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Proof.

(1). By Proposition 4.8, the maximal number of vertices of a type in $\mathcal{F}_d$ is 2^d, and it is easily verified that all polytope types of dimension $d \leq 2$ with at most 2^d vertices, namely the point, the segment, the triangle, and the quadrangle, have both a Delaunay embedding and a $[0,1]$-embedding.

(2). Let

$$U = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}, \qquad V = \{(0,0,1),(1,0,1),(2,1,1)\}$$

be two sets of integral points in $\mathbb{R}^3$. Thus, $conv(U)$ is an integral-free square in the hyperplane $\{x : x_3 = 0\}$, and $conv(V)$ is an integral-free triangle in the hyperplane $\{x : x_3 = 1\}$, and so $P = conv(U \cup V)$ is an integral-free polytope with type $\Delta \in \mathcal{F}_3$. By Theorem 5.1 in [8], no 3-dimensional Delaunay polytope has 7 vertices as does P, so $\Delta \notin \mathcal{D}_3$. Now, P has a quadrangle facet $conv(U)$ and a triangular facet $conv(V)$ which are disjoint. It is not hard to see that any $[0,1]$-polytope in $\mathbb{R}^3$ with this property is affinely equivalent to

$$Q = conv(U \cup \{(0,0,1),(1,0,1),(1,1,1)\}).$$

It is then easy to verify that Q has 3 quadrangular facets whereas P has only 2, so P and Q are not of the same type. Hence $\Delta \notin \mathcal{I}_3$ as well. Thus, $\Delta \in \mathcal{F}_3 \setminus (\mathcal{D}_3 \cup \mathcal{I}_3)$. The claim follows from Proposition 5.8 with $\mathcal{C}_d = \mathcal{F}_d$, $\mathcal{M}_d = \mathcal{D}_d \cup \mathcal{I}_d$, and $d_0 = 3$.

(3). The $[0,1]$-polytope Q in (2) has 7-vertices, so its type is in $\mathcal{I}_3 \setminus \mathcal{D}_3$, and the claim follows again from Proposition 5.8 with $\mathcal{C}_d = \mathcal{I}_d$, $\mathcal{M}_d = \mathcal{D}_d$, and $d_0 = 3$.

(4). Theorems 5.1 and 6.2 in [8] describe all five types in $\mathcal{D}_3$ and all nineteen types in $\mathcal{D}_4$. We have verified that each of these types has a $[0,1]$-embedding. The details are omitted.

(5). The layer-number of any $[0,1]$-polytope is at most 1, and so the width of any $[0,1]$-type is at most 1. By Proposition 5.2 and Example 4.2, we conclude that the type of the Schläfli polytope P_{SCH} is in $\mathcal{D}_6 \setminus \mathcal{I}_6$. The claim then follows from Proposition 5.8 with $\mathcal{C}_d = \mathcal{D}_d$, $\mathcal{M}_d = \mathcal{I}_d$, and $d_0 = 6$. □

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Proposition 5.6 *For $d \leq 1$ we have $\mathcal{F}_d = \mathcal{L}_d$. For $d \geq 2$, the class $\mathcal{L}_d$ of all d-dimensional lattice polytope types is infinite, whereas its subclass $\mathcal{F}_d$ of lattice-free types is finite.*

As a last preparation step before proving Theorem 5.9 below, we need to discuss the product of types. If Σ is the type of a face of an embedding of a type Δ, then Σ is a *face* of Δ. The *product* $\Delta_1 \times \Delta_2$ of two types Δ_1 and Δ_2 is the type of $P_1 \times P_2$ where P_i is any embedding of Δ_i $(i = 1, 2)$. It is well defined since, as mentioned before, the faces of $P_1 \times P_2$ are the products $F_1 \times F_2$ where F_i is a face of P_i $(i = 1, 2)$, so the type of $P_1 \times P_2$ depends only on the types of P_1 and P_2. A class $\mathcal{C}$ is closed under *taking products* if $\Delta_1 \times \Delta_2$ is in $\mathcal{C}$ whenever Δ_1 and Δ_2 are. It is closed under *taking faces* if any face Σ of $\Delta \in \mathcal{C}$ is also in $\mathcal{C}$. Observations 2.3 and 4.7 yield the following analogue.

Observation 5.7 *The classes of lattice-free types, Delaunay types, and $[0, k]$-types are closed under taking products and under taking faces.*

Proposition 5.8 *Let $\mathcal{C} = \cup_{d \geq 0} \mathcal{C}_d$ be a type class closed under taking products, and let $\mathcal{M} = \cup_{d \geq 0} \mathcal{M}_d$ be a type class closed under taking faces. Assume further that the segment type belongs to $\mathcal{C}$. If $\mathcal{C}_{d_0} \not\subseteq \mathcal{M}_{d_0}$ for some d_0, then $\mathcal{C}_d \not\subseteq \mathcal{M}_d$ for all $d \geq d_0$.*

Proof. Let Σ denote the segment, so that $\Sigma \in \mathcal{C}_1$. Pick any $\Delta \in \mathcal{C}_{d_0} \setminus \mathcal{M}_{d_0}$. Given $d > d_0$, let

$$\Delta' = \Delta \times \Sigma \times \cdots \times \Sigma$$

be the product of Δ with the $(d - d_0)$-fold product of Σ. Then $\Delta' \in \mathcal{C}_d$. But Δ is face of Δ' which is not in $\mathcal{M}$, so Δ' can not be in $\mathcal{M}$ either. □

Let $\mathcal{D}$ and $\mathcal{I}$ denote, respectively, the classes of Delaunay types and $[0, 1]$-types. Delaunay types and $[0, 1]$-types are lattice-free, so $\mathcal{D}_d \cup \mathcal{I}_d \subseteq \mathcal{F}_d$ for all $d \geq 0$. We now prove the main theorem of this section, which compares the three lattice-free type classes $\mathcal{F}$, $\mathcal{D}$ and $\mathcal{I}$. The proof will make use of the complete lists of types in $\mathcal{D}_3$ and $\mathcal{D}_4$ provided in Theorems 5.1 and 6.2, respectively, of [8]. Our theorem below asserts, in particular, that there are infinitely many lattice-free types which are neither Delaunay nor $[0, 1]$, Delaunay types which are not $[0, 1]$, and $[0, 1]$-types which are not Delaunay. The only relation unsettled by the theorem is whether or not $\mathcal{D}_5 \subset \mathcal{I}_5$.

Theorem 5.9 *The following relations among lattice-free types, Delaunay types, and $[0, 1]$-types, hold:*

(1)	$\mathcal{D}_d = \mathcal{I}_d = \mathcal{F}_d$	$(d \leq 2)$
(2)	$\mathcal{D}_d \cup \mathcal{I}_d \neq \mathcal{F}_d$	$(d \geq 3)$
(3)	$\mathcal{I}_d \not\subseteq \mathcal{D}_d$	$(d \geq 3)$
(4)	$\mathcal{D}_d \subset \mathcal{I}_d$	$(d = 3, 4)$
(5)	$\mathcal{D}_d \not\subseteq \mathcal{I}_d$	$(d \geq 6)$.

between their vertex sets such that $conv(F)$ is a face of P if and only if $conv(\phi(F))$ is a face of Q for all $F \subseteq vert(P)$. If two polytopes are affinely equivalent then they are also of the same type. We will use Δ, Σ to denote types. An *embedding* of a type is any polytope of that type. Combinatorial invariants of polytopes are defined naturally for types, where the invariant for a type is taken as the corresponding invariant of any embedding of the type. For example, with a type Δ are associated its *dimension* $dim(\Delta)$ and the numbers of faces of various dimensions. The index of a polytope (Definition 3.3) is also a combinatorial invariant. Another invariant of interest to us here is introduced below.

Definition 5.1 The *width* $W(\Delta)$ of a type Δ is the smallest value of the layer-number $w(P)$ (Definition 3.1) over all embeddings P of Δ. The *width* $W(P)$ of a polytope P is the width of its type.

Proposition 5.2 *The width of the Schläfli polytope satisfies* $W(P_{\rm SCH}) = 2$.

Proof. By Proposition 3.5 we have $W(P_{\rm SCH}) \leq w(P_{\rm SCH}) = 2$. Now, let P be a polytope of the same type as $P_{\rm SCH}$ achieving $W(P_{\rm SCH}) = W(P) = w(P)$. Since the index of a polytope is a type invariant, we have $i(P) = i(P_{\rm SCH}) > 2$ (see proof of Proposition 3.5). Thus, by Proposition 3.4, we have $w(P) \neq 1$, so $W(P_{\rm SCH}) = W(P) = w(P) \geq 2$ and we are done. □

Definition 5.3 A *lattice-free type* (respectively, *Delaunay type*, $[0,k]$*-type*) is a type that admits a lattice-free (respectively, Delaunay, $[0,k]$) embedding.

The Schläfli polytope provides an example of a lattice-free type whose width is larger than one. Let $g(d)$ be the maximum value of $W(\Delta)$ over all d-dimensional lattice-free types Δ. Note that $g(d) \leq f(d)$ for all d, where $f(d)$ is the function discussed in Question 3.9. Thus, $g(d) \leq cd^2$ for some constant c, $g(d) \leq 1$ for $d \leq 3$ and $g(d) \geq 2$ for $d \geq 6$, and the exact value of $g(d)$ is unknown for all $d \geq 4$.

Question 5.4 *How fast does* $g(d)$ *grow with* d*?*

Propositions 2.5, 4.3, and 4.6 yield the following statement.

Proposition 5.5 *Any lattice-free (respectively, Delaunay,* $[0,k]$*) type has a full dimensional lattice-free (respectively, Delaunay,* $[0,k]$*) embedding.*

We write $\mathcal{C}_d$ for the subclass of d-dimensional types in a type-class $\mathcal{C}$, so that $\mathcal{C} = \cup_{d\geq 0}\mathcal{C}_d$. Denote by $\mathcal{L} = \cup_{d\geq 0}\mathcal{L}_d$ the class of all lattice (equivalently, rational) polytope types and by $\mathcal{F} = \cup_{d\geq 0}\mathcal{F}_d$ the class of all lattice-free types. Thus, for example, $\mathcal{L}_1$ and $\mathcal{F}_1$ consist of the *segment* type only. However, already $\mathcal{L}_2$ consists of all infinitely many *n-gon* types ($n \geq 3$), while, by Proposition 4.8, the number of vertices of any d-dimensional lattice-free type is bounded above by 2^d, implying that there can be only finitely many d-dimensional lattice-free types. We summarize this in the proposition below. It contrasts Theorem 4.9 which asserted that, up to affine equivalence, there are infinitely many lattice-free polytopes in each sufficiently large dimension.

Proof. By Voronoi's theorem (Proposition 4.4), for each d, there are only finitely many d-dimensional affinely nonequivalent Delaunay polytopes.

The analogous statement for $[0,k]$-polytopes follows from Proposition 4.6: for any d, there are only finitely many polytopes with vertex set contained in $\{0,\cdots,k\}^d$.

Next, we consider low dimensional lattice-free polytopes. The situation for dimensions $d \leq 1$ being obvious, we consider 2-dimensional polygones in the plane. The triangle $conv\{0,e_1,e_2\}$ and the parallelogram $conv\{0,e_1,e_2,e_1+e_2\}$ are $[0,k]$-polytopes for all positive k, and are integral Delaunay polytopes. It remains to show then, that up to affine equivalence, there are no other integral-free polygones. By Proposition 4.8, a lattice-free polygon has at most 4 vertices. We need to show that any lattice quadrangle is a parallelogram. Without loss of generality assume that $P = conv\{0,v_1,u,v_2\}$ is an integral-free quadrangle with u lying in the convex cone generated by v_1 and v_2. Now, the triangle $conv\{0,v_1,v_2\}$ is integral-free, which implies ([13, Theorem 4, p. 20]) that $\{v_1,v_2\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^2$ (note that the analogue to this statement in higher dimensions is false). Thus, $u = n_1v_1 + n_2v_2$ for some positive integers n_1 and n_2, so $v_1 + v_2 \in conv\{0,v_1,n_1v_1+n_2v_2,v_2\}$. Since P is integral-free, we must have that $v_1 + v_2 = u$, and so P is a parallelogram.

Finally, we construct infinitely many affinely nonequivalent integral-free polytopes in each dimension $d \geq 3$. Consider first the case $d = 3$. For $n \geq 1$, let

$$S_n = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,n)\} \subset \mathbb{R}^3.$$

Then S_n is a set of points in general position and $D(S_n) = n$, and $P_n = conv(S_n)$ is a 3-dimensional integral-free polytope. Since $D(S_m) \neq D(S_n)$ for all $1 \leq m < n$, there is no nonsingular affine transformation that takes S_m to S_n, and so P_m and P_n are affinely nonequivalent. Now, define $P_{d,n}$ for all $d \geq 3$ and $n \geq 1$ recursively by putting $P_{3,n} = P_n$ and setting $P_{d+1,n} = conv(P_{d,n} \cup \{e_{d+1}\}) \subset \mathbb{R}^{d+1}$, a pyramid over $P_{d,n}$. By induction on d, we claim that all $P_{d,n}$ are integral-free and that $P_{d,m}$ and $P_{d,n}$ are affinely nonequivalent for all $1 \leq m < n$. Indeed, $P_{d+1,n}$ is integral-free since $P_{d,n}$ is; also, an affine bijection of $P_{d+1,m}$ onto $P_{d+1,n}$ must take the face $P_{d,m}$ of $P_{d+1,m}$ bijectively onto the face $P_{d,n}$ of $P_{d+1,n}$, which is impossible by induction. This completes the proof. □

5 Type-Classes of lattice-free polytopes

We turn now to compare the class of *combinatorial types* of lattice-free polytopes and its subclasses of Delaunay polytopes and $[0,1]$-polytopes. We start with a few definitions and propositions and with a comparison of the class of lattice-free types and the class of all integral polytope types.

The *type* of a polytope $P \subset \mathbb{R}^d$ is its equivalence class under isomorphism of face lattices (cf. [14]). Equivalently, two polytopes Q and P are of the same type if and only if there exists a bijection

$$\phi : vert(P) \longrightarrow vert(Q)$$

$B_i = \{x_i \in \mathbb{R}^{n_i} : \ ||x_i - c_i|| \leq r_i\}$ be a Λ_i-hole, and let $P_i = conv(B_i \cap \Lambda_i)$ be a Delaunay Λ_i-polytope. Now let

$$B = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \ ||(x_1, x_2) - (c_1, c_2)||^2 \leq r_1^2 + r_2^2\}.$$

Then it is not hard to check that B is a $(\Lambda_1 \times \Lambda_2)$-hole and that $(B_1 \cap \Lambda_1) \times (B_2 \cap \Lambda_2) = B \cap (\Lambda_1 \times \Lambda_2)$. Thus,

$$P_1 \times P_2 = conv(B_1 \cap \Lambda_1) \times conv(B_2 \cap \Lambda_2) = conv((B_1 \cap \Lambda_1) \times (B_2 \cap \Lambda_2)) = conv(B \cap (\Lambda_1 \times \Lambda_2))$$

is a Delaunay $(\Lambda_1 \times \Lambda_2)$-polytope. □

We need the following proposition of [27].

Proposition 4.8 *A d-dimensional lattice-free polytope has at most 2^d vertices.*

Proof. Let $P \subset \mathbb{R}^d$ be an integral polytope with more than 2^d vertices. Then it has two vertices $v = (v_1, \cdots, v_d)$ and $v = (v_1, \cdots, v_d)$ such that the integers v_i and u_i have the same parity for all i. Thus, $\frac{1}{2}(v + u)$ is an integral point in P which is not a vertex, so P is not integral-free. Since any lattice-free polytope is affinely equivalent to a full-dimensional integral-free one (Proposition 2.5), the claim follows. □

Remark. In solving a problem of Erdös, Danzer and Grünbaum [5] showed that the maximum number of points in $\mathbb{R}^d$ no three of which form an obtused triangle is 2^d. The bound for the class of Delaunay polytopes follows from this result as well as from Proposition 4.8 above.

As a final preparation for Theorem 4.9 below, we need to discuss the *spread* of a simplicial point configuration [10]. Given an affinely independent set of points $S = \{v_0, \cdots, v_d\} \subset \mathbb{R}^d$, let $V(S)$ denote the (positive) volume of the simplex $conv(S)$, given, up to a normalizing constant, by the absolute value of the determinant of $[v_1 - v_0, \cdots, v_d - v_0]$. For a finite set $T \subset \mathbb{R}^d$ of points in general position, i.e., no $d + 1$ of which are affinely dependent, define (see [10]) the *spread* $D(T)$ of T as the ratio

$$D(T) = \frac{max_S V(S)}{min_S V(S)},$$

where S ranges over all $(d+1)$-subsets of T. The spread is invariant under nonsingular affine transformations on $\mathbb{R}^d$, since the ratio $\frac{V(\phi(S))}{V(S)}$ is the same for any d-simplex $S \subset \mathbb{R}^d$.

Theorem 4.9 *For any d, up to affine equivalence, there are only finitely many d-dimensional Delaunay polytopes and $[0, k]$-polytopes for each fixed positive k. Up to affine equivalence, the class of lattice-free polytopes of dimension $d \leq 2$ coincides with each of its subclasses above, and consist of the point, the segment, the triangle and the parallelogram. However, for all $d \geq 3$, there are infinitely many affinely nonequivalent d-dimensional lattice-free polytopes.*

Definition 4.5 A $[0,k]$-*polytope* is one whose vertices lie in $\{0,\cdots,k\}^n$ for some n.

Of course, any rational polytope is homothetic to a $[0,k]$-polytope for some k, but for a fixed k, the $[0,k]$-polytopes form a proper subclass. Note also that the layer-number of a $[0,k]$-polytope is at most k. A class of special interest is that of $[0,1]$-polytopes. Clearly, $[0,1]$-polytopes are integral-free.

The definition of a $[0,k]$-polytope is not invariant under affine transformations, so the following question arises: does any d-dimensional $[0,k]$-polytope in $\mathbb{R}^n$ have a full dimensional affinely equivalent $[0,k]$-polytope in $\mathbb{R}^d$? Proposition 4.6 below provides a positive answer, which will be used in Theorem 4.9. The argument is essentially the same as the one used in [23] for a special case regarding 1-skeletons; the second author thanks W. Pulleyblank for drawing his attention to this fact.

Proposition 4.6 *Any $[0,k]$-polytope is affinely equivalent to a full dimensional $[0,k]$-polytope.*

Proof. We prove by induction on $n-d$ that any d-dimensional $[0,k]$-polytope $P \subset \mathbb{R}^n$ has an affinely equivalent full dimensional $[0,k]$-polytope in $\mathbb{R}^d$. The statement holding trivially for $n-d=0$, assume that $n-d \geq 1$ and let $P \subset \mathbb{R}^n$ be a d-dimensional $[0,k]$-polytope. Let $a=(a_1,\cdots,a_n) \in \mathbb{R}^n$ be a nonzero element in the orthogonal complement of $lin(P-P)$. Without loss of generality assume that $a_n \neq 0$, and consider the projection

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1} : \ (x_1,\cdots,x_n) \mapsto (x_1,\cdots,x_{n-1}).$$

We now show that π is injective on P. Suppose $x,y \in P$ have the same image under π. Then

$$\mathbf{0} = \pi(x) - \pi(y) = \sum_{i=1}^{n-1} (x_i - y_i)e_i,$$

so $x_i - y_i = 0$ for $i = 1,\cdots,n-1$. Since $a \in (lin(P-P))^{\perp}$, we get

$$0 = \langle a, x-y \rangle = \sum_{i=1}^{n} a_i(x_i - y_i) = a_n(x_n - y_n).$$

Since $a_n \neq 0$, we find that $x_n - y_n = 0$ as well, and so $x=y$. Thus, $\pi(P) \subset \mathbb{R}^{n-1}$ is affinely equivalent to P, and is clearly a $[0,k]$-polytope. By induction, $\pi(P)$, and hence P, has an affinely equivalent full dimensional $[0,k]$-polytope in $\mathbb{R}^d$, and we are done. □

We have an analogue of Observation 2.3 for Delaunay polytopes and $[0,k]$-polytopes.

Observation 4.7 *The classes of Delaunay polytopes and $[0,k]$-polytopes are closed under taking products and under taking faces.*

Proof. The only part that requires some justification is that the product of two Delaunay polytopes is a Delaunay polytope again. By Proposition 4.3 we may assume the two polytopes we start with to be full dimensional. For $i=1,2$ let then $\Lambda_i \subset \mathbb{R}^{n_i}$ be a lattice, let

Definition 4.1 A *Delaunay polytope* is one which is a Delaunay Λ-polytope for some lattice Λ.

Examples of Delaunay polytopes in suitable lattices include, for all d, the d-dimensional simplex, cross-polytope, and cube. Of special interest to us is the Schläfli polytope.

Example 4.2 *Schläfli polytope.* Let $E_8 \subset \mathbb{R}^8$ be the root lattice having the $\mathbb{Z}$-basis as in Example 2.4, so that the Schläfli polytope $P_{\rm SCH}$ is E_8-free. Let

$$A = \{x = (x_1, \cdots, x_8) \in \mathbb{R}^8 : \ x_1 + x_2 = 1, \ \ \frac{1}{2}\sum_{i=1}^{8} x_i = 1\},$$

and let $B = \{x \in A : \ ||x|| \leq \sqrt{2}\}$. Then A is a 6-dimensional affine subspace of $\mathbb{R}^8$ and B is a 6-ball, and it is easy to verify that the vertex set of $P_{\rm SCH}$ lies on the relative boundary of B in A. It can be verified that, moreover, $B \cap E_8 = vert(P_{\rm SCH})$ (for more details see [7]). Thus, the Schläfli polytope $P_{\rm SCH}$ is a Delaunay polytope.

If $\Lambda \subset \mathbb{R}^n$ is a lattice and $B \subset \mathbb{R}^n$ a d-dimensional Λ-hole, then there is an affine map $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ which maps $\Lambda \cap aff(B)$ bijectively onto $\mathbb{Z}^d$ and the Delaunay Λ-polytope $P = conv(B \cap \Lambda)$ onto a full dimensional integral-free polytope (cf. Proposition 2.5). Letting $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a nonsingular affine map which maps the ellipsoid $\phi(B)$ onto a ball, we find that $\psi(\mathbb{Z}^d)$ is a lattice, $\psi(\phi(B))$ is a $\psi(\mathbb{Z}^d)$-hole, and $\psi(\phi(P))$ is a Delaunay polytope. Thus, the following statement holds.

Proposition 4.3 *Any Delaunay polytope is affinely equivalent to a full dimensional Delaunay polytope.*

Delaunay polytopes are closely related to *Voronoi polytopes* [30]. Recall that, given a full dimensional lattice $\Lambda \subset \mathbb{R}^d$, the Λ-Voronoi polytope of $x \in \Lambda$ is the set of all points in $\mathbb{R}^d$ the distance of which from x is no larger than from any other lattice point (for example, the 24-cell mentioned in Example 2.1 is the Voronoi polytope of the lattice F_4 [4]). Then the vertices of the Λ-Voronoi polytope of x are precisely the centers of the full dimensional Λ-holes that contain x. Both the collection of all Λ-Voronoi polytopes and the collection of all full dimensional Delaunay Λ-polytopes form tilings of $\mathbb{R}^d$. In [30], Voronoi established the following fundamental result.

Proposition 4.4 *For any d there are only finitely many affinely nonequivalent Delaunay tilings of $\mathbb{R}^d$.*

This implies, in particular, that there are only finitely many d-dimensional affinely nonequivalent Delaunay polytopes. For a more accessible proof of this later fact see [6].

Next, we consider $[0,k]$-polytopes - those which are embeddable on a fixed grid of side length k. Here we loose no generality by assuming the lattice to be the integral lattice, and we let $\{0, \cdots, k\}^n$ be the set of integral points in $\mathbb{Z}^n$ all coordinates of which (with respect to the standard basis) are in $\{0, 1, \cdots, k\}$.

Question 3.12 *Is the diameter of a lattice-free polytope at most its dimension?*

Let $\Delta(n,d)$ be the largest diameter of any d-polytope with at most n facets. The *Hirsch conjecture* is that $\Delta(n,d) = n-d$, but it is yet unknown whether $\Delta(n,d)$ is bounded from above by any polynomial in n and d (see [19] for more information). Existing upper bounds on $\Delta(n,d)$ include $n2^{d-3}$ [21] and n^{logd+2} [17]. The first of these can be combined with the results above to obtain the following more general bound for lattice polytopes.

Theorem 3.13 *There exists a constant c for which the following holds: the diameter of any d-dimensional lattice polytope P with no more than n facets and the property that for some $k \leq d$ no face of P of dimension larger than k contains a lattice point in its relative interior, satisfies*

$$\delta(P) \leq cd^2(d-k) + (n-(d-k))2^k.$$

Proof. First note that any facet F of a polytope P has fewer facets than P since each facet of F is the intersection of F and another facet of P. We use induction on $d-k$. If $d-k=0$ then the bound is $n2^d$ and is guaranteed by [21]. Assume then that $d-k \geq 1$. Any proper face F of P has at most $n-1$ facets. By induction and using Proposition 3.10 and Theorem 3.8 we thus obtain

$$\delta(P) \leq cd^2 + (c(d-1)^2(d-1-k) + (n-1-(d-1-k))2^k) \leq cd^2(d-k) + (n-(d-k))2^k. \quad \square$$

Applying this theorem with $k = log(d)$, we obtain a polynomial bound on the diameter for a larger class of lattice polytopes.

Corollary 3.14 *There exists a constant c for which the following holds: the diameter of any d-dimensional lattice polytope P with no more than n facets and the property that no face of P of dimension larger than $log(d)$ contains a lattice point in its relative interior, satisfies*

$$\delta(P) \leq nd + cd^3.$$

4 Delaunay polytopes and $[0,k]$-polytopes

In this section we discuss two important classes of lattice polytopes: those which are embeddable on an empty sphere, and those which are embeddable on a fixed size grid. We derive some properties of such embeddings which will be used also in the following section, and investigate the finiteness of the number of affinely nonequivalent polytopes in these classes.

We start with lattice-free polytopes which are embeddable on a sphere - Delaunay polytopes. A *ball* in $\mathbb{R}^n$ is any subset $\{x \in A : ||x-c|| \leq r\}$, where r is a nonnegative real number, $c \in \mathbb{R}^n$, and A is an affine subspace of $\mathbb{R}^n$. A *d-ball* is a ball whose affine dimension is d. Given a lattice $\Lambda \subset \mathbb{R}^n$, a d-ball in $\mathbb{R}^n$ is a *Λ-hole* if it contains $d+1$ affinely independent lattice points in its relative boundary (in its affine hull), whereas its relative interior contains none. A *Delaunay Λ-polytope* is the convex hull of the lattice points in some Λ-hole. In particular, a Delaunay Λ-polytope is Λ-free.

We now turn to discuss the diameter of lattice-free polytopes. Denote by $d(u,v)$ the distance between two vertices u and v of P on its 1-skeleton, i.e. the smallest number of edges in a walk from u to v. The *diameter* of P is

$$\delta(P) = max\{d(u,v) : \ u,v \in vert(P)\}.$$

The following result is from [20]. We include the proof for completeness.

Proposition 3.10 *Let P be a polytope of positive dimension. Then*

$$\delta(P) \leq w(P) + max\{\delta(F) : \ F < P\}.$$

Proof. Pick $a \in lin(P-P)$ satisfying $w(P) = w(P,a)$, and let $F' = F(P,-a)$ and $F'' = F(P,a)$ be the two distinct proper faces of P in directions $-a$ and a, respectively. Pick two vertices u and v of P such that $\delta(P) = d(u,v)$. There exists a (possibly empty) path on the 1-skeleton of P from u to some vertex $u' \in F'$, where the vertices along the path attain strictly decreasing values under $\langle a, \cdot \rangle$. Similarly, there exists a strictly decreasing path from v to some $v' \in F'$, and strictly increasing paths from u and v to some vertices u'' and v'' in F'', respectively. Now

$$\begin{aligned} min\{d(u,u') + d(v',v),\ d(u,u'') + d(v'',v)\} &\leq \frac{1}{2}(d(u,u') + d(v',v) + d(u,u'') + d(v'',v)) \\ &\leq \frac{1}{2}(w(P,a) + w(P,a)) = w(P,a), \end{aligned}$$

since $d(u',u) + d(u,u'') \leq w(P,a)$ and $d(v',v) + d(v,v'') \leq w(P,a)$. Assuming then that, say, $d(u,u') + d(v',v) \leq w(P,a)$ and noting that every edge of F' is also an edge of P, we find

$$\begin{aligned} \delta(P) &= d(u,v) \leq d(u,u') + d(u',v') + d(v',v) \\ &\leq w(P,a) + \delta(F') \leq w(P) + max\{\delta(F) : \ F < P\}. \quad \square \end{aligned}$$

We now prove that the diameter of any lattice-free polytope is bounded by a low degree polynomial in its dimension.

Theorem 3.11 *There exists a constant c such that the diameter of any d-dimensional lattice-free polytope P satisfies $\delta(P) \leq cd^3$.*

Proof. We use induction on the dimension d of P. If $d = 0$ then $\delta(P) = 0$ as well. Assume then that $d \geq 1$. Any proper face F of P is also lattice-free and has smaller dimension than d, so by induction $\delta(F) \leq c(d-1)^3$. By Proposition 3.10 and Theorem 3.8 we then obtain

$$\delta(P) \leq w(P) + max\{\delta(F) : \ F < P\} \leq cd^2 + c(d-1)^3 \leq cd^3. \quad \square$$

It is known (see [22] and [20]) that the diameter of a d-dimensional $[0,1]$-polytope (see next section) is at most d. As we shall see, $[0,1]$-polytopes form a strict subclass of lattice-free polytopes, and so the following question remains.

We can now establish the following theorem.

Theorem 3.8 *There exists a constant c such that the layer-number of any d-dimensional lattice-free polytope P satisfies $w(P) \leq cd^2$. For $d \leq 3$, all d-dimensional lattice-free polytopes satisfy $w(P) \leq 1$, but there are infinitely many lattice-free polytopes of dimension $d \geq 6$ that satisfy $w(P) \geq 2$.*

Proof. Let P be any lattice-free d-polytope, $d \geq 1$. Since $w(P)$ is an affine invariant, by Proposition 2.5 we may assume that $P \subset \mathbb{R}^d$ is a full dimensional integral-free polytope. Let c be the constant guaranteed in Proposition 3.7 and choose λ in the range

$$\frac{cd^2}{\lfloor cd^2 \rfloor + 1} < \lambda < 1.$$

Let v be any point in the interior of P and let K be obtained by shrinking P by factor λ about v, that is

$$K = \lambda P + (1-\lambda)v = \{\lambda x + (1-\lambda)v : \ x \in P\}.$$

Then $K \subset int(P)$ so that K is a convex body containing no integral points. Thus, by Proposition 3.7 there exists $a \in \mathbb{Z}^d$ such that

$$\begin{aligned} z &:= max\{\langle a, x\rangle : \ x \in P\} - min\{\langle a, x\rangle : \ x \in P\} \\ &= \frac{1}{\lambda}(max\{\langle a, x\rangle : \ x \in K\} - min\{\langle a, x\rangle : \ x \in K\}) \leq \frac{1}{\lambda} cd^2 < \lfloor cd^2 \rfloor + 1. \end{aligned}$$

Now, the maximum and minimum over P are attained at vertices, which are integral points, and $a \in \mathbb{Z}^d$, so z is an integer, and so in fact $z \leq \lfloor cd^2 \rfloor \leq cd^2$. Moreover, for any two vertices u, v of P, if $\langle a, v\rangle > \langle a, u\rangle$ then in fact $\langle a, v\rangle \geq \langle a, u\rangle + 1$, and so

$$w(P) \leq w(P, a) = |\{\langle a, v\rangle : \ v \in vert(P)\}| - 1 \leq z \leq cd^2$$

and we are done.

The statement about dimensions $d \leq 3$ is provided by Proposition 3.2.

Finally, the Schläffi polytope is a 6-dimensional lattice-free polytope with $i(P_{\text{SCH}}) > 2$. By Observation 2.3 and Proposition 3.6, for any $k \geq 1$ the k-fold product

$$P_k = P_{\text{SCH}} \times \cdots \times P_{\text{SCH}}$$

of the Schläffi polytope is a $6k$-dimensional lattice-free polytope of index $i(P_k) = i(P_{\text{SCH}}) > 2$. Thus, by Proposition 3.4, $w(P_k) \geq 2$. □

Let $f(d)$ be the maximum value of $w(P)$ over all d-dimensional lattice-free polytopes P. Theorem 3.8 asserts that $f(d) \leq cd^2$ for some constant c, that $f(d) \leq 1$ for $d \leq 3$, and that $f(d) \geq 2$ for $d \geq 6$. The exact value of $f(d)$ is unknown for all $d \geq 4$, and in particular it is unknown whether $f(4) > 1$ and $f(5) > 1$. Of most interest is the following question.

Question 3.9 *How fast does $f(d)$ grow with d?*

Let $v(P)$ denote the number of vertices of a polytope P.

Definition 3.3 The *index* of a polytope P is the ratio

$$i(P) = \frac{v(P)}{max\{v(F):\ F < P\}}$$

of the number of vertices of P to the maximal number of vertices of a proper face of P (it is infinite when P is 0-dimensional).

Proposition 3.4 *If the layer-number of a polytope P satisfies $w(P) = 1$, then there are two proper faces of P that partition the vertex set of P, and the index of P satisfies $i(P) \leq 2$.*

Proof. Pick $a \in lin(P - P)$ such that $w(P, a) = w(P) = 1$. Then each vertex of P is in exactly one of the two proper faces $F(P, a)$ and $F(P, -a)$. One of these faces contains at least half the number of vertices of P, so the index of P is at most two. □

Proposition 3.5 *The layer-number of the Schläfli polytope satisfies $w(P_{\rm SCH}) = 2$.*

Proof. The vector $e_1 - e_2$ lies in $lin(P_{\rm SCH} - P_{\rm SCH})$, so $w(P_{\rm SCH}) \leq w(P_{\rm SCH}, e_1 - e_2) = 2$. Now, it is known (see [3, Section 4]), and can be verified by direct computation, that all proper faces of $P_{\rm SCH}$ are either simplices of dimension less than or equal to 5, or 5-dimensional cross-polytopes. Thus, the maximum number of vertices in a proper face of $P_{\rm SCH}$ is 10, and so $i(P_{\rm SCH}) = \frac{27}{10} > 2$. Then, by Proposition 3.4, we have $w(P_{\rm SCH}) \neq 1$. The claim follows. □

Proposition 3.6 *The index of the product of two polytopes satisfies*

$$i(P_1 \times P_2) = min\{i(P_1),\ i(P_2)\}.$$

Proof. The faces of the product $P_1 \times P_2$ are exactly the products $F_1 \times F_2$ where F_i is a face of P_i $(i = 1, 2)$. For $i = 1, 2$, let F_i be a proper face of P_i with maximum number of vertices. Then either $F_1 \times P_2$ or $P_1 \times F_2$ is a proper face of $P_1 \times P_2$ with maximum number of vertices. Thus, as claimed,

$$\begin{aligned} i(P_1 \times P_2) &= \frac{v(P_1 \times P_2)}{max\{v(F_1 \times P_2),\ v(P_1 \times F_2)\}} \\ &= min\{\frac{v(P_1)v(P_2)}{v(F_1)v(P_2)},\ \frac{v(P_1)v(P_2)}{v(P_1)v(F_2)}\} = min\{i(P_1),\ i(P_2)\}. \quad \square \end{aligned}$$

We need the following result of [18].

Proposition 3.7 *There exists a constant c with the following property: for any convex body K in $\mathbb{R}^d$ containing no integral points, there exists an element $a \in \mathbb{Z}^d$ such that*

$$max\{\langle a, x\rangle :\ x \in K\} - min\{\langle a, x\rangle :\ x \in K\} \leq cd^2.$$

Any $\mathbb{Z}$-submodule of $\mathbb{Z}^d$ is free (see e.g. [15, Chapter 2, Theorem 1.6]), and is moreover a lattice, since any $\mathbb{Z}$-basis for it is also $\mathbb{R}$-linearly independent. Thus, more generally, any $\mathbb{Z}$-submodule of any lattice is a lattice. In particular, the intersection $L \cap \Lambda$ of a lattice $\Lambda \subset \mathbb{R}^n$ and an $\mathbb{R}$-linear subspace $L \subset \mathbb{R}^n$ is a lattice. Thus, if P is a d-dimensional Λ-polytope then there is an isomorphism of affine spaces $aff(P) \longrightarrow \mathbb{R}^d$ which maps $\Lambda \cap aff(P)$ bijectively onto $\mathbb{Z}^d$. In particular, P is mapped bijectively onto an integral polytope which is integral-free if and only if P is Λ-free.

Proposition 2.5 *Any lattice-free polytope is affinely equivalent to a full dimensional integral-free polytope.*

Remark. With each integral d-polytope there are associated two integral vectors $h = (h_0, \cdots, h_d)$ and $h^* = (h_0^*, \cdots, h_d^*)$, the first being obtained from the faces enumerating vector of P and the second from the Ehrhart polynomial of P (see e.g. [29]). While $h_0 = h_0^*$ always, the polytope is integral-free if and only if $h_1 = h_1^*$ as well.

3 Layer-number and diameter of lattice-free polytopes

Let $lin(P - P)$ denote the $\mathbb{R}$-linear span of $P - P = \{x - y : \ x, y \in P\}$, that is, the linear subspace parallel to the affine hull of P. We use $F \leq P$ to denote that F is a face of P, possibly the empty face. For a polytope $P \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$ let

$$F(P, a) = \{x \in P : \ \langle a, y \rangle \leq \langle a, x \rangle \ \text{ for all } \ y \in P\}$$

be the face of P in direction a, and let

$$w(P, a) = |\{\langle a, v \rangle : \ v \in vert(P)\}| - 1$$

be the number of layers formed by the partition of the vertices of P according to their values under $\langle a, \cdot \rangle$.

Definition 3.1 The *layer-number* $w(P)$ of a polytope P is defined to be zero if $dim(P) = 0$ and otherwise by

$$w(P) = min\{w(P, a) : \ a \in lin(P - P) \setminus \{0\}\}.$$

It is invariant under affine equivalence.

Remark. A related, but different, notion of layer-number was considered in [9] in the context of $\mathbb{Z}$-equivalence. It is, though, of little relevance to our study here, and will not be discussed.

We shall use the following statement, a stronger version of which was proved in [27].

Proposition 3.2 *For $d \leq 3$, all d-dimensional lattice-free polytopes satisfy $w(P) \leq 1$.*

of any $\mathbb{Z}$-basis for it, and it is *full dimensional* if it is d-dimensional. The *integral lattice* is the lattice $\mathbb{Z}^d$ consisting of all *integral points* in $\mathbb{R}^d$, i.e., points all of which coordinates are integers. A *polytope* will always mean a convex polytope in some affine space $\mathbb{R}^n$. We denote by $vert(P)$ the vertex set of a polytope P and by $dim(P)$ its affine dimension. If $dim(P) = d$ then P is a *d-polytope*, and if P is a d-polytope in $\mathbb{R}^d$ then it is *full dimensional*. Two polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are *affinely equivalent* if there is an affine transformation $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ which maps P bijectively onto Q. Given a lattice $\Lambda \subset \mathbb{R}^d$, a polytope $P \subset \mathbb{R}^d$ is a Λ-*polytope* if all its vertices are in Λ, and is Λ-*free* if, moreover, it contains no lattice points other than its vertices, i.e., if $P \cap \Lambda = vert(P)$. A $\mathbb{Z}^d$-polytope (respectively, $\mathbb{Z}^d$-free polytope) will be called *integral* (respectively, integral-free).

Example 2.1 *Regular integral polytopes.* Out of all regular polytopes (see [4]), the ones which admit an integral embedding are the following ([25]): the regular d-dimensional simplex α_d, cross polytope β_d, and cube γ_d for all d, and, in addition, the regular hexagon and the 4-dimensional so-called 24-cell (see [4]). Not all of the above admit a full dimensional regular integral embedding: for example, the d-simplex has a regular embedding in $\mathbb{Z}^d$ if and only if $d+1$ is an odd square, the sum of two odd squares, or a multiple of four.

Definition 2.2 A *lattice-free* polytope is one which is Λ-free for some lattice Λ.

Given two sets $S_1 \subset \mathbb{R}^{n_1}$ and $S_2 \subset \mathbb{R}^{n_2}$, their product

$$S_1 \times S_2 = \{(x_1, x_2) : \ x_1 \in S_1, \ x_2 \in S_2\} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2}$$

is a polytope if S_1 and S_2 are and is a lattice if S_1 and S_2 are. The affine dimension of $S_1 \times S_2$ equals the sum of the affine dimensions of S_1 and S_2. If $\Lambda_i \subset \mathbb{R}^{n_i}$ is a lattice and P_i is a Λ_i-free polytope ($i = 1, 2$), then $P_1 \times P_2$ is $(\Lambda_1 \times \Lambda_2)$-free. Also, any face of a lattice-free polytope is lattice-free. We record this below.

Observation 2.3 *The class of lattice-free polytopes is closed under taking products and taking faces.*

We now give an example of an important lattice-free polytope, which will be used later on. It had been studied by several authors in connection with the 27 lines on the cubic surface. For a more detailed account of this polytope, see [3] and [4], and also [7].

Example 2.4 *Schläfli polytope.* Let P_{SCH} be the 6-dimensional *Schläfli polytope* (2_{21} in Coxeter notation [4]) in $\mathbb{R}^8$, with the 27 vertex set

$$\{e_i + e_j : \ i = 1, 2, \ \ 3 \le j \le 8\} \cup \{\frac{1}{2}\sum_{i=1}^{8} e_i - (e_j + e_k) : \ 3 \le j < k \le 8\} \subset \mathbb{R}^8.$$

It is an E_8-free polytope (moreover, it is a Delaunay polytope of E_8 - see next section), where E_8 is the root lattice having the $\mathbb{Z}$-basis

$$\{e_i - e_{i-1} : \ 3 \le i \le 8\} \cup \{e_1 + e_2, \ \frac{1}{2}(e_1 + \sum_{i=6}^{8} e_i - \sum_{i=2}^{5} e_i)\} \subset \mathbb{R}^8.$$

In the next section we fix some terminology and give an example of a classical lattice-free polytope - the *Schläfli polytope* - which will be of use throughout the article. Next, in Section 3 we consider the *layer-number* and *diameter* of lattice-free polytopes. The layer-number, an affine invariant of a polytope, is the smallest number of layers formed by its vertices when ordered under a linear functional. We show (Theorem 3.8) that the layer number of a d-dimensional lattice-free polytope is $O(d^2)$, and that, while it is known to be one in dimensions $d \leq 3$, it can be larger in dimension 6 and higher. The exact growth rate remains unsettled. We proceed to prove (Theorem 3.11) that the (combinatorial) diameter of a d-dimensional lattice-free polytope is $O(d^3)$, and deduce (Corollary 3.14) a polynomial upper bound $O(nd + d^3)$ on the diameter, in terms of the dimension d and number of facets n, for a large class of rational polytopes. This is of interest in connection with the well known Hirsch conjecture on the diameter of convex polytopes (see [19]).

In Section 4 we turn to discuss two important subclasses of lattice-free polytopes already mentioned above: Delaunay polytopes and $[0,1]$-polytopes (and more generally, $[0,k]$-polytopes for any fixed k). After establishing a few elementary properties of such polytopes (Proposition 4.6 being of interest in its own right), we prove Theorem 4.9. It asserts that for any d there are only finitely many d-dimensional affinely nonequivalent $[0,k]$-polytopes (for any fixed k) and Delaunay polytopes (the last statement being a fundamental theorem of Voronoi [30]), but for all $d \geq 3$ there are infinitely many affinely nonequivalent lattice-free polytopes. Lattice-free polytopes of dimension $d \leq 2$ are characterized up to affine equivalence.

Finally, in Section 5 we discuss and compare the combinatorial types of lattice-free polytopes and of the subclasses of $[0,1]$-polytopes and Delaunay polytopes. Theorem 5.9 establishes the inclusion relations among these type classes in various dimensions, and, in particular, asserts that there are infinitely many lattice-free types which are neither Delaunay nor $[0,1]$, Delaunay types which are not $[0,1]$, and $[0,1]$-types which are not Delaunay. An important ingredient in the proof is provided by the classification of three and four dimensional Delaunay polytopes in [8]. Another ingredient is a combinatorial invariant which we introduce - the *width* of a polytope (Definition 5.1), defined to be the smallest layer-number over all possible embeddings of the polytope. As for the layer-number, the rate of growth, as a function of the dimension, of the largest width of a lattice-free polytope type, is an open and seemingly hard question.

2 Lattice-free polytopes

Here we briefly discuss some basics of lattices and polytopes, and give some definitions and examples. As references for the geometry of lattices we suggest [13] and the survey article [11], and for convex polytopes [14]. Let $\mathbb{R}^d$ be Euclidean space equipped with the standard basis $\{e_1, \cdots, e_d\}$ and inner product $\langle e_i, e_j \rangle = \delta_{i,j}$. Thus, $\langle e_i, \cdot \rangle$ is the i-th coordinate function on $\mathbb{R}^d$ and an element $\sum_{i=1}^d x_i e_i \in \mathbb{R}^d$ is also the d-tuple $(x_1, \cdots, x_d)$ of its coordinates. A *lattice* in $\mathbb{R}^d$ is a $\mathbb{Z}$-submodule of $\mathbb{R}^d$ generated by a set of $\mathbb{R}$-linearly independent vectors, hence, in particular, is a free $\mathbb{Z}$-module. The *dimension* of the lattice is its rank, i.e. the size

1 Introduction

A convex polytope in real Euclidean space $\mathbb{R}^d$ is *lattice-free* if it intersects some lattice in $\mathbb{R}^d$ exactly in its vertex set. Lattice-free polytopes form a large class of rational polytopes, and arise in many combinatorial and algorithmic contexts. In this article we study affine and combinatorial properties of lattice-free polytopes. We give bounds, shared by all such polytopes, on some of their combinatorial and affine invariants. We also study important subclasses of lattice-free polytopes, and examine the inclusion relations among these subclasses.

Before describing the content of the article in more detail, we mention a few situations where lattice-free polytopes appear.

A first example is provided by $[0,1]$-polytopes - those with all vertices having all coordinates $\{0,1\}$-valued. Such polytopes arise when polyhedral methods are applied to combinatorial optimization problems (see, e.g., [12] for more information on this subject). They also provide a link between convex polytopes and finite set systems, a recent example being the resolution of Borsuk's problem [16].

Another example is given by complexes of maximal lattice free bodies and test sets for integer programming [1]. With a finite set $\mathcal{A} = \{a_0, \cdots, a_n\} \subset \mathbb{R}^d$ is associated the collection $\Delta(\mathcal{A})$ of all integral-free polytopes of the form

$$conv\{x \in \mathbb{Z}^d : \ a_i x \leq r_i, \quad i = 0, \cdots, n\},$$

obtained as $r_0, \cdots, r_n$ range over all real numbers. The points $x \in \mathbb{Z}^d$ for which $\{0, x\}$ is contained in a polytope of $\Delta(\mathcal{A})$, constitute a finite *test set*, which enables one to deduce global optimality from local optimality for each of the integer programs

$$min\{a_0 x : \ x \in \mathbb{Z}^d, \quad a_i x \leq s_i, \quad i = 1, \cdots, n\},$$

where $s_1, \cdots, s_n$ are any real numbers. For details see [1].

A third example comes from lattice tilings of Euclidean space. A *Delaunay* polytope is a lattice-free polytope whose vertices lie on a sphere. Unlike the collection $\Delta(\mathcal{A})$ above, whose polytopes may have intersecting interiors and is an extremely complex object, the collection of all Delaunay polytopes of a given lattice forms a tiling of Euclidean space since any two of its members intersect in a common face (see, e.g., [13] for more information).

Finally we mention that lattice-free polytopes are related to *unimodular* polytopes, which correspond to smooth projective toric varieties (see, e.g., [24]).

Lattice-free polytopes are, computationally, hard objects. Given a polytope with vertices in a lattice, deciding whether it is lattice-free (which is essentially integer programming) is NP-complete, though polynomial time solvable in fixed dimension (see [28]). Analogously, counting the number of lattice points contained in such a polytope is $\#P$-complete, but polynomial time solvable when the dimension is fixed, an algorithm having been found only very recently [2]. In fact, the decision and counting questions are hard even for simplices, and a good characterization of lattice-free simplices exists only for dimension at most three [26].

ABSTRACT

A convex polytope in real Euclidean space is *lattice-free* if it intersects some lattice in space exactly in its vertex set. Lattice-free polytopes form a large and computationally hard class, and arise in many combinatorial and algorithmic contexts.

In this article, affine and combinatorial properties of such polytopes are studied. First, bounds on some invariants, such as the diameter and width, are given. It is shown that the diameter of a d-dimensional lattice-free polytope is $O(d^3)$. A bound of $O(nd + d^3)$ on the diameter of a d-polytope with n facets is deduced for a large class of integral polytopes. Second, Delaunay polytopes and $[0,1]$-polytopes, which form major subclasses of lattice-free polytopes, are considered. It is shown that, up to affine equivalence, for any $d \geq 3$ there are infinitely many d-dimensional lattice-free polytopes but only finitely many Delaunay and $[0,1]$-polytopes. Combinatorial types of lattice-free polytopes are discussed, and the inclusion relations among the subclasses above are examined. It is shown that the classes of Delaunay types and $[0,1]$-types are mutually incomparable starting in dimension 6, and that both are strictly contained in the class of all lattice-free types.

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On Lattice-Free Polytopes [1]

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