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8 Disclaimer

There are many open problems concerning reliability polynomials that haven't been discussed herein. Connections with the chromatic polynomial [13, 18, 59, 61, 67] have been extensively studied, as have generalizations to properties of the Tutte polynomial [28, 32, 38, 67, 68, 72, 73]. Extensions to directed graphs (and polyhedral complexes) [6, 7, 52] and other reliability problems on matroids (e.g., [25]) have been studied.

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More generally, characterising where such roots can lie has the potential to give important information about the coefficients of reliability polynomials.

7 Derivatives and Crossings

In this section, we mention some miscellaneous problems on the "shape" of the reliability polynomial in the interval $p \in [0,1]$. Evidently $\mathrm{Rel}_\phi(G;0) = 0$, and $\mathrm{Rel}_\phi(G;1) = 1$, provided that $\phi(\emptyset) = 0$ and $\phi(E) = 1$. When ϕ is coherent, the curve is sigmoid (S–shaped) in the range $p \in [0,1]$ [8]. Moore and Shannon [49] proved the elegant theorem that for coherent structure functions, the reliability polynomial crosses the function p at most once in the range $p \in (0,1)$. This suggests two directions that remain open. The function p is itself a reliability polynomial, so we may ask about the number of times that two reliability polynomials can cross. More concretely, we have:

Open Problem 7.1 *For fixed n and m, how many times can the (all-terminal or two-terminal) reliability polynomials of two n–node, m–edge graphs cross in the range $0 < p < 1$?*

Naturally, it may happen that two reliability polynomials do not cross. They can cross once [40], and indeed they can cross twice [23]. Is there is constant k so that two such reliability polynomials cannot cross more than k times in the range of probabilities? If we consider crossings outside the interval of probabilities, the crossings are frequent as expected. Crossings when p is a probability seem relatively infrequent.

Another direction from the Moore-Shannon theorem is motivated by the observation that as p ranges from 0 to 1, we typically see three phases. Consider a small increase Δ in the value of p. The corresponding increase in the network reliability typically behaves as follows: at first, it is smaller than Δ, then it overtakes Δ, and near $p = 1$ is again less than Δ. The single crossing of p permitted by the Moore-Shannon theorem occurs in the middle portion when present. This suggests the following problem:

Open Problem 7.2 *Is it true that an (all-terminal or two-terminal) reliability polynomial has at most one point of inflection in the range $p \in (0,1)$? Equivalently, does its first derivative have a unique maximum? its second derivative at most one zero?*

Open Problem 5.5 *Is it true that for all–terminal reliability,*

$$H_0 \leq H_1 \leq \cdots \leq H_{\lfloor d/2 \rfloor}?$$

Using the interval partition based on passive and active elements of a matroid without coloops under an element ordering $\prec$, we find that an element that is active in a basis B is passive under the ordering $\succ$. This observation underlies a simple proof [16] of a theorem of Hibi [36]: if $(H_0, \ldots, H_d)$ is the H-vector of a matroid, then for all $j \leq \lfloor d/2 \rfloor$, $\sum_{i=0}^{j} H_i \leq \sum_{i=0}^{j} H_{d-i}$. Small examples suggest that a stronger condition may hold, namely that this inequality holds coefficientwise:

Open Problem 5.6 *Is it true for all-terminal reliability that for all $i \leq \lfloor d/2 \rfloor$, $H_i \leq H_{d-i}$ provided that G is loopless and connected?*

The statement $F_i \leq F_{d-i}$ for all $i \leq \lfloor d/2 \rfloor$ does hold [47]; this can be seen easily for all shellable complexes using the interval partition.

Dawson [28] remarks that $H_j \geq 0$ implies that $\sum_{i=0}^{j} (-1)^i F_{j-i} \geq 0$. Brown and Colbourn [15] establish a similar statement for the H-vector itself. This suggests the examination of another sequence:

Open Problem 5.7 *Is there a natural combinatorial interpretation of the sequence $(D_0, \cdots, D_d)$, where $D_j = \sum_{i=0}^{j} (-1)^i H_{j-i}$?*

6 Roots

Brown and Colbourn [15] prove that all real roots of the all-terminal reliability polynomial are at $p = 0$ and in the interval $p \in (1, 2]$. They advance a conjecture about the location of all complex roots:

Open Problem 6.1 *Is it true that all complex roots of all–terminal reliability polynomials lie in the unit circle in the complex plane centred at $1 + 0i$?*

Open Problem 5.1 *For matroids, is there a (conceptually) simple combinatorial construction for the order ideal of monomials in Stanley's theorem? In particular, is there a natural bijection from spanning trees to monomials in the order ideal?*

Stanley's results give a complete characterization of H-vectors (and hence F-vectors) of shellable complexes. Since for all–terminal reliability, $\mathcal{F}$ is a matroid, we might hope for tighter inequalities yet:

Open Problem 5.2 *Characterize H-vectors or F-vectors of matroids (or binary matroids, regular matroids, cographic matroids) of dimension d with ground set of size m.*

In the absence of such a characterization, let us ask some more modest, but still difficult questions. A sequence of nonnegative integers $x_0, \ldots, x_d$ is *unimodal* if there is an index j for which $x_0 \leq x_1 \leq \ldots \leq x_j \geq x_{j+1} \geq \ldots \geq x_d$. It is *log concave* if $x_i^2 \geq x_{i-1}x_{i+1}$ for all i. Log concavity implies unimodality when the sequences contain only positive integers. Mason [47] and Welsh [72, 73] conjecture unimodality and log concavity for F-vectors of matroids in general.

Open Problem 5.3 *Is the sequence $(F_0, \ldots, F_{m-n+1})$ for the all-terminal reliability polynomial unimodal? log concave?*

Dawson [28] conjectured log concavity for H-vectors of matroids in general:

Open Problem 5.4 *Is the sequence $(H_0, \ldots, H_{m-n+1})$ for the all-terminal reliability polynomial unimodal? log concave?*

See [14, 28, 34, 46] for some progress on these questions. Mason [47] shows that the F_i coefficients are nondecreasing for $i = 0, 1, \ldots, \lfloor d/2 \rfloor$. It appears plausible that the same may hold for the H-vector:

For multigraphs, it is easy to verify that the least reliable n–vertex m–edge graph contains $n-2$ bridges and one bundle of $m-n+2$ edges in parallel.

Brown, Colbourn and Devitt [17] establish that a number of transformations have predictable effect on coefficients of all–terminal reliability polynomials. Perhaps transformations can be useful in the following:

Open Problem 4.2 *For all-terminal reliability, is it true that if G has a cycle basis containing cycles of lengths $c_1, \cdots, c_{m-n+1}$, then*

$$H_i \leq \sum_{I \subseteq \{1,\ldots,m-n+1\}, |I|=i} \prod_{j \in I} (c_j - 1)?$$

5 Inequalities

In this section, we consider some problems on the relations among the coefficients. Sperner [63] established a famous lemma which implies that $(m-i)N_i \leq (i+1)N_{i+1}$ for coherent structure functions. This is not at all surprising, since it just states that the fraction of k-edge subgraphs that are connected is a nondecreasing function of k. However, the use of inequalities such as this *is* surprising. Notice that, given F_i, we can now obtain easily a *lower* bound on F_{i-1} and an *upper* bound on F_{i+1}. Together with exact calculations, we can then produce lower and upper bounds on every coefficient in the F–form.

One might ask whether Sperner's result is the best possible result. Kruskal [43] and Katona [39] independently improved on Sperner's result, and obtained the best possible inequality for complexes in general. Daykin [29] and Frankl [31] give much simpler proofs, and Clements and Lindström [20] prove a generalization. The application of the Kruskal–Katona theorem to reliability was first studied by Van Slyke and Frank [70], and leads naturally to the question: for the two–terminal or all-terminal complexes, are the bounds of the Kruskal–Katona theorem still tight?

In the all-terminal case, $\mathcal{F}$ is the cographic matroid, so it is reasonable to expect that it exhibits much more structure than complexes in general. The fact that matroids are shellable enable one to employ Stanley's characterization [64, 65, 66] of the H–vectors of shellable complexes. The application to reliability has been studied in [3, 4, 17]. Stanley's characterization rests on a mapping from a shelling of the complex $\mathcal{F}$ to an order ideal of monomials. Each interval corresponds to a monomial. The techniques used are primarily from commutative algebra; see Billera [6] for a clear précis. We have remarked that Tutte's notion of active and passive elements allows us to obtain a combinatorial bijection from spanning trees to intervals; the last step is sorely needed:

Open Problem 3.3 *For two–terminal reliability, is there an efficient algorithm to compute F_{m-n+1}, the number of subgraphs on $n-1$ edges containing an s,t–path?*

4 Transformations

A further strategy is to find transformations that have a predictable effect on each coefficient, for example by never increasing a coefficient. A basic device for studying transformations is the *factoring theorem* [48, 50], also called *pivotal decomposition*. Consider an edge e of G. Deleting e gives a new graph $G-e$. Contracting e by identifying its endpoints, removing e, but saving all multiple edges and other loops that arise, gives a multigraph $G \cdot e$. It is easy to see that

$$\mathrm{Rel}(G;p) = p \cdot \mathrm{Rel}(G \cdot e;p) + (1-p) \cdot \mathrm{Rel}(G-e;p).$$

This implies simple recursive formulae for the coefficients ("simple" here means conceptually simple, not efficient). For example, $F_i(G) = F_i(G \cdot e) + F_{i-1}(G-e)$; $P_i(G) = P_{i-1}(G \cdot e) + P_i(G-e) - P_{i-1}(G-e)$; and $H_i(G) = H_i(G \cdot e) + H_{i-1}(G-e)$. These simple relations permit one to determine the effect of transformations on coefficients in many cases.

A transformation, roughly speaking, is just the replacement of one subgraph by another. For example, the operation of deleting an edge $\{x,y\}$ and adding a pair of edges $\{\{x,z\},\{y,z\}\}$ where z is a target is a *Lomonosov join* on the graph. It never decreases the reliability, and never decreases any coefficient in the F-vector or the H-vector of the graph [17].

One main problem on which progress has been made by transformation methods is the determination of the least reliable all–terminal network having no loops or multiple edges. Given a number of edges m and a number of vertices n, let s be the largest integer satisfying $\binom{s}{2} + n - s - 1 \le m$. Form a complete graph on s vertices. Add one more vertex adjacent to $m - \binom{s}{2} - s + 2$ nodes in this complete graph. Finally add the $n-s-1$ remaining nodes, each connected by a bridge to the graph constructed. Call this graph $L_{n,m}$. It has been conjectured that $L_{n,m}$ is the least reliable simple graph on n nodes and m edges [11]. Some progress in proving this has been made using transformations; see [41, 17]. The problem remains open, however:

Open Problem 4.1 *Is $L_{n,m}$ the least reliable connected simple graph on n nodes and m edges for all edge operation probabilities? Is the H–vector of any such graph bounded below by that of $L_{n,m}$ coefficientwise?*

These complexity results establish that exact calculation of all of the coefficients (in any form) is #P-hard. However, the calculation of *some* of the coefficients is feasible. Let us consider the F–form for the all–terminal operation. Evidently if fewer than $n-1$ edges operate, the graph is disconnected. Thus $F_i = 0$ for $i > m-n+1$. If the smallest edge cutset has size c (the edge–connectivity is c), there is no way to remove fewer than c edges and disconnect the graph. Thus $F_i = \binom{m}{i}$ for $i < c$. For any k, Ramanathan and Colbourn [57] give an algorithm for determining F_{c+k} whose running time is polynomial in the size of the graph but exponential in k (and so, for fixed k, is polynomial). The coefficient F_{m-n+1} is exactly the number of spanning trees of the graph, which by the Kirchoff Matrix–Tree theorem [42] can be calculated efficiently (although the number itself can be exponential in the size of the graph). Little else is known here. In particular, for the coefficient F_{m-n}, we have:

Open Problem 3.1 *Can the number of spanning connected unicyclic subgraphs of a graph be computed efficiently?*

Liu and Chow [44] show that an efficient algorithm does exist for planar graphs, but their method relies heavily on planar duality.

Turning to the H-form, assume that the graph is loopless and let $d = m-n+1$. Then $F_k = \sum_{j=0}^{k} \binom{d-j}{d-k} H_j$ and $H_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{d-k} F_j$. It follows that we can calculate $H_0, \ldots, H_{c+k}$ for fixed k efficiently, and also that we can calculate $\sum_{j=0}^{d} H_j$, which is again the number of spanning trees. One coefficient here has been extensively studied. H_d is sometimes called the *Crapo β-invariant* or the *reliability domination* [10, 11, 27, 37, 51]. Some bounds for H_d are known, as we see later. However, the following basic question seems to be open:

Open Problem 3.2 *Can the reliability domination of a graph be calculated efficiently?*

Calculating any of the unknown coefficients efficiently is of interest, but from a structural point of view, H_d would provide substantial information.

Essentially nothing is known about calculating coefficients in other forms that is not trivially implied by the results mentioned here.

For two–terminal reliability, the situation is somewhat similar. Let ℓ be the length of a shortest s,t-path, and c be the s,t–edge–connectivity. Then $F_i = \binom{m}{i}$ for $i < c$, and $F_i = 0$ for $i > m-\ell$. Brecht and Colbourn [12] show that $F_{m-\ell-k}$ can be calculated in time polynomial in the size of the graph, but exponential in k. However, computing F_c itself is already #P-complete [55]. As a result of Chari's decomposition, one very important coefficient is F_{m-n+1}.

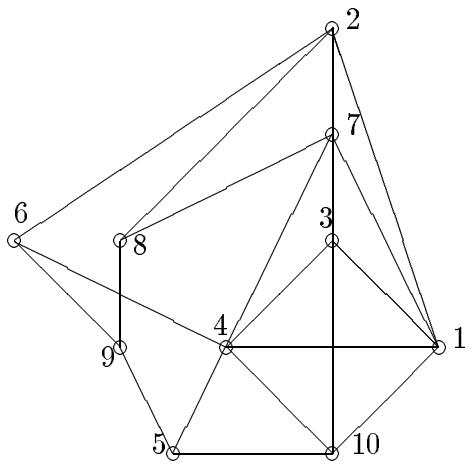

Figure 1: An example graph

i	F_i	C_i	H_i	P_i
0	1	0	1	0
1	20	0	9	0
2	190	0	45	0
3	1136	4	161	0
4	4771	74	453	0
5	14861	643	1050	0
6	35285	3475	2056	0
7	64480	13040	3435	0
8	90052	35918	4856	0
9	93177	74783	5588	24704
10	65793	118963	4764	-205951
11	24704	143256	2286	793967
12	0	125970	0	-1864016
13	0	77520	0	2955596
14	0	38760	0	-3318205
15	0	15504	0	2688033
16	0	4845	0	-1569511
17	0	1140	0	646718
18	0	190	0	-178958
19	0	20	0	29910
20	0	1	0	-2286

Figure 2: Coefficients of the Reliability Polynomial

there is a unique index i for which $1 \leq i \leq t$ and $L_i \subseteq S \subseteq U_i$, and $|U_i| = m - \ell$ for all i. When such an interval partition is found, the number of intervals whose lower set has size i is exactly H_i.

The coefficients of the split H-form can be interpreted similarly. Let $\mathcal{F}_x$ the the complex obtained from $\mathcal{F}$ by removing all sets of cardinality more than x, and define $\mathcal{M}_x$ from $\mathcal{M}$ similarly. When $\mathcal{F}_{m-n+1}$ and $\mathcal{M}_{n-2}$ both admit interval partitions, the coefficients $H_i^{(1)}$ and $H_i^{(2)}$ are the numbers of intervals with lower set of size i in these partitions.

In this paper, we focus on two structure functions, the all-terminal operation and the two-terminal operation. For the all-terminal operation, $\mathcal{F}$ is precisely the cographic matroid of the graph. Hence $\mathcal{F}$ is a shellable complex [56], and admits an interval partition [2]. In fact, a partition into intervals can be obtained by considering all bases of the cographic matroid (complements of spanning trees), and for each taking the lower set of the interval to be the "internally passive" elements of this basis. Under an arbitrary order $\prec$, an element e of a basis B is *passive* if it is *not* the first element under $\prec$ of the fundamental cocircuit of e with respect to B. The notion of passive and active edges was introduced by Tutte [68], extended to matroids by Crapo [27], and the relation to interval partitioning has been employed extensively [9, 26, 28, 32, 67]. The importance here is that while interval partitions are something that every shellable complex exhibits, partitions that arise from Tutte's notion of passive elements offer more structure.

For the two-terminal operation, the complex $\mathcal{F}$ is not in general shellable. However, using a related partition into intervals [26], Chari [19] establishes that $\mathcal{F}_{m-n+1}$ and $\mathcal{M}_{n-2}$ are shellable. Neither is matroidal, however.

This thumbnail sketch of forms and their interpretation suggests the main direction taken in studying reliability polynomials: to interpret the coefficients in some form of the polynomial combinatorially, and then to employ the combinatorial structure to characterize the possible sequences of coefficients in that or another form of the polynomial.

For concreteness, we give the actual coefficients in a number of forms for a small all-terminal problem. The graph is depicted in Figure 1, and coefficients for four forms of its reliability polynomial are given in Figure 2.

3 Calculating Coefficients

The goal of "understanding the coefficients" can be sought in a number of ways. One method is to determine relationships among the coefficients. A second is to determine inequalities that all graphs with specified parameters (number of nodes, number of edges, edge connecticity, and so on) must satisfy. And a third is to determine which coefficients we can calculate efficiently given a graph under either the all-terminal or the two-terminal operation. We consider the computational question in this section.

Valiant [69] established that calculating two–terminal reliability, even when all edges have probability $p = \frac{1}{2}$ is #P-complete. Provan and Ball [55] proved the analogous result for all-terminal reliability, and Provan [54] and Vertigan [71] established similar results for planar graphs.

These are all simple translations, using the fact that $N_i + M_i = \binom{m}{i}$, $F_i + C_i = \binom{m}{i}$, and $N_i = F_{m-i}$.

Of course, a further form is simply to expand the reliability polynomial in powers of p, the *P–form*:

$$\mathrm{Rel}_\phi(G;p) = \sum_{i=0}^{m} P_i p^i.$$

One form of particular interest here is the *H-form*. Let ℓ be the smallest number of edges whose operation suffice to make the graph operational under ϕ. Then we can factor p^ℓ from the reliability polynomial, and collect the remaining terms in powers of $1-p$ to obtain the *H–form*:

$$\mathrm{Rel}_\phi(G;p) = p^\ell \sum_{i=0}^{m-\ell} H_i (1-p)^i$$

Finally, we mention two forms studied by Chari [19]. By rewriting the coefficients F_i in the F–form having $i > m-n+1$ as $\binom{m}{i} - M_{m-i}$, we obtain:

$$\mathrm{Rel}_\phi(G;p) = \sum_{i=0}^{m-n+1} F_i p^{m-i}(1-p)^i + \sum_{i=0}^{n-2} \binom{m}{i} p^i (1-p)^{m-i} - \sum_{i=0}^{n-2} M_i p^i (1-p)^{m-i}.$$

Call this the *split F–form*. Collecting powers of $1-p$ in the first part, and of p in the second, we obtain:

$$\mathrm{Rel}_\phi(G;p) = p^{n-1} \sum_{i=0}^{m-n+1} H_i^{(1)} (1-p)^i + \sum_{i=0}^{n-2} \binom{m}{i} p^i (1-p)^{m-i} - (1-p)^{m-n+2} \sum_{i=0}^{n-2} H_i^{(2)} p^i.$$

Call this the *split H-form*.

The forms defined here are the main ones that have been useful in determining properties of reliability polynomials thus far, in part because for various types of structure functions, each has a combinatorial interpretation. The interpretations of N_i, F_i, C_i and M_i are all immediate. The remaining forms require some explanation.

Let us first consider the P–form. For a coherent structure function, let $A_1, \ldots, A_s$ be the set of all minimal operating states. The reliability can then be written as

$$\sum_{I \subseteq \{1,\ldots,s\}} (-1)^{|I|-1} \Pr[\bigcup_{i \in I} A_i].$$

A state S occurs in this sum only when S is the union of minimal operating states. The number of times it contributes to this sum depends upon the difference between the number of times it arises as the union of an odd number of minimal operating states and the number of times it arises as the union of an even number. Summing these differences over all i–edge states gives P_i [60].

For the H-form, a natural interpretation arises when the complex $\mathcal{F}$ is shellable [66]. For our purposes, it is enough to know that every maximal set in $\mathcal{F}$ has the same cardinality $m-\ell$, and that $\mathcal{F}$ has a partition into intervals $[L_i, U_i] : i = 1, \cdots, t$; that is for every $S \in \mathcal{F}$

2 Forms

Let N_i be the number of operating states of cardinality i under structure function ϕ. Then

$$\mathrm{Rel}_\phi(G;p) = \sum_{i=0}^{m} N_i p^i (1-p)^{m-i}.$$

Of course, from a network design standpoint, what is interesting is the *value* that this polynomial takes on when evaluated at a point. However, suppose that we were able to find lower and upper bounds, $\underline{N_i} \le N_i \le \overline{N}_i$, on each *coefficient*. It is then easy to see that

$$\sum_{i=0}^{m} \underline{N_i} p^i (1-p)^{m-i} \le \mathrm{Rel}_\phi(G;p) \le \sum_{i=0}^{m} \overline{N}_i p^i (1-p)^{m-i}.$$

This motivates one to understand the *coefficients* of reliability polynomials. Of course, for arbitrary structure functions, essentially nothing can be said beyond the trivial bounds that $0 \le N_i \le \binom{m}{i}$. However, constraints on the structure function impose constraints on the coefficients.

Often these constraints are best seen (and found) by writing the reliability polynomials in different forms. One natural restriction to make is that the structure function should be *coherent*, that is, if $S \subseteq T$, then $\phi(S) \le \phi(T)$. Define a family $\mathcal{F}_\phi$ of sets as follows:

$$\mathcal{F}_\phi = \{S : S \subseteq E,\ \phi(E \setminus S) = 1\}.$$

When ϕ is coherent, $\mathcal{F}_\phi$ is an hereditary family of sets, or *complex*. Letting $F_i = |\{F \in \mathcal{F}_i : |F| = i\}|$, we obtain the *F-form* of the reliability polynomial:

$$\mathrm{Rel}_\phi(G;p) = \sum_{i=0}^{m} F_i (1-p)^i p^{m-i}.$$

Call this the *F–form*. Evidently, $F_i = N_{m-i}$.

We can similarly consider the reliability as the probability that G is *not* in a failed state. If M_i is the number of failed states on i edges, we have

$$\mathrm{Rel}_\phi(G;p) = 1 - \sum_{i=0}^{m} M_i p^i (1-p)^{m-i},$$

and if C_i is the number of sets of i edges whose removal renders the graph failed (*cutsets*, not necessarily minimal), we have:

$$\mathrm{Rel}_\phi(G;p) = 1 - \sum_{i=0}^{m} C_i (1-p)^i p^{m-i}.$$

In this case again, when ϕ is coherent, M_i is the number of failed states on i edges in the complex $\mathcal{M}$ of all failed states.

1 The Basics

Our goal here is to state some tantalizing open problems concerning reliability polynomials, with just enough context to motivate and explain the problems. A full treatment of the subject of reliability polynomials would run to great length; as we proceed, we therefore give some pointers to the literature to those interested in pursuing some particular avenue. We refer the reader to [1, 21, 62] for background on network reliability, [73] for related material on matroids, [5, 9] for information on Tutte polynomials, and [6, 66] for background on complexes.

Let $G = (V, E)$ be a multigraph on n vertices and m edges; loops and parallel edges are permitted. Associate with each edge of E a probability p that the edge is *operating*; it is *failed* with the complementary probability $q = 1 - p$. A *structure function* is a function $\phi : 2^E \mapsto \{0, 1\}$. For a set $S \subseteq E$, we call S a *state* of the network. State S occurs when all edges of S are operating, and all edges of $E \setminus S$ are failed. A state S is *operating* when $\phi(S) = 1$, and *failed* when $\phi(S) = 0$. The basic question of interest here is: if every edge operates independently, what is the probability $\mathrm{Rel}_\phi(G; p)$ that the graph is in an operating state (under ϕ)? This can be viewed as the *reliability* of the graph at performing the "operation" specified by the structure function ϕ. An example of some importance is obtained as follows. Let $K \subseteq V$ be a set of *target* vertices. For each state S, define $\phi_K(S) = 1$ if and only if the subgraph (V, S) has all target vertices in the same connected component. Then $\mathrm{Rel}_{\phi_K}(G; p)$ is the *k-terminal reliability* of G for target set K and operation probability p; we typically abbreviate this as $\mathrm{Rel}_K(G; p)$. When $K = V$, this is just the *all-terminal reliability* or *connectedness probability* of the graph. When $K = \{s, t\}$, this is a *two–terminal reliability*, namely the probability that at least one s, t-path is operating.

Numerous other natural structure functions are of interest. Each shares an important property. Consider a fixed state S containing i edges. If each edge operates independently with probability p, the probability of being in state S is exactly $\Pr[S] = p^i(1 - p)^{m-i}$ (each of i edges operates, the remaining edges fail, and all edge events are independent). Now consider the reliability with respect to an arbitrary structure function ϕ:

$$\mathrm{Rel}_\phi(G; p) = \sum_{S \subseteq E} \Pr[S]\phi(S), \tag{1}$$

since the 2^m states are disjoint events covering all possibilities.

Equation (1) expresses the reliability with respect to ϕ as a *polynomial* of degree at most m in one variable, the edge operation probability p. This is a *reliability polynomial* for ϕ. Here we shall be concerned primarily with reliability polynomials for k-terminal operation, and the restriction to all– and two–terminal operations.

An enormous amount of research has been invested in studying reliability polynomials and their close relatives. Moore and Shannon [49] employed the two–terminal operation on multigraphs to model switching networks; since that time, the model has been used extensively as a simple model for use in network design and analysis [1]. Of course, reliability polynomials tell only a small part of the story of a network subject to random failures, but when used in context they afford useful guides to reliable network design [22, 24].

ABSTRACT

Some open questions, both computational and combinatorial, concerning reliability polynomials are discussed.

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Some Open Problems on Reliability Polynomials

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