{x, y\}$, and x and y are adjacent vertices of C (i.e. $(x, y) \in E(C)$).

Proof.

Let G' denote the graph G/C obtained from G by contracting each edge of C. Then all vertices of C in G are identified with a single vertex in G', say, the vertex c. Obviously $E(G') = E(G) \setminus E(C)$ so that by $(h4)$, each edge of G' is of one of two colors 1 or 2, and there are edges of distinct colors. By $(h1)$, G is 2-connected. By $(h2)$, C is a non-separating circuit of G. Therefore G' is also 2-connected.

Let $\mathcal{R}$ denote the set of two-colored paths r_1Rr_2 in G with the property $C \cap R = \{r_1, r_2\} \subseteq C'$.

Let us first prove that $\mathcal{R} \neq \emptyset$. From $(h1) - (h3)$ it follows that there exist two edges $x_1a_1y_1$ and $x_2a_2y_2$ of G such that $C \cap a_i = x_i$, $i = 1, 2$, and $x_1 \neq x_2$. Clearly $a_1, a_2 \in E(G')$. Since G' is 2-connected, there exists a circuit A of G' containing both a_1 and a_2 . If A is two-colored then x_1Ax_2 is a required path of G. So suppose that A is of color 1, that is $E(A) \subseteq E_1$. Let z_1ez_2 be an edge form E_2. Since G' is 2-connected, there exist two disjoint paths $z_1Z_1s_1$ and $z_2Z_2s_2$ from $\{z_1, z_2\}$ to A in G'. Then the subgraph $A \cup Z_1z_1ez_2Z_2$ of G contains a required path.

Given $r_1Rr_2 \in \mathcal{R}$, let $t(R)$ denote the number of edges of the subpath $r_1C'r_2$ of the path C'. Let q_1Qq_2 be a path from $\mathcal{R}$ such that $t(Q) \leq t(R)$ for every $r_1Rr_2 \in \mathcal{R}$. We are going to prove that q_1 and q_2 are adjacent in C, i.e. that q_1Qq_2 is a required path.

If $t(Q) = 1$ then q_1Qq_2 is a required path. So let us assume that $t(Q) \geq 2$. Then there exists an inner vertex x of $q_1C'q_2$. Since $x \in C'$, it follows from $(h3)$ that x is of degree at least 3 in G. Hence $G \setminus E(C)$ contains an edge u with x as an end-vertex. Therefore u is an edge of $G' \setminus E(Q)$ and c is an end-vertex of u in G'. Let Q' denote the circuit in G' with the edge set $E(Q)$. Thus u does not belong to Q', and u and Q' have a common vertex c in G'. Since G' is 2-connected, there exists a path cPp of G' containing u and such that $Q' \cap P = \{c, p\}$ in G'. Consider two paths $q_iQ_ix = q_iQpPx$, $i = 1, 2$, in G. Then $C \cap Q = \{q_i, x\}$, $t(Q_i) < t(Q)$ for $i = 1, 2$ and at least one of the paths Q^1 and Q_2 is two-colored, a contradiction. $\square$

Lemma 5.2 (on transversal cocircuits with one common edge).

Suppose that

(h1) G is a 2-connected graph,

(h2) K is a separating cocircuit of $M(G)$ (i.e. $G \setminus K$ consists of at least two blocks), and

(h3) L is a cocircuit of $M(G)$ which contains exactly one edge, say e, from K and edges from at least two blocks of $G \setminus K$.

Then $K \Delta L = K \cap L \setminus e$ is not a cocircuit of $M(G)$.

Proof.

By $(h1)$, G is connected, and by $(h2)$, K is a cocircuit of $M(G)$. Therefore the graph $G \setminus K$ consists of exactly two components. Similarly $G \setminus L$ consists of exactly two components because L is a cocircuit of $M(G)$, by $(h3)$. Now $G \setminus (K \cup L)$ consists of exactly four components, since L contains edges from distinct blocks of $G \setminus K$. Hence $G \setminus (K \cup L \setminus e)$ consists of exactly three components. Therefore $K \Delta L = K \cup L \setminus e$ is not a cocircuit of $M(G)$. □

Theorem 5.1 (on a circuit semi-duality of graphs).

Suppose that

(h1) G is a 3-connected graph or is obtained from a 3-connected graph by subdividing at most one edge, and

(h2) $e : EG \rightarrow EG^*$ is a bijection such that if A is a circuit of $M(G)$ then $e(A)$ is a cocircuit of $M(G^*)$ (i.e. e is a circuit semi-duality of G onto G^*).

Then G and G^* are planar graphs and A is a circuit of $M(G)$ if and only if $e(C)$ is a cocircuit of $M(G^*)$. In other words, e is a circuit duality of G onto G^*, and the matroid of G is dual to that of G^*: $M(G) = M^*(G^*)$.

Proof (uses Theorem 4.1, and Lemmas 5.1 and 5.2).

We may assume that G^* has no isolated vertices. By $(h1)$, G is 2-connected. Therefore by $(h2)$, G^* is also 2-connected. If $e(A)$ is a non-separating cocircuit of $M(G^*)$ for every non-separating circuit A of $M(G)$ then by Theorem 4.1, we are done. Therefore we may assume that there exists a non-separating circuit A of $M(G)$ such that $C^* = e(A)$ is not a non-separating cocircuit of $M(G^*)$. By $(h2)$, C^* is a cocircuit of $M(G^*)$. Hence C^* is a separating cocircuit of $M(G^*)$. Therefore $G^* \setminus C^*$ consists of at least two blocks. Let B_1^* be a block of $G^* \setminus C^*$. Let $E_1^* = E(B_1^*)$, and $E_2^* = E(G^*) \setminus (C^* \cup E_1^*)$. Clearly $E_1^* \neq \emptyset$ and $E_2^* \neq \emptyset$.
Put $E_i = e^{-1}(E_i^*)$. Then $E(G) \setminus A = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, and $E_1 \neq \emptyset$, $E_2 \neq \emptyset$.

Let G' denote the graph G/A obtained from G by contracting each edge of A. Then $E(G') = E(G) \setminus A$. Let C be the circuit of G with the edge set A. By $(h1)$ and by the assumption on A and on E_1, E_2 we have: (G, C, E_1, E_2) satisfies the hypotheses $(h1)$ – $(h4)$ of Lemma 5.1. So by Lemma 5.1, there exists a path $q_1 Q q_2$ of G such that $Q \cup E_1 \neq \emptyset$, $Q \cup E_2 \neq \emptyset$, $C \cap Q = \{q_1, q_2\}$ and $q = (q_1, q_2)$ is an edge of C. Then $R = Q \cup q$ is a circuit of G such that $C \cap R = \{q_1, q, q_2\}$ and $R \cap E_1 \neq \emptyset$, and $R \cap E_1 \neq \emptyset$. Put $D = C \cup R \setminus q$. Put $R^* = e(R)$, $q^* = e(q)$, and $D^* = e(D)$. Then $C^* \cap R^* = q^*$, $D^* = C^* \cup R^* \setminus q^*$, and $R^* \cap E_1^* \neq \emptyset$, and $R^* \cap E_2^* \neq \emptyset$. By $(h2)$, R^* is a cocircuit of G^*. Therefore the graph G^* and its cocircuits C^* and R^* satisfy the hypothesis of Lemma 5.2. So by Lemma 5.2, D^* is not a cocircuit of G^*. Since the circuits C and R have in common only the edge q and its end-vertices q_1 and q_2, it follows that D is a circuit of G. By $(h2)$, $D^* = e(D)$ is a cocircuit of $M(G^*)$, a contradiction. $\square$

6 On an $\mathcal{N}^*$-semi-isomorphism of graphs

Theorem 6.1 (on an $\mathcal{N}^*$-semi-isomorphism of graphs).

Suppose that

(h1) G is a 3-connected graph or can be obtained from a 3-connected graph by replacing some of the edges by parallel edges and

(h2) $e : EG \to EG'$ is a bijection such that if K is a non-separating cocircuit of G then $K' = e(K)$ is a non-separating cocircuit of G' (i.e. e is an $\mathcal{N}^*$-semi-isomorphism of G onto G').

Then G and G' are isomorphic, and moreover there exists an isomorphism of G onto G' which induces e.

Proof.

Since G sayisfies $(h1)$, every vertex star of G is a non-separating cocircuit of G, by Proposition 2.5. Therefore every edge of G belongs to exactly two non-separating cocircuits of G. By $(h2)$, every edge of G' belongs to at least two non-separating cocircuit of G'. By Proposition 2.4, every non-separating cocircuit of G' is a vertex star of G'. Therefore every edge edge of G' belongs to at most two non-separating cocircuit of G'. Thus every edge of G' belongs to exactly two non-separating cocircuit of G', and an edge set of G' is a non-separating cocircuit of G' if and only if it is a vertex star of G'. Therefore e induces a bijection between the vertex stars of G and G'. This bijection gives the corresponding bijection between the vertices of G and G' which is an isomorphism of G onto G' inducing e. □

7 On a cocircuit semi-isomorphism of graphs

We say that a cocircuit $[V_1, V_2]$ of G *separates* disjoint vertex subsets X_1 and X_2 of G if $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$. In particular if X_i consists of a single vertex x_i, $i = 1, 2$, then we say that a cocircuit $[V_1, V_2]$ of G *separates* the vertices x_1 and x_2.

Lemma 7.1 (on a 2-colored $\{x, y\}$-cocircuit).

Suppose that

(h1) G *is a 2-connected graph, and*

(h2) the edge set $E(G)$ *is colored arbitrarily in two colors:* $E(G) = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, $E_1 \neq \emptyset$ *and* $E_2 \neq \emptyset$ *(we say that an edge from* E_i *is of color* i*).*

Then for any two distinct vertices x *and* y *of* G *there exists a two-colored cocircuit of* G *separating* x *and* y*.*

Proof. Let G^+ denote the graph obtained from G by adding an additional edge f with ends x and y. By $(h1)$, G^+ is 2-connected.

Let us assume that G has a one-colored path xPy, say, of color 1. By $(h2)$, G has an edge p_2 of color 2. Since G^+ is 2-connected, there exists a cocircuit K^+ of G containing both f and p_2. Then $K = K^+ \setminus f$ is a cocircuit of G separating x and y. Therefore K contains an edge (say, p_1) of the path P, and p_1 is of color 1. Thus $p_i \in K \cap E_i$ for $i = 1, 2$, and we are done.

Now let us assume that G has no one-colored paths from x to y. By $(h1)$, G is 2-connected. Therefore there exist two inner disjoint paths xP_1y and xP_2y of G connecting x and y. Then $R = P_1 \cup P_2$ is a circuit of G.

It is easy to show that for any circuit C in G and for any two distinct edges e_1 and e_2 of C there exists a cocircuit C^* in G such that $C \cap C^* = \{e_1, e_2\}$.

By the assumption, both P_1 and P_2 are two-colored paths. Therefore there exists an edge e_i of color i in P_i, $i = 1, 2$. By the above statement, there exists a cocircuit R^* in G such that $R \cap R^* = \{e_1, e_2\}$. Hence R^* is a two-colored cocircuit. Since $e_i \in P_i$, $i = 1, 2$, it follows that the cocircuit R^* separates x and y. □

A cocircuit $[V_1, V_2]$ of G *separates* disjoint vertex subsets X_1 and X_2 of G if $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$.

Lemma 7.2 (on a 2-colored $\{x, Y\}$-cocircuit).

Suppose that

(h1) G is a 2-connected graph, and

(h2) the edge set $E(G)$ is colored arbitrarily in two colors: $E(G) = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, $E_1 \neq \emptyset$ and $E_2 \neq \emptyset$ (we say that an edge from E_i is of color i).

Then for an arbitrary vertex-subset X of G with at least two vertices there exist two pairs (v_i, R_i), $i = 1, 2$, such that $v_1 \neq v_2$, $v_i \in X$ and R_i is a two-colored cocircuit separating v_i from $X \setminus v_i$ in G.

Proof (uses Lemma 7.1).

Given a cocircuit K of G let $G_1(K)$ and $G_2(K)$ be the two components of $G \setminus K$.

For $|X| = 2$, the lemma coincides with Lemma 7.1. So we may assume that $|X| \geq 3$. Let $x_1, x_2 \in X$. By Lemma 7.1, there exists a two-colored cocircuit K of G separating x_1 and x_2. Put $X_i = G_i(K^1) \cap X$, and so $x_i \in X$, $i = 1, 2$. If $|X_i| = 1$ then the pair (x_i, K^1) is a required pair (v_i, R_i). So we may assume that $|X_1| \geq 2$.

It is sufficient to prove that for some subset $S \subset X_1$ there exists a two-colored cocircuit separating S and $X \setminus S$. Then by induction on $|X \setminus S|$ we can find a vertex v_1 in X_1 and a two-colored cocircuit R_1 such that R_1 separates v_1 and $X \setminus v_1$, so that (v_1, R_1) is a required pair. By using the same argument, we can find another required pair (v_2, R_2) with $v_2 \in X_2$.

Let $y_1, y_2 \in X_1$. By Lemma 7.1, there exists a two-colored cocircuit K^2 separating y_1 and y_2 . Put $G_{i,j} = G_i(K^1) \cap G_j(K^2)$, $R^j = [G_{1,j}, G \setminus G_{1,j}]$ and $X_{i,j} = G_{i,j} \cap X$, $i, j = 1, 2$. We may assume that $y_j \in X_{1,j}$, $j = 1, 2$. If $X_{2,k} = \emptyset$ for some $k \in \{1, 2\}$ then K^2 is a required cocircuit and $S = X_{1,k}$. So we assume that $X_{i,j} \neq \emptyset$ for any $i, j = 1, 2$. Since both K^1 and K^2 are two-colored, either R^1 or R^2 is also two-colored, say, R^1 . If R^1 is a cocircuit then R^1 is a required cocircuit. So we can assume that R^1 is not a cocircuit. Since K^i is a cocircuit, $G_2(K^i)$ is connected, $i = 1, 2$. We have:

$\emptyset \neq X_{2,j} \subseteq G_{2,j} = G_2(K^1) \cap G_j(K^2)$.

Therefore $G_2(K^1) \cup G_j(K^2)$ is connected for $j = 1, 2$.

Since $G \setminus G_{1,j} = G_2(K^1) \cup G_j(K^2)$, it follows that $G \setminus G_{1,j}$ is connected.

Since $R^1 = [G_{1,1}, G \setminus G_{1,1}]$ is not a cocircuit, it follows that $G_{1,1}$ is not connected. Let $\mathcal{H}^j$ denote the set of components of $G_{1,j}$, $j = 1, 2$, and put $\mathcal{H} = \mathcal{H}^1 \cup \mathcal{H}^2$. For $H \in \mathcal{H}$ put $K(H) = [H, G \setminus H]$. Since $G_j(K^2)$ is connected, $[H, G_{2,j}] \neq \emptyset$ for any $H \in \mathcal{H}^j$. Therefore $K(H)$ is a cocircuit for any $H \in \mathcal{H}$.

Suppose that $K(D)$ is two-colored and $D \cap X \neq \emptyset$ for some $D \in \mathcal{H}$. Then $K(D)$ is a required cocircuit with $S = D \cap X$ because $D \cap X \subseteq X_{1,1} \subset X_1$. Hence we can assume that

(a) for every $H \in \mathcal{H}$ either $K(H)$ is one-colored or $H \cap X = \emptyset$.

Consider the following factor-graph F of the graph $G_1(K^1)$: $\mathcal{H}$ is the set of vertices of F and a pair $\{A, B\}$ of vertices of F forms an edge in F if and only if $[A, B] \neq \emptyset$. We say that a *vertex H of F is one-colored (of color i)* if $K(H)$ is one-colored (of color i), and *two-colored* otherwise. A *vertex H of F is* said to be *empty* if $H \cap X = \emptyset$, and *non-empty* otherwise. We say that an *edge (A, B) of F is one-colored (of color i)* if $[A, B]$ is one-colored (of color i), and *two-colored* otherwise. Then (a) is equivalent to

(a') every vertex of F is either one-colored or empty.

Let $T(F)$ and $N(F)$ denote the sets of two-colored and non-empty vertices of F, respectively. From (a') we have: $T(F) \cap N(F) = \emptyset$. Since $G_1(K^1)$ is connected and K^1 is two-colored, there exists $Q \in \mathcal{H}$ such that $K(Q)$ is two-colored. Hence F has a two-colored vertex Q, $Q \in T(F)$. Since $X_{1,i} \neq \emptyset$, there exists H^i in $\mathcal{H}^i$ such that

$H^i \cap X \neq \emptyset$, $i = 1, 2$. Hence F has at least two non-empty vertices, say, H^1 and H^2: $H^1, H^2 \in N(F)$. Since $G_1(K^1)$ is connected, F is also connected. Let P be a shortest path from $T(F)$ to $N(F)$ in F, and let T be the end-vertex of P in $T(F)$ and N be the end-vertex of P in $N(F)$.

Let W denote the subgraph of G induced by $\cup\{H : H \in V(P)\}$. Our goal is to prove that $Z = [W, G \setminus W]$ is a required cocircuit with $S = W \cap X$. In other words, we should prove that Z is a two-colored cocircuit and that $|S| < |X|$. Since P is a path and each vertex H of F is a connected subgraph of G, it follows that W is connected. Since $G_2(K^1)$ is connected and $[H, G_2(K^1)] \neq \emptyset$ for any $H \in \mathcal{H}$, we have: $G \setminus W$ is connected. Hence Z is a cocircuit. Since P is a shortest path from $T(F)$ to $N(F)$, all the vertices of P, except for T, are one-colored. Hence all the edges of P are one-colored and have the same color, say, 1. Then the cocircuit $K(N)$ of G is also of color 1. Since $\emptyset \neq [N, G_2(K^1)] \subseteq Z$, it follows that $Z \cap K(N) \neq \emptyset$, and so Z contains an edges of color 1. Since T is a two-colored vertex of F, we have: $K(T)$ is two-colored. Since $[T, W \setminus T]$ is of color 1, then $[T, G \setminus W]$ contains an edge of color 2. But $[T, G \setminus W] \subseteq [W, G \setminus W] = Z$. Hence Z contains an edge of color 2. Thus Z is a two-colored cocircuit of G. Now $W \cap X = N \cap X_1$ since N is the only non-empty vertex of P in F. Moreover $|N \cap X_1| < |X_1|$ because F has a non-empty vertex distinct from N. Therefore for $S = W \cap X$ we have: $|S| < |X_1|$. $\square$

Theorem 7.1 (on a cocircuit semi-isomorphism of graphs).

Suppose that

(h1) G is a 3-connected graph or is obtained from a 3-connected graph by replacing one edge by a number of parallel edges, and

(h2) $e : EG \to EG'$ is a bijection such that if K is a cocircuit of $M(G)$, then $K' = e(K)$ is a cocircuit of $M(G')$ (i.e. e is a cocircuit semi-isomorphism of G onto G').

Then G and G' are isomorphic, and moreover there exists an isomorphism of G onto G' which induces e.

Proof (uses Theorem 6.1, and Lemmas 7.2 and 5.2).

According to Theorem 6.1, it is sufficient to prove that if K is a non-separating cocircuit of $M(G)$ then $K' = e(K)$ is a non-separating cocircuit of $M(G')$. Let us assume the contrary, i.e. that there exists a non-separating cocircuit K of $M(G)$ such that $K' = e(K)$ is not a non-separating cocircuit of $M(G')$. By $(h2)$, K' is a cocircuit of $M(G')$. Thus K' is a separating cocircuit of $M(G')$. Therefore $G' \setminus K'$ consists of at least two blocks. Let B'_1 be a block of $G' \setminus K'$. Put $E'_1 = E(B'_1)$, and $E'_2 = E(G') \setminus (K' \cup E'_1)$. Clearly $E'_1 \neq \emptyset$, and $E'_2 \neq \emptyset$. Put $E_i = e^{-1}(E'_i)$. Clearly $EG \setminus K = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$ and $E_1 \neq \emptyset$, $E_2 \neq \emptyset$. By Proposition 2.4, a non-separating cocircuit of a graph is a vertex star of the graph. By the assumption, K is a non-separating cocircuit of $M(G)$. Therefore there exists a vertex x of G such that K is the vertex star of x, i.e. $K = E(x)$. Put $V(x) = X$ and $G^- = G \setminus x$. Then $E(G^-) = E(G) \setminus K$. By (h1), G is 3-connected. Therefore $|X| \geq 3$, and G^- is 2-connected. By Lemma 7.2, there exist two pairs (v_i, R_i), $i = 1, 2$, such that $v_1 \neq v_2$, $v_i \in X$ and R_i is a cocircuit of G^- separating v_i from $X \setminus v_i$ and having common edges with both E_1 and E_2. Put $K_i = R_i \cup [v_i, x]$. Since $|X| \geq 2$ and R_i separates v_i from $X \setminus v_i$, the set K_i is a cocircuit of G. Since by $(h1)$, G has at most one pair of vertices connected with more than one edge, at least one of vertices v_1, v_2, say v_1, is linked with x by a single edge. Denote this edge by d, i.e. $d = (v_1, x)$, and put $L = K_1$. Then $K \cap L = d$. Since R_1 is a cocircuit of G^- separating v_1 from $X \setminus v_1$, it follows that $D = K \cup L \setminus d$ is a cocircuit of G. Put $L' = e(L)$, $D' = e(D)$, and $d' = e(d)$. Then $K' \cap L' = d'$ and $D' = K' \cup L' \setminus d'$. Now by $(h2)$, D' is a cocircuit of G', because D is a cocircuit of G. From $(h1)$ and $(h2)$ it follows that G' is 2-connected. Since $R_1 \cap E_i \neq \emptyset$ and $L = R_1 \cup d$, we have $L \cap E_i \neq \emptyset$ for $i = 1, 2$. Hence $L' \cap E'_i \neq \emptyset$ for $i = 1, 2$. Thus G' and its cocircuits K' and L' satisfy the hypothesis of Lemma 5.2. Therefore by Lemma 5.2, $D' = K' \cup L' \setminus d'$ is not a cocircuit of G', a contradiction. □

8 On a cocircuit semi-duality of graphs

A *wheel* is a graph obtained from a graph-circuit (*the rim of the wheel*) by adding a new vertex (*the center of the wheel*) and the edges (*the spokes of the wheel*) connecting the new vertex with all the vertices of the circuit.

A *Möbius ladder* is a graph obtained from an even circuit (*the rim of the ladder*) by adding all main diagonals (*the spokes of the ladder*) of the circuit. A *Möbius graph* is a graph obtained from a Möbius ladder by contracting and/or subdividing some edges of the rim of the ladder. *The spokes of a Möbius graph* are the spokes of the corresponding ladder, and *the rim of a Möbius graph* is the subgraph obtained from the rim of the ladder.

Given a circuit C of a graph G, put $R(C) = [C, G \setminus C]$, $B(C) = R(C) \cup C$, $R'(C) = e(R(C))$, $C' = e(C)$ and $B'(C) = e(B(C))$. Let $H'(C) = H'$ be the subgraph of G' induced by $B'(C)$.

Lemma 8.1 *Suppose that*

(h1) G is a 3-connected graph,

(h2) $e : EG \to EG'$ is a bijection such that if K is a cocircuit of $M(G)$ then $K' = e(K)$ is a circuit of $M(G')$ (i.e. e is a cocircuit semi-duality of G onto G'), and

(h3) C is a non-separating circuit of G.

Then there are two possibilities for $H'(C)$:

(p1) H' is a wheel W, $R'(C)$ forms the rim and C' is the set of spokes of the wheel W, and

(p2) H' is a Möbius graph M, R' forms the rim of M and C' is the set of spokes of M.

Proof.

By $(h1)$, $G \setminus x$ is connected for every vertex x of G. Hence $E(x,G) = E(x)$ is a cocircuit of G. Put $E'(x) = e(E(x))$. Then by $(h2)$, $E'(x)$ is a circuit of $M(G')$. Obviously $B(C) = \cup\{E(x) : x \in V(C)\}$ and $R(C) = \Delta\{E(x) : x \in V(C)\}$. By $(h1)$, G has no parallel edges. Hence $|E(x) \cap E(y)|$ is either 0 or 1 for any distinct vertices x, y of G. Since by (h3), C is a non-separating circuit of G, it follows that C has no edge-chords. Therefore for any distinct vertices x, y of C we have: $E(x) \cap E(y) = (x,y)$, and $E'(x) \cap E'(y) = e(x,y)$ if (x,y) is an edge of C, and $E(x) \cap E(y) = E'(x) \cap E'(y) = \emptyset$ if (x,y) is not an edge of C. Since by $(h3)$, C is a non-separating circuit of G, it follows that $R(C)$ is a cocircuit of $M(G)$. By $(h2)$, $R'(C)$ is a circuit of $M(G')$. Now it is easy to see that there are only two possibilities $(p1)$ and $(p2)$ for $H'(C)$. □

Theorem 8.1 *Suppose that*

(h1) G is a 3-connected graph,

(h2) $e : EG \to EG'$ is a bijection such that if K is a cocircuit of $M(G)$ then $K' = e(K)$ is a circuit of $M(G')$ (i.e. e is a cocircuit semi-duality of G onto G'), and

(h3) for any non-separating circuit C of G the subgraph $H'(C)$ of G' is a wheel.

Then

(a1) G and G' are planar graphs, and

(a2) $M(G) = M^*(G')$.

Proof.

Put $C' = e(C)$. Let C be a non-separating circuit of G. Then by $(h3)$, the subgraph $H'(C)$ of G' is a wheel W'. Let c' be the vertex of G' which is the center of W'.

Then $C' \subseteq E(c', G')$. Let us prove that $C' = E(c', G')$. Assume the contrary, that is there exists an edge u' with an end-vertex c' which does not belong to C', i.e. $u' \in E(c', G') \setminus C'$. Put $u = e^{-1}(u')$. Then $u \in E(G) \setminus C$. Let v be an edge of C. Since by $(h1)$, G is 2-connected, there exists a cocircuit K of $M(G)$ which contains both u and v. Put $v' = e(v)$. Then $v' \in C' \subseteq E(c', G')$. By $(h2)$, $K' = e(K)$ is a circuit of G', and K' contains u' and $v' = e(v)$. Since $u', v' \in E(c', G')$ and K' is a circuit, it follows that $E(c', G') \cap K' = \{u', v'\}$. Then $K' \cap C' = \{v'\}$. Therefore $K \cap C = \{v\}$, and so $|K \cap C| = 1$. But $|K \cap C| \neq 1$ since C is a circuit and K is a cocircuit of G, a contradiction. Thus $C' = e(C)$ is a vertex star of G' for every non-separating circuit C of G. Since each edge of G' belongs two at most two vertex stars, each edge of G belongs to at most two non-separating circuits of G. By $(h1)$, G is 3-connected. Therefore by Theorem 6.2 in [8], each edge of G belongs to at least two non-separating circuits of G. Therefore each edge of G belongs to exactly two non-separating circuits of G, and e induces a bijection between the set of non-separating circuits of G and the set of vertex stars of G'. Since G is 3-connected, it follows from Theorem 6.7 in [8] that G is planar, G is uniquely embedable into the sphere, the set of non-separating circuits of G is the set of circuit-faces of G, and G' is the planar graph dual to G. Therefore $M(G) = M^*(G')$. □

Lemma 8.2 *Suppose that*

(h1) G is a 3-connected graph,

(h2) $e : EG \to EG'$ is a bijection such that if K is a cocircuit of $M(G)$ then $K' = e(K)$ is a circuit of $M(G')$ (i.e. e is a cocircuit semi-duality of G onto G'), and

(h3) C is a non-separating circuit of G such that $H'(C)$ is not a wheel.

Then each vertex of C is of degree at least 4 in G.

Proof (uses Lemma 8.1).

By $(h1)$, each vertex of G is of degree at least 3. By Lemma 8.1, $R'(C) = e(R(C)) = R'$ is a circuit of $M(G')$, and $C' = e(C)$ is the set of edge-hords of the circuit $G'[R']$ induced by R' in G'. Assume the contrary, that is there exists a vertex x in C of degree 3 in G, namely $E(x, G) = \{u_1, u_2, u_3\}$. Let $u_1, u_2 \in C$ and $u_3 \notin C$, i.e. $u_3 \in R(C)$. Then $e(u_1)$ and $e(u_2)$ are chords of the circuit $G'[R']$ and $e(u_3) \in R'$. By $(h2)$, the edge set $\{e(u_1), e(u_2), e(u_3)\}$ forms a triangle T' in G'. Therefore $e(R(C))\Delta e(E(x)) = e(R(C)\Delta E(x))$ is not a circuit of $M(G')$. But it is easy to see that $R(C)\Delta E(x)$ is a cocircuit of G. This contradicts $(h2)$. $\square$

Theorem 8.2 (on a cocircuit semi-duality of cubic graphs).

Suppose that

(h1) G is a 3-connected graph,

(h2) $e : EG \to EG^*$ is a bijection such that if K is a cocircuit of $M(G)$ then $K^* = e(K)$ is a circuit of $M(G^*)$ (i.e. e is a cocircuit duality of G onto G'), and

(h3) each non-separating circuit of G contains a vertex of degree 3.

Then G and G^* are planar graphs, and $M(G) = M^*(G')$.

Proof (uses Theorem 8.1 and Lemma 8.2).

By Theorem 8.1, it is sufficient to prove that for every non-separating circuit C of G the subgraph $H'(C)$ of G' is a wheel. Assume the contrary, that is there exists a non-separating circuit of G such that $H'(C)$ is not a wheel in G'. Then by Lemma 8.2, each vertex of C is of degree at least 4 in G. This contradicts $(h3)$. $\square$

Let $g(G)$ denote the length of a shortest circuit of a graph G. Clearly if A is a subset of $|V(G)| - g(G) + 1$ vertices of G then every circuit of G contains a vertex from A. Therefore we have from Theorem 8.2

Corollary 8.1 *Suppose that the hypotheses* $(h1)$ *and* $(h2)$ *of Theorem 8.2 hold. Then if* G *has at least* $|V(G)| - g(G) + 1 (\leq |V(G)| - 2)$ *vertices of degree 3 then* G *and* G^* *in Theorem 8.2 are planar graphs, and* $M(G) = M^*(G')$.

Theorem 8.3 .

Suppose that G *is a triangulation of the projective plane,* G^p *is the graph dual to* G *in the projective plane, and* $e : E(G) \to EG^p$ *is the corresponding bijection. Then*

(a0) G *is a 3-connected graph,*

(a1) if K *is a cocircuit of* $M(G)$ *then* $e(K)$ *is a circuit of* $M(G)$ *(i.e.* e *is a cocircuit semi-duality of* G *onto* G^p*), and*

(a2) there exists a circuit C^p *of* $M(G^p)$ *such that* $e^{-1}(C^p)$ *is not a cocircuit of* $M(G)$*, i.e.* e *is not a circuit duality (or cocircuit duality) of* G *onto* G^p*.*

Proof.

By Barnette's theorem [1], every triangulation of the projective plane can be obtained from one of the two special triangulations T_1 and T_2 by means of a sequence of operations of splitting a vertex along two given edges incident to the vertex. The graph of T_1 is the complete graph on 6 vertices, and the graph of T_2 is obtained from the complete graph on 7 vertices by deleting the three edges of a triangle. It can be checked that T_i and the graph T_i^p dual to T_i on the projective plane satisfy $(a0)$ and $(a1)$, $i = 1, 2$. Using Barnette's theorem it can be proved that $(a0)$ and $(a1)$ holds for every triangulation of the projective plane.

It is easy to prove directly that every graph-triangulation of the projective plane is not planar. (This fact also follows from Barnette's theorem because the graph of T_1 and T_2 are not planar.) Therefore by Theorem 3.2 (Whitney), $(a2)$ holds. □

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