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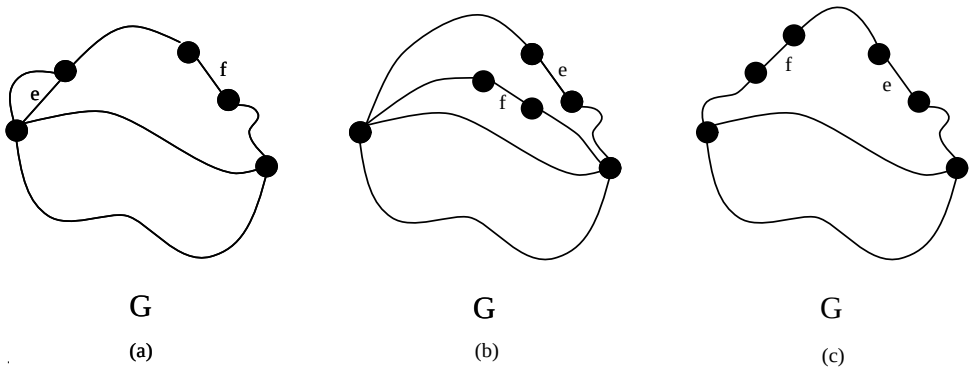

Figure 10:

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and we are done. □

In fact, equality holds only in (b), and then only if $a = b = c$, i.e. G is a θ–graph with 3 equal length paths.

Such a bound pulls up to produce non–Stanley bounds for all fixed dimensions. For example,

Corollary 19 *If G is a connected graph of dimension $d \leq 3$, then*

$$H_3(G) \leq \frac{1}{9}(H_1^3(G) + 3H_1^2(G) + 5H_1(G) + 3).$$

Proof We proceed by induction on n, the number of vertices of G. Clearly, we can assume $d = 3$ and that G has no cut edges. If $n = 1$ we are done, so we can assume that $n \geq 2$. Now if $e \in E(G)$, then

$$\begin{aligned} H_3(G) &= H_2(G - e) + H_3(G \cdot e) \\ &\leq \frac{1}{3}(H_1(G - e) + 2)^2 + (H_1(G - e) + 2) + 1) + \frac{1}{9}(H_1^3(G \cdot e) + 3H_1^2(G \cdot e) + \\ &\quad 5H_1(G \cdot e) + 3). \end{aligned}$$

Now $H_1(G - e) \leq H_1(G) = n - 1$ and $H_1(G \cdot e) = H_1(G) - 1 = n - 2$, so

$$\begin{aligned} H_3(G) &\leq \frac{1}{3}((n-1)^2 + (n-1) + 1) + \frac{1}{9}((n-2)^3 + 3(n-2)^2 + 5(n-2) + 3) \\ &\leq \frac{1}{9}((n-1)^3 + 3(n-1)^2 + 5(n-1) + 3) \\ &\leq \frac{1}{9}(H_1^3(G) + 3H_1^2(G) + 5H_1(G) + 3). \end{aligned}$$

In fact, if equality were to hold, then we must have equality holding everywhere above, so in particular, $H_2(G - e) = \frac{1}{3}(H_1(G - e) + 2)^2 + (H_1(G - e) + 2) + 1)$. From the previous theorem, this means $G - e$ is a θ–graph with 3 equal length paths of length $a \geq 1$. Also, $\dim(G \cdot e) = 3$, so (as e was arbitrary), G must be a simple graph. Now $a = 1$ implies that G is a bundle of parallel edges, so $H_2(G) = H_1(G) = 1$ and $\frac{1}{9}(H_1^3(G) + 3H_1^2(G) + 5H_1(G) + 3) = \frac{4}{3}$, so strict inequality holds. Thus $a \geq 2$ and G is of one of the forms in Figure 10.

If we choose f instead of e in (a) or (b), $G - e$ (after contracting bridges) is not a θ–graph. If we choose f instead of e in (c), then $G - f$ is a θ–graph with path lengths $1, a \geq 2, a \geq 2$, so we can see in any event, we can choose an edge e so that $H_2(G - e) < \frac{1}{3}(H_1^2(G - e) + H_1(G - e) + 1)$. Thus equality never holds in the corollary.

We expect that the results for lower dimensions can be drawn up to higher dimensions to improve bounds between terms that are close to one another in the H–vector of a graph.

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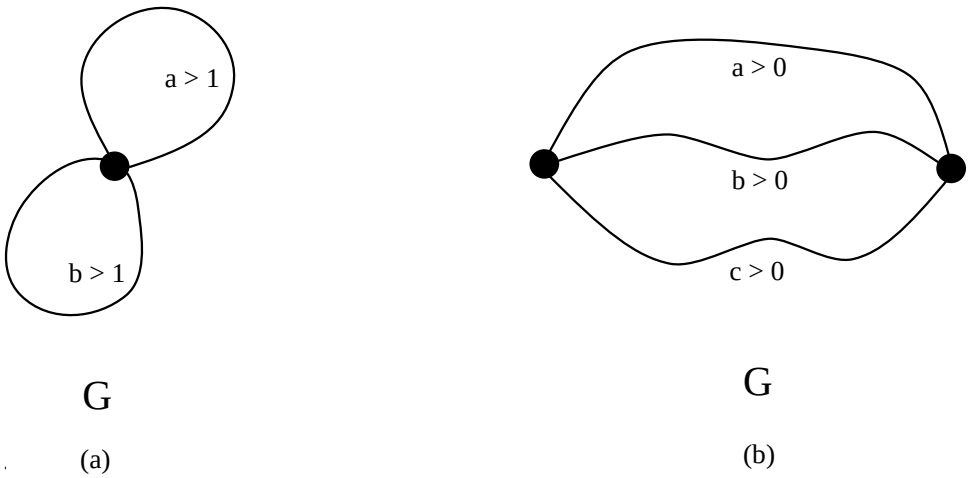


Figure 9:

has degree at least 2, and if there are x vertices of degree ≥ 3, and let G have order n and size m (so $d = m - n + 1$). Then

$$\begin{aligned} 2m &= \sum_{v \in V(G)} \deg(v) \\ 2(d+n-1) &\geq 3x + 2(n-x) \\ 2n+2 &\geq 2n + x \\ x &\leq 2. \end{aligned}$$

Thus either $x = 1$ and the only vertex of degree greater than two has degree 4, or $x = 2$ and the two vertices of degree greater than two have degree 3. It follows that G is of one of the two forms shown in Figure 9 .

In (a), G had H–vector $(1, a-1) \cdot (1, b-1) = (1, a+b-2, ab-a-b+1)$. Now for fixed $a+b$ (and hence fixed $H_1(G) = a+b-2$), $ab-a-b+1$ is maximized when ab is maximized, i.e. when $a = b = \frac{H_1+2}{2}$. Therefore,

$$\begin{aligned} H_2(G) &\leq \left(\frac{H_1(G)+2}{2}\right)^2 - 2\left(\frac{H_1(G)+2}{2}\right) + 1 \\ &\leq \frac{H_1^2}{4} \\ &< \frac{1}{3}(H_1^2(G) + H_1(G) + 1). \end{aligned}$$

In (b), G has H–vector $(1, a+b+c-3, ab+ac+bc-a-b-c+1)$. Now for fixed $a+b+c$ (and hence for fixed $H_1(G) = a+b+c-2$), $ab+ac+bc-a-b-c+1$ is maximized when $ab+ac+bc$ is, i.e. when $a = b = c = \frac{H_1(G)+2}{3}$. Thus

$$\begin{aligned} H_2(G) &\leq \frac{1}{3}(H_1(G)+2)^2 + (H_1(G)+2) + 1) \\ &\leq \frac{1}{3}(H_1^2(G) + H_1(G) + 1). \end{aligned}$$

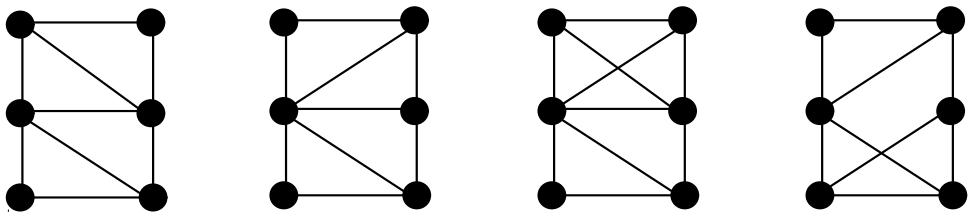


Figure 8:

Thus we can assume that G is a block. If $G \cdot e$ is simple for some edge e of G, i.e. e is not in a triangle of G, then as $G - e$ and $G \cdot e$ are of girth at least g and $g-1$ respectively,

$$\begin{aligned} H_d(G) &= H_{d-1}(G-e) + H_d(G \cdot e) \\ &\geq (d-1)(g-1) + d(g-2) \\ &\geq d(g-1) \end{aligned}$$

as $d \geq 2$ and $g \geq 3$.

The only remaining case is that G is a block and every edge of G belongs to a triangle. Thus $g = 3$, and if e is any edge of G,

$$\begin{aligned} H_d(G) &= H_{d-1}(G-e) + H_d(G \cdot e) \\ &\geq (d-1)2 + 2 \\ &= d(3-1) \end{aligned}$$

by Theorem 14 unless the underlying simple graph of $G \cdot e$ is $C_3 \circ C_3$. As G is a block with every edge in a triangle, one can verify that the only possibilities for G are shown in Figure 8. In each case the inequality holds, so we are done. □

5 Non–Stanley Bounds for Graphs of Low Dimension

According to the Stanley bounds, the best upper bounds for H_d of a shellable complex of fixed dimension d is

$$H_d \leq \binom{H_1 - 1 + d}{d} \sim \frac{H_1}{d!}.$$

with equality obtained by taking a uniform matroid. However, for cographic matroids, the results are different.

Theorem 18 *If G is a connected graph of dimension $d \leq 2$, then*

$$H_2(G) \leq \frac{1}{3}(H_1^2(G) + H_1(G) + 1).$$

Proof If $d < 2$ then $H_2(G) = 0$, so the result is trivial. Assume now $d = 2$. We can also assume G has no cut edges, as contracting a cut edge leaves the H–vector unchanged. Thus every vertex

Proof We can clearly assume $d \geq 2$ (as for $d = 1$, G is a cycle of length at least 3). If G has blocks $G_1, \ldots, G_k$ ($k \geq 2$) with G_i of dimension $d_i \geq 1$, then

$$\begin{aligned} H_d(G) &= \prod_{i=1}^{k} H_{d_i}(G_i) \\ &\geq \prod_{i=1}^{k} 2d_i \\ &\geq \sum_{i=1}^{k} 2d_i \\ &= 2d. \end{aligned}$$

Thus we can assume that G is a block. If e is an edge of G, then

$$\begin{aligned} H_d(G) &= H_d(G \cdot e) + H_{d-1}(G - e) \\ &\geq H_d(G \cdot e) + 2(d-1) \\ &\geq 2d \end{aligned}$$

unless $H_d(G \cdot e) = 1$. By corollary 14, this implies that $G \cdot e$ is a multiple of a tree. In this case, as G is a simple block, one can verify that G must be $K_2 + \overline{K_{n-2}}$, which has $H_d(G) = 2^{n-2}$, and $d = n - 2$. The result follows as $2^{n-2} > n - 2$ for $n \geq 3$. □

Corollary 16 *If G' is the underlying simple graph of a connected graph G, then*

$$H_d(G) \geq 2\,dim(G').$$

Finally, we strengthen our results in terms of the girth of the graph.

Theorem 17 *Suppose that G is a simple connected graph of dimension d and girth g, $g \geq 3$. Then*

$$H_d(G) \geq d(g-1).$$

Proof We proceed by induction on the size of G. Clearly we can assume that $d \geq 1$, and since $H_d(C_n) = n - 1$, we can assume in fact that $d \geq 2$. We can also as usual assume that G has no bridges.

If G is not a block, i.e. $G = G_1 \circ G_2$, then without loss $\gamma(G_1) \geq \gamma(G_2)$. Let G_i have dimension $d_i \geq 1$. Then $\gamma(G_1) \geq g$ and we have

$$\begin{aligned} H_d(G) &= H_{d_1}(G_1) H_{d_2}(G_2) \\ &\geq d_1(g-1) H_{d_2}(G_2) \\ &\geq d_1(g-1)(2d_2) \\ &\geq (d_1 + d_2)(g-1) \\ &= d(g-1) \end{aligned}$$

where we use Theorem 15 to go from line 2 to line 3.

since $G \cdot e$ has dimension d as G is simple (and hence $H_d(G \cdot e) > 0$). Otherwise, $G - e$ has a bridge, say $f = wz$. If e and f are non-adjacent, then $G \cdot e$ is simple, so

$$\begin{aligned} H_d(G) &= H_d(G \cdot e) + H_{d-1}(G - e) \\ &\geq H_1(G \cdot e) + H_{d-1}(G - e) \\ &> H_1(G) - 1 + H_{d-1}(G - e) \\ &= H_1(G) \end{aligned}$$

since $d - 1 \geq 1$ implies that $G - e$ has a cycle of length at least 3, and hence $H_{d-1}(G - e) \geq H_1(G - e) \geq 2$. Finally, if e and f are adjacent, say $x = w$ (and as G is simple, $y \neq z$), then as G is a block, x is of degree 2 in G. If $yz \notin E(G)$, then $G \cdot e$ is simple and we proceed as in the previous case. If $yz \in E(G)$, then $G - x$ has no bridges and

$$\begin{aligned} H_d(G) &= 2H_{d-1}(G - x) \\ &\geq 2(H_1(G) - 1) \\ &> H_1(G) \end{aligned}$$

and for equality to hold, G must have order 3, i.e. $G = K_3$; this implies that $d = 1$, a contradiction. Thus we are done. □

Corollary 14 *Let G be a connected loopless graph of dimension $d \geq 1$. Then*

$$H_d(G) = 1 \qquad \textit{if and only if } G \textit{ is a multiple of a tree}$$

and

$$H_d(G) = 2 \qquad \textit{if and only if } G \textit{ is} \equiv \textit{to a multiple of } K_3.$$

Proof Let G' be the underlying graph of G, and let $d' = \dim(G')$. Without loss, G' has no cut edges. If G' is a block that is not K_1 or K_3, then from Theorem 13, $H_d(G) = H_{d'}(G') \geq H_1(G') \geq 3$. If G' has blocks G_i of dimension $d_i \geq 1$ $(i = 1, \ldots, r \geq 2)$ with $G_i = K_3$ if and only if $i \leq \ell$, then

$$\begin{aligned} H_d(G) &= H_{d'}(G') \\ &= 2^\ell \prod_{i=\ell+1}^{r} H_{d_i}(G_i) \\ &\geq 2^\ell \prod_{i=\ell+1}^{r} H_1(G_i) \\ &\geq 2^\ell 3^{r-\ell} \\ &> 2. \end{aligned}$$

This completes the proof. □

We are now ready to prove a new lower bound for the domination number of a graph in terms of its dimension.

Theorem 15 *Let G be a simple graph. Then*

$$H_d(G) \geq 2d.$$

If $G = 2T$ for some tree T, then $H_{d-1} = n - 1 = H_1$, so we are done.

For equality to hold, we need either $d = 2$ or $d \geq 3$ and either (i) e is not parallel to any other edges and $H_{d-2}(G - e) = 1$, which implies by induction $H_1(G - e) = 1$, so $G - e$ is a tree with one other edge parallel to some edge of G, and this implies $d = 2$, a contradiction, or (ii) every edge is parallel to another edge, at least one edge e is such that $G - e$ has no bridges and equality holds for $G - e$. By induction applied to $G - e$, either $d = 3$ or the underlying simple graph of $G - e$ is a tree. In either case, G must be a multiple of a tree (as the extra edge e must be parallel to some other), and we are done.

Finally, as mentioned earlier, the H–vector of any graph whose underlying simple graph is a tree is symmetric, and hence if the dimension is at least two, then $H_1 = H_{d-1}$. □

4 New Bounds on the Domination Number

The **domination number** (c.f. [4]) is the last non–zero term of the H-vector, namely H_d, where d is the dimension of the graph; as noted earlier, H_d is also a topological invariant of the matroid, the absolute value of its Euler characteristic. While some results are known for domination, we prove here some new bounds. These have implications to the Ball–Provan method for bounding network reliability.

$H_d(G) = H_{d'}(G')$, where G' is the underlying simple graph of G and d' is its dimension.

Theorem 13 *Let G be a connected simple graph of dimension $d \geq 1$. Then*

$$H_d(G) \geq H_1(G)$$

with equality if and only if $d = 1$ or $G \equiv K_3 \circ K_3$.

Proof We proceed by induction on the size of G. We may assume that G has no bridges. If $d = 1$ then the result trivially holds, so we can assume that $d \geq 2$.

First assume that G has blocks $G_1, \ldots, G_k$ with $k \geq 2$. Let the dimension of G_i be d_i, which is at least 1 as G has no bridges. Then as G simple implies that $H_1(G_i) = |V(G_i)| - 1 \geq 2$,

$$\begin{aligned} H_d(G) &= \prod_{i=1}^{k} H_{d_i}(G_i) \\ &\geq \prod_{i=1}^{k} H_1(G_i) \\ &\geq \sum_{i=1}^{k} H_1(G_i) \\ &= H_1(G) \end{aligned}$$

where equality holds if and only if $k = 2$ and $H_1 = H_2 = 2$, i.e. $G \equiv K_3 \circ K_3$.

Now assume G is a block. If $e = xy$ is an edge of G such that $G - e$ has no bridges, then

$$\begin{aligned} H_d(G) &= H_d(G \cdot e) + H_{d-1}(G - e) \\ &> H_1(G - e) \\ &= H_1(G) \end{aligned}$$

Now by induction on m, we derive from (8) and (9) that

$$\begin{aligned} H_{d-1}(G) &\leq d'H_{d'}(F) + (n-1)H_{d'}(F) \\ &\leq (d' + (n-1))H_{d'}(F) \\ &= \frac{m}{2}H_d(G) \\ &\leq (m-(n-1))H_d(G) \qquad \text{(as } F \text{ connected implies } \tfrac{m}{2} \geq (n-1)) \\ &= dH_d(G) \end{aligned} \tag{10}$$

and the inequality is proved.

Now if G is a connected and loopless graph for which equality holds in $(*)$, then from above either G is a tree, or $G = 2F$ for some simple graph F and we must have equality in all the steps leading up to (10). That is,

$$|E(F)| = 2|E(F)| - (n-1)$$

which implies

$$|E(F)| = n-1$$

so G is a tree. It is not hard to see that in this case G has dimension $n-1$ and H–vector $(1, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-2}, 1)$, and thus

$$H_{d-1}(G) = (n-1) = (n-1)\cdot 1 = dH_d(G)$$

and equality holds. □

On the other hand, H_{d-1} cannot be smaller than H_1.

Theorem 12 *For any graph G of dimension $d \geq 2$, $H_{d-1} \geq H_1$, and equality holds if and only if $d = 2$ or the underlying simple graph is a tree.*

Proof We proceed by induction on $m = |E(G)|$. If $m = n-1$, the result is trivial, so assume that $m > n-1$. We can assume that G is loopless and bridgeless. Now

$$H_{d-1}(G) = H_{d-2}(G-e) + H_{d-1}(G\cdot e)$$

so if $\dim(G\cdot e) = d$, that is, e is not parallel to another edge, then

$$\begin{aligned} H_{d-1}(G) &\geq H_{d-2}(G-e) + H_1(G\cdot e) \\ &= H_{d-2}(G-e) + (n-2) \qquad \text{(as } G\cdot e \text{ has no bridges)} \\ &\geq n-1 \end{aligned}$$

as $\dim(G-e) = d-1$.

The only remaining case is that every edge is parallel to at least one other edge. Now if G is not twice some tree, then there is an edge e of G such that $G-e$ has no cut edges. Also, $d = 2$ is trivial, so we can assume $d \geq 3$, so by induction

$$\begin{aligned} H_{d-1}(G) &= H_{d-2}(G-e) + H_{d-1}(G\cdot e) \\ &\geq n-1 + H_{d-1}(G\cdot e) \\ &\geq H_1(G). \end{aligned}$$

Now $\dim(G-e) = \dim(G) - 1 = d-1$ since e is not a bridge or loop, so by induction on the number of edges,

$$H_{d-2}(G-e) \quad = \quad (d-1)H_{d-1}(G-e). \tag{3}$$

From (2) and (3),

$$H_{d-1}(G) \quad \leq \quad (d-1)H_{d-1}(G-e) + H_{d-1}(G \cdot e). \tag{4}$$

Let e be parallel to μ other edges of G; it is not hard to see that

$$\dim(G \cdot e) \quad = \quad d - \mu. \tag{5}$$

Case 1: $\mu = 0$.
Then $\dim(G \cdot e) = d$ by (5), so by induction on m,

$$H_{d-1}(G \cdot e) \quad \leq \quad dH_d(G \cdot e). \tag{6}$$

Case 2: $\mu \geq 2$.
Then $\dim(G \cdot e) \leq d-2$ by (5), so $H_{d-1}(G \cdot e) = H_d(G \cdot e) = 0$, and thus

$$H_{d-1}(G \cdot e) \quad \leq \quad dH_d(G \cdot e). \tag{7}$$

Thus in either case 1 or 2, by (4), (6) and (7) we get

$$\begin{aligned} H_{d-1}(G) &\leq (d-1)H_{d-1}(G-e) + dH_d(G \cdot e) \\ &< d(H_{d-1}(G-e) + H_d(G \cdot e)) \\ &= dH_d(G) \end{aligned}$$

since as G is loopless with no bridges, $d = m-(n-1) > 0$ and $H_{d-1}(G-e) > 0$ as $\dim(G-e) = d-1$.
Case 3: $\mu = 1$. As e is any edge of G, $G = 2F$ for some simple graph F. Now F has no loops as G has no loops, so

$$d' \equiv \dim(F) = |E(F)| - (|V(F)| - 1) = \frac{m}{2} - (n-1).$$

Now

$$\begin{aligned} \mathrm{Rel}(G,p) &= \mathrm{Rel}(F, 1-(1-p)^2) \\ &= p^{n-1}(1+z)^{n-1} \sum_{i=0}^{d'} H_i(F) z^{2i} \end{aligned}$$

where $z = 1-p$, and so

$$H_d(G) \quad = \quad H_{d'}(F) \tag{8}$$

and

$$H_{d-1}(G) \quad = \quad H_{d'-1}(F) + (n-1)H_{d'}(F). \tag{9}$$

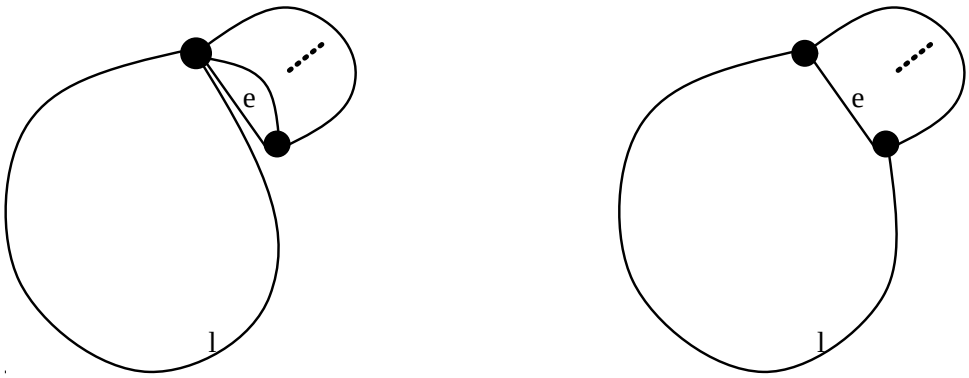


Figure 6:

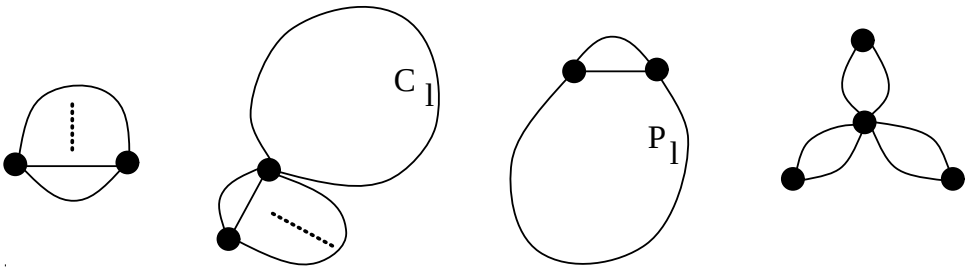


Figure 7:

In fact, equality holds if and only if G is of one of the forms in Figure 7.

We turn our attention now to the second last term, H_{d-1}. The Stanley bounds imply an upper bound for H_d in terms of H_{d-1}, but for shellable complexes, H_d may be arbitrarily small. We show below that for cographic matroids, H_d is bounded below by H_{d-1}/d.

Theorem 11 *Let a connected graph G have dimension d. Then*

$$H_{d-1}(G) \leq dH_d(G) \qquad (*)$$

and equality holds if and only if (except for loops) $G = kT$, where T is a tree and $k = 1$ or 2.

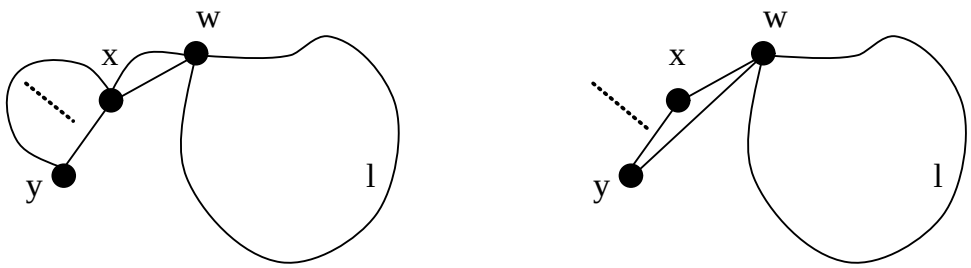
Proof We proceed by induction on $m = |E(G)|$. If $m = n - 1$, then G is a tree, so $d = 0$ and

$$0 = H_{d-1}(G) \leq dH_d(G) = 0 \cdot 1,$$

so equality holds.

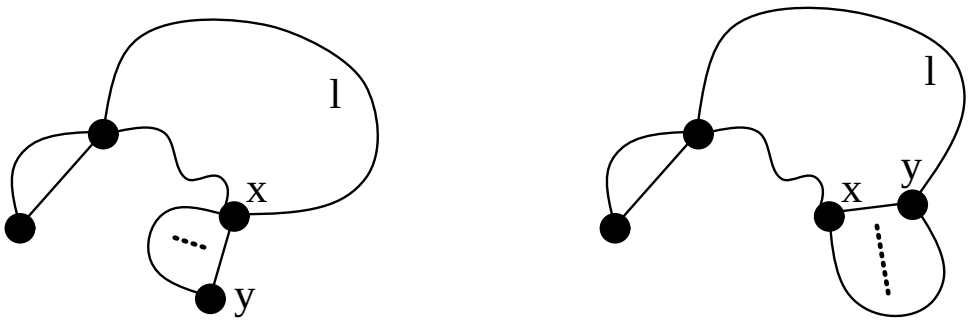
Now assume $m > n - 1$. Without loss, G has no bridges (as if e is a bridge, then G and $G \cdot e$ have the same H–vectors but $G \cdot e$ has one fewer edge, so we are done by induction). Similarly, we can assume that G is loopless. Let $e \in E(G)$. Then

$$H_{d-1}(G) \quad = \quad H_{d-2}(G - e) + H_{d-1}(G \cdot e). \qquad (2)$$



G (a) G (b)

Figure 4:



G (a) G (b)

Figure 5:

In (a)–(d), it is easy to verify that $H_2(G) \geq H_1(G)$, with equality if and only if G is the K_1–bond of 3 cycles of length 2. In case (e), $H_2(G) \geq H_1(G)$ (unless $\ell = 2$ and $\mu(x, y) = 1$, but this implies that G is the K_1–bond of a 2–cycle and a 3–cycle, a contradiction), so here $H_2(G) > H_1(G)$.

Subcase II G is of one of the forms in Figure 4.

In (a), $H_2(G) \geq H_1(G)$ (with equality if and only if $\ell = 2$ and G is the K_1–bond of three 2–cycles). In (b), $H_2(G) \geq H_1(G)$ (unless $\ell = 2$ and $\mu(x, y) = 1$, which implies G is the K_1–bond of a 2–cycle and a 3–cycle, a contradiction). Thus here $H_2(G) \geq H_1(G)$, with equality if and only if G is the K_1–bond of two 3–cycles.

Subcase III G is of one of the forms in Figure 5.

In (a) or (b), $H_2(G) \geq H_1(G)$.

We turn now to case (ii). First assume that $\dim(G \cdot e) = 1$. As $G \cdot e$ is 2–edge connected, $G \cdot e$ is a cycle (with a loop at $\cdot e$), so G is of one of the the forms in Figure 6.

In (a) the H–vector is $(1, \ell, \ell, \ldots, \ell, \ell - 1)$, so $H_2(G) = H_1(G)$. In (b), we have seen that its H–vector is $(1, \ell, \ell, \ldots, \ell)$, so $H_2(G) = H_1(G)$.

If $\dim(G \cdot e) = 0$, then G is a group of at least 3 parallel edges, so $H_2(G) = H_1(G) = 1$. □

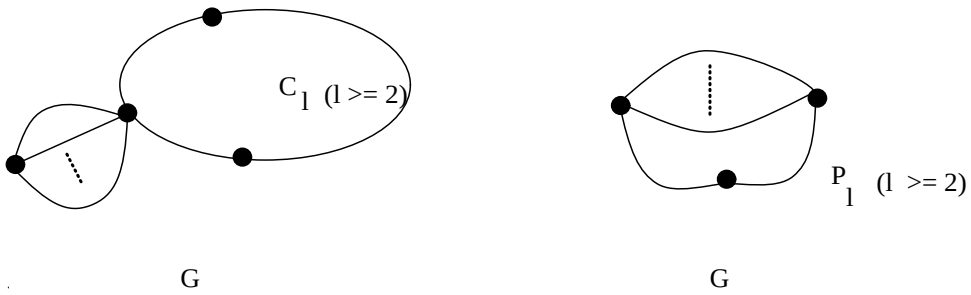


Figure 1:

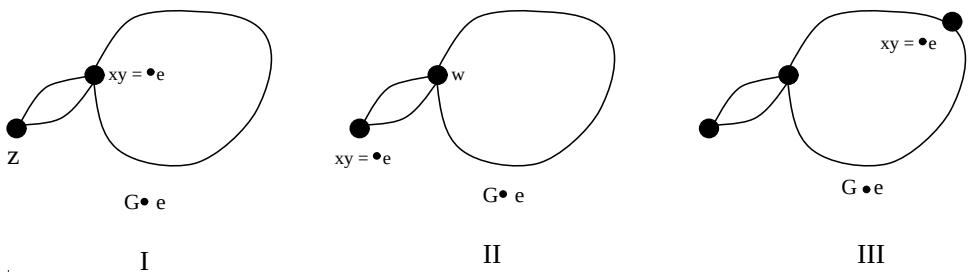


Figure 2:

In fact, equality can hold here if and only if $H_2(G \cdot e) = H_1(G \cdot e)$ and $H_1(G - e) = 1$, i.e. $G - e$ (up to contraction of bridges) is a set of at least 2 parallel edges, so G is of one of the two forms in Figure 1.

The H–vectors are respectively $(1, \ell, \ell, \ldots, \ell, \ell - 1)$ and $(1, \ell, \ldots, \ell, \ell)$. By assumption, G is not of the forbidden form, so equality holds in this case if and only if G if one of the graphs in Figure 1.

The remaining cases are

(i) $\dim(G \cdot e) = 2$ and $G \cdot e$ is of the forbidden form, or

(ii) $\dim(G \cdot e) \leq 1$.

We begin with case (i). There are essentially 3 choices for which vertex is $\cdot e$ (see Figure 2).

Subcase I G is of one of the following forms (since G is not of the forbidden form and G has no bridges); consider Figure 3.

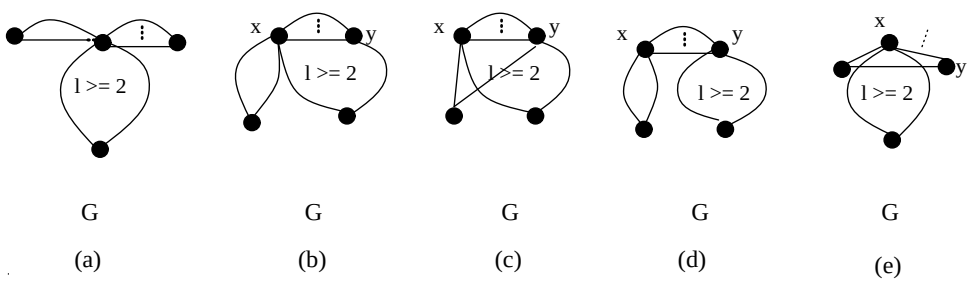


Figure 3:

By the previous theorem, this sum has sign $(-1)^{i+1}$ or 0, and hence

$$\sum_{j=0}^{\lfloor i/2 \rfloor} H_{i-2j} \leq \sum_{j=0}^{\lfloor (i+1)/2 \rfloor} H_{i+1-2j},$$

and since $\sum_{j=0}^{d} H_j = t$, we have

$$\sum_{j=0}^{\lfloor i/2 \rfloor} H_{i-2j} \leq t/2.$$

The Stanley bound and the elementary properties of pseudopowers mentioned in the first section ensure that $H_{i-2j} \geq H_i^{<(i-2j)/i>}$, to complete the proof. □

This corollary is of use in bounding individual terms in the H–vector of a graph knowing only the number of spanning trees. One striking example is obtained by examining the H–vector of the 1979 Arpanet (see [8]). That network has 272816563831 trees, and the best upper bounds found by network transformations for H_{11} is 185684104966. Corollary 9, on the other hand, ensures that H_{11} cannot exceed 103425922079. Naturally, knowing lower bounds on coefficients H_i for $i \geq 12$ in this case can be used to obtain a tighter inequality yet.

3 Other General Non–Stanley Bounds

Recall that the H–vector of any graph has no internal zeros, and the last nonzero term is H_d, where d is the dimension of the graph. It is clear that $H_1 \geq 1 = H_0$ when the dimension is at least 1, and in fact if G has order n and ℓ bridges, then $H_1 = n - 1 - \ell$. We now characterize when $H_2 \geq H_1$.

Theorem 10 *If G is a loopless graph of dimension at least 2 then*

$$H_2(G) \geq H_1(G)$$

unless $G \equiv C_p \circ C_q$ where $p = 2$ or $q = 2$.

Proof Without loss, G is 2–edge connected, as the contraction of any bridge leaves the H–vector unchanged, but reduces the number of bridges. We proceed by induction on $n = |V(G)|$. Assume that $G \not\equiv C_p \circ C_q$ where $p = 2$ or $q = 2$.

If $n = 1$, the result is trivial as $\dim(G) = 0$, so we can assume $n \geq 2$. Let e be an edge of G, not a bridge. Now

$$H_2(G) = H_1(G - e) + H_2(G \cdot e).$$

If $\dim(G \cdot e) \geq 2$ and $G \cdot e$ is not the K_1–bond of two cycles, one of which has length 2, then by induction on n,

$$\begin{aligned} H_2(G) &\geq H_1(G - e) + H_1(G \cdot e) \\ &\geq 1 + (H_1(G) - 1) \\ &= H_1(G) \end{aligned}$$

(as $\dim(G - e) = \dim(G) - 1 \geq 1$ since G has no bridges).

so we have

$$\sum_{i=0}^{d} \binom{i}{2} H_i \geq t \cdot \binom{d}{2} \cdot \min\{(1-\frac{1}{g})^2, 1-\frac{2}{g}+\frac{3}{c^2}\}$$

and

$$\sum_{i=0}^{d} \binom{i}{2} H_i \leq t \cdot \binom{d}{2} \cdot \max\{(1-\frac{1}{c})^2, 1-\frac{2}{c}+\frac{3}{g^2}\}.$$

□

Considering instead externally active pairs of edges, we can also show the following result (the details are omitted).

Theorem 7 *If a connected loopless graph G of dimension d has order n, t spanning trees, girth at least g and circumference at most c then*

$$\sum_{i=0}^{d} \binom{d-i}{2} H_i \geq t \cdot \binom{d}{2} \cdot \frac{1}{c^2}$$

and

$$\sum_{i=0}^{d} \binom{d-i}{2} H_i \leq t \cdot \binom{d}{2} \cdot \frac{3}{g^2}.$$

We point out one other inequality (though not related to external activity) of H–vectors of cographic matroids, which was proved in [7] in conjunction with an investigation of the roots of reliability polynomials.

Theorem 8 *For any H–vector $(H_0, \ldots, H_d)$ of a connected loopless multigraph, any real number $b \geq 1$, and any integer j,*

$$\sum_{i=0}^{j} (-b)^i H_i \begin{cases} \geq 0 & \text{if } \min(j,d) \text{ is even} \\ \leq 0 & \text{if } \min(j,d) \text{ is odd} \end{cases}$$

Equality holds only if $b = 1$ or $j < 0$.

We derive the following result.

Corollary 9 *If $(H_0, \ldots, H_d)$ is the H–vector of a connected loopless multigraph with exactly t spanning trees, then for all $i < d$,*

$$\sum_{j=0}^{\lfloor i/2 \rfloor} H_i^{\langle (i-2j)/i \rangle} \leq \frac{t}{2}.$$

Proof Consider the sum

$$\sum_{j=0}^{i+1} (-1)^j H_j.$$

where $m(T,e_1,e_2) = \ell_1+\ell_2$ is the number of edges in $C_1 \cup C_2$.

Case 2. $T+e_1+e_2$ contains a θ–graph.

Let the θ–subgraph of $T+e_1+e_2$ be denoted by Θ and have vertex disjoint paths P_1,P_2,P_3 of lengths ℓ_1,ℓ_2,ℓ_3 respectively. For $i \neq j$, $g \leq \ell_i+\ell_j \leq c$. Without loss we can assume that $e_1 \in P_1$ and $e_2 \in P_2$. Now the number of orders of $E(\Theta)$ such that e_1,e_2 are not externally active is

$$\begin{aligned} g(T,e_1,e_2) &\equiv (\ell_1+\ell_2+\ell_3)! - (\ell_1+\ell_3-1)!\ell_2!\binom{\ell_1+\ell_2+\ell_3}{\ell_2} - (\ell_2+\ell_3-1)!\ell_1!\binom{\ell_1+\ell_2+\ell_3}{\ell_1} \\ &\quad +(\ell_2+\ell_3-1)!(\ell_1)!\binom{\ell_1-1+\ell_2+\ell_3}{\ell_1-1} + (\ell_1+\ell_3-1)!(\ell_2)!\binom{\ell_2-1+\ell_1+\ell_3}{\ell_2-1} \\ &= m(T,e_1,e_2)!\left(1-\frac{1}{\ell_1+\ell_3}-\frac{1}{\ell_2+\ell_3}+\frac{1}{m(T,e_1,e_2)}\left(\frac{1}{\ell_2+\ell_3}+\frac{1}{\ell_1+\ell_3}\right)\right). \end{aligned}$$

where $m(T,e_1,e_2) = \ell_1+\ell_2+\ell_3$ is the number of edges in Θ.

Now we have

$$\begin{aligned} \sum_{T\in\mathcal{T}}\sum_{e_1,e_2\in E(G)-T}\sum_{\sigma} f(e_1,e_2,T,\sigma) &= \sum_{T\in\mathcal{T}}\sum_{e_1,e_2\in E(G)-T} g(T,e_1,e_2)\cdot(m-m(T,e_1,e_2))!\cdot \\ &\qquad \binom{m}{m(T,e_1,e_2)} \\ &= m!\sum_{T\in\mathcal{T}}\sum_{e_1,e_2\in E(G)-T}\frac{g(T,e_1,e_2)}{(m(T,e_1,e_2))!}. \end{aligned}$$

In case 1,

$$\left(1-\frac{1}{g}\right)^2 \leq \frac{g(T,e_1,e_2)}{(m(T,e_1,e_2))!} \leq \left(1-\frac{1}{c}\right)^2$$

and in case 2,

$$1-\frac{2}{g}+\frac{3}{c^2} \leq \frac{g(T,e_1,e_2)}{(m(T,e_1,e_2))!} \leq 1-\frac{2}{c}+\frac{3}{g^2}.$$

Thus

$$\sum_{T\in\mathcal{T}}\sum_{e_1,e_2\in E(G)-T}\sum_{\sigma} f(e_1,e_2,T,\sigma) \geq \binom{d}{2}\cdot t\cdot m!\cdot\min\{\left(1-\frac{1}{g}\right)^2, 1-\frac{2}{g}+\frac{3}{c^2}\}$$

and

$$\sum_{T\in\mathcal{T}}\sum_{e_1,e_2\in E(G)-T}\sum_{\sigma} f(e_1,e_2,T,\sigma) \leq \binom{d}{2}\cdot t\cdot m!\cdot\max\{\left(1-\frac{1}{c}\right)^2, 1-\frac{2}{c}+\frac{3}{g^2}\}.$$

By reversing the sums,

$$\sum_{T\in\mathcal{T}}\sum_{e_1,e_2\in E(G)-T}\sum_{\sigma} f(e_1,e_2,T,\sigma) = m!\sum_{i=0}^{d}\binom{i}{2}H_i,$$

Corollary 4 *If a connected loopless graph G of dimension d has order n, t spanning trees, girth at least g and circumference at most c then*

$$\frac{g-1}{g}td \leq \sum_{i=0}^{d} iH_i \leq \frac{c-1}{c}td.$$

In particular, if G is simple then

$$\frac{2}{3}td \leq \sum_{i=0}^{d} iH_i \leq (1-\frac{1}{n})td.$$

Again, the arguments yield a result on the H–vectors of matroids.

Corollary 5 *If a matroid M with no coloops has order n (i.e. its base set has n elements), t bases and smallest and largest lengths of cocircuits g and ℓ respectively, then*

$$\frac{g-1}{g}td \leq \sum_{i=0}^{d} iH_i \leq \frac{\ell-1}{\ell}td.$$

By a similar argument, we prove the following:

Theorem 6 *If a connected loopless graph G of dimension d has order n, t spanning trees, girth at least g and circumference at most c then*

$$\sum_{i=0}^{d}\binom{i}{2}H_i \geq t\cdot\binom{d}{2}\cdot min\{(1-\frac{1}{g})^2, 1-\frac{2}{g}+\frac{3}{c^2}\}$$

and

$$\sum_{i=0}^{d}\binom{i}{2}H_i \leq t\cdot\binom{d}{2}\cdot max\{(1-\frac{1}{c})^2, 1-\frac{2}{c}+\frac{3}{g^2}\}.$$

Proof For a given linear order σ of $E(G)$ and spanning tree T of G, let $\mathrm{EA}_\sigma(T)$ denote the set of externally active of G with respect to T and the ordering σ, and let $\mathrm{NEA}_\sigma(T) = E(G)-T-\mathrm{EA}_\sigma(T)$. Let $f(e_1,e_2,T,\sigma)=1$ if $e_1,e_2\in \mathrm{NEA}_\sigma(T)$, and let $f(e_1,e_2,T,\sigma)=0$ otherwise. Consider the sum

$$\sum_{T\in\mathcal{T}}\sum_{e_1,e_2\in E(G)-T}\sum_{\sigma} f(e_1,e_2,T,\sigma).$$

For a given spanning tree $T\in\mathcal{T}$ and edges $e_1,e_2\in E(G)-T$, for how many σ's are $e_1,e_2\notin NEA_\sigma(T)$? There are two cases for $T+e_1+e_2$.

Case 1. $T+e_1+e_2$ contains two edge disjoint cycles.

Let the two cycles be C_1 and C_2 with lengths ℓ_1 and ℓ_2 respectively. Note that $g\leq \ell_1,\ell_2\leq c$. Now the number of orders of $E(C_1)\cup E(C_2)$ such that e_1,e_2 are not externally active is

$$\begin{aligned} g(T,e_1,e_2) &= (\ell_1-1)\cdot(\ell_1-1)!\cdot(\ell_2-1)\cdot(\ell_2-1)!\cdot\binom{\ell_1+\ell_2}{\ell_1} \\ &= m(T,e_1,e_2)!\cdot\left(1-\frac{1}{\ell_1}\right)\left(1-\frac{1}{\ell_2}\right) \end{aligned}$$

We return to graphs. It is easy to see that $\sum H_i$ is simply the number of spanning trees of G (and hence is computable in polynomial time from the well known Matrix Tree Theorem). By considering all possible linear orderings of the edges we can obtain a new equation involving $\sum iH_i$.

Theorem 3 *For any connected loopless graph G of dimension d,*

$$\sum_{i=0}^{d} iH_i = \sum_{T\in\mathcal{T}} \sum_{e\in E(G)-T} \frac{c(e,T)-1}{c(e,T)}$$

where $\mathcal{T}$ is the set of all spanning trees of G, and for a spanning tree T of G and an edge e of G not in T, $c(e,T)$ is the length of the unique cycle in $T+e$.

Proof For each spanning tree T of G, each edge e of G not in T and each linear order σ of $E(G)$, let $f(T,e,\sigma)$ be 1 if e is not externally active with respect to T under the ordering σ, and 0 otherwise. Now for a fixed tree T and edge $e \in E(G)-T$, how many orderings σ is e externally active with respect to T? There are

$$\binom{m}{c(e,T)}$$

positions for the edges of the unique cycle of $T+e$; there are

$$(m-c(e,T))!$$

ways to arrange the remaining edges, and there are

$$(c(e,T)-1)!$$

ways to arrange the edges of the cycle so that e is externally active. Moreover, for the remaining

$$(c(e,T)-1)\cdot(c(e,T)-1)!$$

ways to order the cycle edges, e is **not** externally active. Thus we have

$$\begin{aligned}\sum_{T\in\mathcal{T}}\sum_{e\in E(G)-T}\sum_{\sigma} f(T,e,\sigma) &= \sum_{T\in\mathcal{T}}\sum_{e\in E(G)-T}\binom{m}{c(e,T)}\cdot(m-c(e,T))!\cdot(c(e,T)-1)\\ &\quad\cdot(c(e,T)-1)!\\ &= \sum_{t\in\mathcal{T}}\sum_{e\in E(G)-T}\frac{m!\cdot(c(e,T)-1)}{c(e,T)}.\end{aligned}$$

On the other hand, by reversing the order of summation, we get

$$\sum_{\sigma}\sum_{T\in\mathcal{T}}\sum_{e\in E(G)-T} f(T,e,\sigma) = m!\sum_{i=0}^{d} iH_i,$$

so we have

$$\sum_{i=0}^{d} iH_i = \sum_{T\in\mathcal{T}}\sum_{e\in E(G)-T}\frac{c(e,T)-1}{c(e,T)}.$$

□

Now for any unicyclic subgraph G' of G with a cycle of length $\ell \geq 2$,

$$|\mathcal{T}_{\prec,0,0}(G')| = \ell - 2$$

for any ordering $\prec$ on the graph. Let $0 \leq k \leq d-1$. Let the edges not in G' be $f_1, \ldots, f_{d-1}$. Order the edges of G such that $f_1, \ldots, f_k$ are smaller than those of G', which are in turn smaller than $f_{k+1}, \ldots, f_{d-1}$. It is easy to verify that if $T \in \mathcal{T}_{\prec,i,j}(G'')$ is a spanning tree of a spanning subgraph G'' of G and e is an edge of G not in T, then by setting e smaller than all edges of G'', $T \in \mathcal{T}_{\prec,i+1,j}(G''+e)$; and conversely if we set e larger than all edges of G''. Thus we see inductively that

$$\mathcal{T}_{\prec,d-k-1,k}(G) \geq |\mathcal{T}_{\prec,0,0}(G')| = \ell - 2$$

and substituting this into (1), we derive

$$\sum_{i \leq k} H_i + (\ell - 2) \leq \sum_{i \geq d-k} H_i.$$

□

The results can be improved. For example, if G has a θ–subgraph G' with path lengths ℓ_1, ℓ_2, and ℓ_3, then

$$|\mathcal{T}_{\prec,0,0}(G')| = \sum_{i \neq j} (\ell_i - 1)(\ell_j - 1)$$

and

$$|\mathcal{T}_{\prec,0,1}(G')| = |\mathcal{T}_{\prec,0,1}(G')| = (\sum_i \ell_i) - 3$$

and if G has a subgraph G' consisting of two vertex–disjoint cycles of lengths n_1 and n_2, then

$$|\mathcal{T}_{\prec,0,0}(G')| = (n_1 - 2)(n_2 - 2)$$

and

$$|\mathcal{T}_{\prec,0,1}(G')| = |\mathcal{T}_{\prec,0,1}(G')| = n_1 + n_2 - 4,$$

we can then pull these up through the ranks to improve the gap between $\sum_{i=0}^{k} H_i$ and $\sum_{i=d-k}^{d} H_i$.

The previous theorem extends to matroids in the following way. Let $\prec$ be a fixed linear order of the ground set E of a matroid M. An element e of a matroid M is *externally active* with respect to basis B of M if $e \in E - B$ and e is the smallest element of the unique circuit in $B + e \equiv B \cup \{e\}$. We let $\mathrm{NEA}_{M,\prec}(B)$ denote the set of edges of $E - B$ that are not externally active with respect to B. Then

$$\{[\mathrm{NEA}_{M,\prec}(B), E - B] : B \text{ is a basis of } M\}$$

forms a partition of M^*, the dual matroid of M (see, for example, [2] or [10]). A similar argument to the proof of Theorem 1 proves an strengthening of Hibi's results on matroids:

Theorem 2 *Let M be a matroid of dimension d without coloops, and suppose that M has a cocircuit of length $\ell \geq 2$. Let the H–vector of M be $(H_0, \ldots, H_d)$. Then for any $k \leq d-1$,*

$$\sum_{i \leq k} H_i + (\ell - 2) \leq \sum_{i \geq d-k} H_i.$$

T. Then

$$\{[\mathrm{NEA}(T), E - T] : T \in \mathcal{T}\}$$

is a partition of the cographic matroid of G.

In [11] the following inequality for H–vectors of matroids was shown by algebraic methods:

$$\sum_{i \le k} H_i \le \sum_{i \ge d-k} H_i.$$

(Here d is the dimension of the matroid minus the number of coloops.) As a corollary, this inequality holds for H–vectors of graphs as well. It is easy to see that equality for holds for graphs G for all k if G is a multiple of a tree, since the H–vector of a set of parallel edges is symmetric, and hence so (by products) is the multiple of any tree. We prove that equality only holds in these cases, and extend Hibi's result with a purely combinatorial argument that considers external activity.

Theorem 1 *Let G be a loopless graph of dimension d with a cycle of length $\ell \ge 2$, and let the H-vector of G be $(H_0, \ldots, H_d)$. Then for any $k \le d-1$,*

$$\sum_{i \le k} H_i + (\ell - 2) \le \sum_{i \ge d-k} H_i.$$

Proof Let $e_1 \prec e_2 \prec \cdots \prec e_m$ be a fixed linear order of $E(G)$. Suppose that T is a spanning tree of G. We let $\mathcal{T}_{\prec,i,j}(G)$ denote the set of spanning trees T of G such that there are exactly i externally active edges in $(G, \prec)$ and exactly j externally active edges in $(G, \succ)$, the reverse ordering; in other words, there are exactly i edges e of G not in T for which e is the least edge in the unique cycle of $T + e$, and there are exactly j edges e of G not in T for which e is the greatest edge in the unique cycle of $T + e$. Since $0 \le i, j \le d$ and G is loopless, $i + j \le d$. Moreover,

$$\begin{aligned} H_i &= \sum_j |\mathcal{T}_{\prec,d-i,j}(G)| \\ &= \sum_j |\mathcal{T}_{\succ,d-i,j}(G)| \\ &= \sum_j |\mathcal{T}_{\succ,j,d-i}(G)| \end{aligned}$$

since the H–vector is independent of the ordering of the graph. Thus

$$\begin{aligned} \sum_{i=d-k}^{d} H_i &= \sum_{i=0}^{k} \sum_j |\mathcal{T}_{\prec,i,j}(G)| \\ &= \sum_{i=0}^{k} \sum_j |\mathcal{T}_{\succ,i,j}(G)| \\ &= \sum_{i=0}^{k} \sum_j |\mathcal{T}_{\prec,j,i}(G)| \\ &= \sum_{j=d-k}^{d} \sum_i |\mathcal{T}_{\prec,j,i}(G)| + \sum_{j=0}^{d-k-1} \sum_{i=0}^{k} |\mathcal{T}_{\prec,j,i}(G)| \\ &= \sum_{i=0}^{k} H_i + \sum_{j=0}^{d-k-1} \sum_{i=0}^{k} |\mathcal{T}_{\prec,j,i}(G)|. \end{aligned} \tag{1}$$

has dimension $m - n + 1$. The sequences $\langle F_i \rangle$ and $\langle H_i \rangle$ from the F and H forms of the reliability polynomial of G are in fact the F–vector and H–vector of the pure complex $\mathcal{F}(G)$, and we simply call these the F– and H–vectors of the graph G. We thus see at once that $\langle H_i \rangle$ is a sequence of nonnegative integers. It is this connection that allowed Ball and Provan to apply Stanley's bounds to estimate reliability efficiently.

One can improve upon the Stanley bounds, as applied to network reliability, in a number of ways. First, one can employ algorithms that calculate bounds on coefficients given a specific input graph. Second, one can employ transformations that have a predictable effect on the H–vector; when such transformations reduce the graph to a simply analyzed structure (such as a series-parallel graph), one can extract bounds on the original H–vector from that of the reduced graph. This approach is taken, for example, in [8] and [4]. The third method is to develop inequalities among the terms of the H–vector that hold for any cographic matroid. It is the third of these approaches that we examine in this paper. We develop here new bounds for H–vectors of cographic matroids, and apply the results to bounding reliability. Björner [3] states that little is known about characterizing the H–vector of matroids. Some of the results in this paper shed light on this problem as they extend also to new general inequalities on the H–vectors of matroids.

We use the following notation. A graph $G = (V(G), E(G))$ may have loops or multiple edges; a **simple** graph has no loops or multiple edges. The **order** of G is its number of vertices, and the **size** of G is its number of edges. Given an edge e of G that is not a loop, $G - e$ denotes the graph formed from G by deleting edge e, and $G \cdot e$ denotes the graph formed from G by contracting edge e (and deleting e). The **girth** of a graph G, $\gamma(G)$, is the length of its smallest cycle (if G is acyclic, we set $\gamma(G) = \infty$). If G and F are graphs then we write $G \equiv F$ if by contracting bridges and deleting loops the two graphs can be made isomorphic; if $G \equiv F$, then G and F have the same H–vector. The K_1**–bond** of graphs G and F is formed from vertex disjoint copies of G and F by identifying a vertex of each; we write $G \circ F$ for any such graph. The reliability of $G \circ F$ is the product of the reliabilities of G and F, and hence its H–vector is the convolution of the H–vectors of G and F. A cycle of length $n \geq 2$ is denoted by C_n (C_2 is a set of two parallel edges). A θ–graph consists of two vertices of degree 3 joined by three internally disjoint paths. A **multiple** of a graph G is a graph G' on the same vertex set formed by replacing each edge of G by a nonempty set of parallel edges. For $k \geq 1$, kG is the multiple of G formed by replacing each edge of G by k parallel edges. We refer the reader to [16] for standard terminology on matroids. Finally, we denote the last nonzero term in the H–vector of a graph G by H_d ($d = d(G)$ is the dimension of the associated cographic matroid if the graph is loopless, and if G is not loopless, the dimension of G is the dimension of its cographic matroid minus the number of loops in G), and say that graph G has dimension d. It is clear from $\mathrm{Rel}(G, p) = (1 - p)\mathrm{Rel}(G - e, p) + p\mathrm{Rel}(G \cdot e, p)$ that for all $i = 1, \ldots, d(G)$,

$$H_i(G) = H_{i-1}(G - e) + H_i(G \cdot e).$$

2 Non–Stanley H–Bounds From External Activity

An explicit partition of a cographic matroid of a connected graph G (see [3, 9]) is as follows. Let $\prec$ be a fixed linear order of $E(G)$, let $\mathcal{T}$ denote the set of all spanning trees of G, and for an edge e of G, let e be externally active with respect to T if $e \notin T$ but e is the least edge of the unique cycle in $T + e$. $\mathrm{NEA}(T)$ denotes the edges of G not in T that are not externally active with respect to

is a purely $(d-1)$–dimensional complex ($\overline{\sigma}$ denotes the set of all subsets of σ); a pure complex $\mathcal{C}$ is *partitionable* if its simplices can be partitioned into intervals

$$[L_1, U_1], \ldots, [L_\ell, U_\ell]$$

where each U_i is a facet of $\mathcal{C}$. It is well known (c.f. [9, 3]) that any shellable complex is partitionable, and in the case of a partitionable complex,

$$H_i = |\{j : |L_j| = i\}|.$$

Tutte's seminal paper [15] on the dichromatic polynomial employed partitions of graphic matroids using internal and external activities of edges; when extended to matroids, Tutte's observations establish that every matroid is partitionable (see also [9, 3]). The H–vector of any matroid has no internal zeros, and the last nonzero term is H_d, where d is the dimension of the matroid minus the number of coloops (see [3, 7]); this last term is the absolute value of the *Euler characteristic* of the matroid [3].

Stanley [13] completely characterizes the H–vectors of shellable complexes. Let $(H_0, \ldots, H_d)$ be the H–vector of such a complex. Fix $k \in \{0, \ldots, d-1\}$. The *k–canonical form* of a positive integer r is the integer–valued vector $(a_k, a_{k-1}, \ldots, a_j)$, where

$$r = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_j}{j}$$

and

$$a_k > a_{k-1} > \ldots a_j \geq j \geq 1$$

(such a representation exists for all r and k and is unique). $r^{<i/k>}$, the *(i,k)–th upper pseudopower of r*, is then defined to be

$$r^{<i/k>} = \binom{a_k - k + i}{i} + \binom{a_{k-1} - k + i}{i-1} + \cdots + \binom{a_j - k + i}{i - k + j};$$

it is not hard to see that $r^{<i/k>}$ is a nondecreasing function of r, $r^{<i+2/i>} = (r^{<i+1/i>})^{<i+2/i+1>}$, and $r = (r^{<i/j>})^{<j/i>}$. Then a necessary [12] and sufficient [13] condition for a nonnegative integer vector $\langle H_0, \ldots, H_d \rangle$ to be an H–vector of some shellable complex is that

$$H_0 = 1$$

and

$$H_{i+1} \leq H_i^{<i+1/i>},$$

for $i = 1, \ldots, d-1$. We call such bounds **Stanley bounds**. Stanley bounds do not put any lower bounds for the tail of an H–vector in terms of the initial sequence.

One of the best techniques for estimating reliability is the Ball–Provan method [1] (see also [9]), which relies on Stanley's bounds for the terms in the H–vectors of shellable complexes (as applied to cographic matroids). For a connected graph G, let $\mathcal{F}(G)$ be the complex on the ground set $E(G)$ of edges of G whose simplices are the subsets of $E(G)$ whose removal does not disconnect G; $\mathcal{F}(G)$ is in fact the *cographic* matroid of G, and if G has n vertices and m edges, then $\mathcal{F}(G)$

1 Introduction

Let G be a connected undirected graph, with possibly multiple edges and loops, of order n and size m. $\mathrm{Rel}(G,p)$ denotes the *reliability* of G, that is, the probability that G is connected, where each edge of G is independently operational with probability p. Much work has been done on calculating and approximating $\mathrm{Rel}(G,p)$ (whose determination is #–P complete); see for example [9] for a survey. $\mathrm{Rel}(G,p)$ is always a polynomial in p (the *reliability polynomial* of G) and can be expressed in a variety of ways (each with its own advantages to the estimation of reliability [9]):

$$\begin{aligned}
\mathrm{Rel}(G,p) &= \sum_{i=n-1}^{m} (-1)^{i-n+1} S_i p^i \quad \text{(S–form)} \\
&= \sum_{i=0}^{m-n+1} F_i p^{m-i} (1-p)^i \quad \text{(F–form)} \\
&= \sum_{i=n-1}^{m} N_i p^i (1-p)^{m-i} \quad \text{(N–form)} \\
&= p^{n-1} \sum_{i=0}^{m-n+1} H_i (1-p)^i \quad \text{(H–form)} \\
&= 1 - \sum_{i=\lambda}^{m} C_i p^{m-i} (1-p)^i \quad \text{(C–form)}
\end{aligned}$$

(λ denotes the *edge connectivity* of G). All the sequences $\langle S_i \rangle$, $\langle F_i \rangle$, $\langle N_i \rangle$, $\langle H_i \rangle$ and $\langle C_i \rangle$ consist of nonnegative integers (c.f. [9]). These sequences have combinatorial significance. N_i is the number of subsets of edges of size i that form a spanning connected subgraph of G. C_i is the number of cut sets of size i. F_i is the number of subsets of edges of size i whose removal do not disconnect G.

In fact, some results on the combinatorial properties of simplicial complexes play a role in reliability, so we introduce some salient notation. A *complex* $\mathcal{C}$ on a 'ground' set S is a non-empty collection of subsets of S that is closed under containment. The subsets of S that belong to $\mathcal{C}$ are called *faces* or *simplices*, and the maximal faces are called *facets* or *bases*. The *face vector* (or simply the F–vector) of $\mathcal{C}$, $\langle F_i \rangle$, is the sequence in which F_i is the number of faces of cardinality i in $\mathcal{C}$. A complex $\mathcal{C}$ is *pure* if all facets are of the same cardinality. The *dimension* of a pure complex is the cardinality of any facet (this definition is one greater than the dimension often defined for pure complexes, but we find this definition simpler here). For a pure complex $\mathcal{C}$ of dimension d with F–vector $(F_0, \ldots, F_d)$, the *H-vector* [2, 14], $(H_0, \ldots, H_d)$ has terms given by

$$H_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} F_j. \qquad (0 \le i \le d).$$

H–vectors have received considerable attention in polytope theory, in particular in connection with the Upper Bound Theorem.

The H–vector can also be interpreted in the following sense. For any sets L and U with $L \subseteq U$, the interval $[L,U]$ is $\{S : L \subseteq S \subseteq U\}$. A purely d–dimensional complex $\mathcal{C}$ is *shellable* if its facets can be ordered as $\sigma_1, \ldots, \sigma_m$ where for any $k \in \{2, \ldots, m\}$,

$$\overline{\sigma_k} \cap \left(\bigcup_{i=1}^{k-1} \overline{\sigma_i} \right)$$

ABSTRACT

Suppose that each edge of a connected graph G of order n is independently operational with probability p; the *reliability* of G is the probability that the operational edges form a spanning connected subgraph. A useful expansion of the reliability is as $p^{n-1} \sum_{i=0}^{d} H_i(1-p)^i$, and the Ball–Provan method for bounding reliability relies on Stanley's combinatorial bounds for the H–vectors of shellable complexes. We prove some new bounds here for the H–vectors arising from graphs, and the results here shed light on the problem of characterizing the H–vectors of matroids.

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Non–Stanley Bounds for Network Reliability

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