# The Russian Option: Reduced Regret

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**ABSTRACT**

We propose a new put asset where the option buyer receives the *maximum* price (discounted) that the option has ever traded at during the time period (which may be indefinitely long) between the purchase time and the exercise time — so that the buyer need look at the fluctuations only occasionally and enjoys having little or no regret that he didn't exercise the option at an earlier time (except for the discounting). We give an exact simple formula for the optimal expected present value (fair price) that can be derived from the option and the (unique) optimal exercise strategy which achieves the optimum value under the assumption that the asset fluctuations follow the Black-Scholes exponential Brownian motion model, widely accepted. It is important to note that the discounting is necessary; if it is omitted or even if it is less than the Black-Scholes drift then the value to the buyer under optimum performance is infinite. We also solve the same problem under a different model — that of the original Bachelier linear Brownian market with linear discounting; this model is no longer accepted, but of course the mathematics is consistent.

To our knowledge no such regretless option is currently traded in any existing market despite its evident appeal. We call it the Russian option partly to distinguish it from the American and European options, where the term of the option is prescribed in advance and where no exact formula for the value has been given.

*The second author is a member of the Steklov Math Institute, Moscow; the work reported here was done while he was visiting Bell Labs.

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## 1 Introduction

Suppose the fluctuations in the price of an asset are given by the geometric Brownian motion model

$$X_t = x e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t} \ , \quad t \geq 0 \tag{1.1}$$

where $x > 0$, $W_0 = 0$ and $W_t$ is a standard Wiener process. The process $X$, which satisfies the stochastic differential equation $dX = \sigma X dW + \mu X dt$, forms the basis for the famous option pricing theory of Black and Scholes [2b, 2c]. The parameters, $\mu$ called the drift, and $\sigma$ the volatility, are assumed known.

We solve the following mathematical problem, where $r > 0$ and $s \geq x$ are given, and we want to find a stopping time $\tau \in [0, \infty)$ (which need be a fixed time but can depend on the fluctuations observed to date in any way) to

$$\underset{\tau}{\text{maximize}} \quad E e^{-r\tau} S_\tau \tag{1.2}$$

where $S$ is the maximum value, starting at $s$, for $X$, i.e.

$$S_t = \max \left\{ s, \ \sup_{0 \leq u \leq t} X_u \right\} \ , \quad t \geq 0 \ . \tag{1.3}$$

The motivation for (1.2) is the study of a new financial option which arose first as a consequence of the probability theory developed to solve (1.2) and to our knowledge is not currently traded in any existing market. This (Russian) option allows its owner to choose an exercise date, represented by the stopping time $\tau$, and then pays the owner either $s$ or the maximum stock price achieved up to this exercise date, whichever is larger, discounted by $\exp(-r\tau)$. In problem (1.2) the owner of the option seeks an exercise strategy which will maximize the expected (present) value of his future reward, where $r$ is the interest rate for

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discounting. Starting with our solution to the mathematical problem (1.2), J. Darrell Duffie and J. Michael Harrison [13] derive an "arbitrage price" for the Russian option. Their pricing analysis parallels the analysis of European call options by Black and Scholes [2c]. Of necessity, this involves a more complete discussion of the interest rate $r$ and drift parameter $\mu$ that are appropriate for arbitrage pricing. In the final analysis, it is not necessary for arbitrage pricing that investors agree on the average rate of return earned by the stock underlying a Russian option, or for that matter, on $\mu$ and $\sigma$; indeed differences may increase the potential for trading.

In this paper the value of the option (i.e. the supremum in (1.2)) will be found exactly, and in particular it will be shown that the maximum in (1.2) is finite if and only if

$$r > \mu \ . \tag{1.4}$$

Assuming (1.4), an explicit formula is given for both the maximal expected present value and the optimal stopping rule in (2.4), which is not a fixed time rule but depends heavily on the observed values of $X_t$ and $S_t$.

We call the financial option described above a "Russian option" for two reasons. First, this name serves to (facetiously) differentiate it from American and European options, which have been extensively studied in financial economics, especially with the new interest in market economics in Russia. Second, our solution of the stopping problem (1.2) is derived by the so-called principle of smooth fit, first enunciated by the great Russian mathematician, A. N. Kolmogorov, cf. [4, 5]. The Russian option is characterized by "reduced regret" because the owner is paid the *maximum* stock price up to the time of exercise and hence feels less remorse at not having exercised at the maximum.

For purposes of comparison and to emphasize the mathematical nature of the contribution here, we conclude the paper by analyzing an optimal stopping problem for the Russian option based on Bachelier's (1900) original linear model of stock price fluctuations,

$$X'_t = x + \sigma W_t + \mu t \ , \quad t \geq 0 \ . \tag{1.5}$$

We again introduce the running maximum as in (1.3)

$$S'_t = \max \left\{ s, \sup_{0 \leq u \leq t} X'_u \right\} \tag{1.6}$$

where $\mu, \sigma, x, s$ are as before except $x, s$ can be negative, $x \leq s$, and solve the problem (in §3)

to

$$\text{maximize}_{\tau} \quad E(S'_\tau - r\tau) \tag{1.7}$$

where discounting is now also applied *linearly* (the case of exponential discounting seems to have no simple solution). The simple explicit value in (1.7) is given in (3.8) along with the optimal stopping rule $\tau$, which is rather different. In the geometric case, $S_\tau / X_\tau = \alpha$, for some $\alpha$ is the form of the stopping time while in the linear case $S_\tau - X_\tau = \theta$, for some $\theta$.

## 2 Derivation of the optimal pricing formula for the Russian option

Let $x, s, \mu, \sigma, r, X_t, S_t$ be as in (1.1) and (1.3) and define

$$V^*(x,s) = V^*(x,s,\mu,\sigma,r) = \sup_\tau E_{x,s} e^{-r\tau} S_\tau \tag{2.1}$$

where the sup is taken over all stopping rules. We will first give a rigorous but rather deus-ex-machinistic proof that $V^*(x,s)$ agrees with $V(x,s)$ below and then supply some motivation, or derivation, from the principle of smooth fit, as to how $V$ was actually guessed. Since the optimal free boundary here turns out so simple, in a sense this example does not show the full power of the principle, although the form of $V$, below, suggests it's not so trivial, after all. If one intuits that the optimal rule $\tau$ is of the form in (2.14) below, one could try to optimize the choice of $\alpha$ and so derive (2.3) and (2.4) below. We do not see how to carry this out (even in the case $\mu = \sigma^2/2$), although maybe it can be done, and this might give an alternate derivation of (2.3) and (2.4) as has been suggested by several readers. (Note: see also our new papers [16], [17].)

So assume $r > \max(0, \mu)$ as in (1.5) and let $\gamma = \gamma_1$, $\gamma = \gamma_2$, $\gamma_1 < 0 < 1 < \gamma_2$ be the two roots of the quadratic equation

$$\frac{1}{2}\sigma^2\gamma^2 + \gamma\left(\mu - \frac{\sigma^2}{2}\right) = r \; ; \qquad \gamma_{1,2} = \frac{\frac{\sigma^2}{2} - \mu \pm \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} \tag{2.2}$$

and set

$$\alpha = \left(\frac{1 - \frac{1}{\gamma_1}}{1 - \frac{1}{\gamma_2}}\right)^{\frac{1}{\gamma_2 - \gamma_1}} \tag{2.3}$$

and

$$V(x,s) = \begin{cases} \dfrac{s}{\gamma_2 - \gamma_1}\left(\gamma_2\left(\dfrac{\alpha x}{s}\right)^{\gamma_1} - \gamma_1\left(\dfrac{\alpha x}{s}\right)^{\gamma_2}\right) \; , & \dfrac{s}{\alpha} \le x \le s \\ s \; , & 0 < x \le \dfrac{s}{\alpha} \; . \end{cases} \tag{2.4}$$

To prove $V^* \equiv V$, verify that $V(x,s)$ in $s/\alpha < x \leq s$ satisfies, by direct observation, $V \in C^2$,

$$rV(x,s) = \mu x V_x(x,s) + \frac{1}{2}\sigma^2 x^2 V_{xx}(x,s) \tag{2.5}$$

$$V(x,s) \geq s \tag{2.6a}$$

$$V_s(s,s) = \frac{\partial V}{\partial s}(x,s)\Big|_{x=s} = 0 \ . \tag{2.6b}$$

Now $X_t$ has Ito differential, from (1.1)

$$dX_t = X_t(\mu dt + \sigma dW_t) \tag{2.7}$$

and so the process

$$Y_t = e^{-rt}V(X_t, S_t) \tag{2.8}$$

is a supermartingale, i.e. in the region $0 < X_t \leq S_t/\alpha$

$$dY_t = de^{-rt}S_t = -re^{-rt}S_t dt \leq 0 \ , \tag{2.9}$$

and in the region $S_t/\alpha \leq X_t \leq S_t$, since $S_t$ grows only when $X_t = S_t$ and $V_s(s,s) = 0$ in (2.6b),

$$\begin{aligned} dY_t &= e^{-rt}[V_x(X_t,S_t)dX_t + \tfrac{1}{2}V_{xx}(X_t,S_t)(dX_t)^2 - rV(X_t,S_t)dt] \\ &= e^{-rt}V_x(X_t,S_t)X_t\sigma dW_t \end{aligned} \tag{2.10}$$

using (2.5), and so in (2.10) $Y_t$ is a positive local martingale, hence again a supermartingale and

$$E_{x,s}dY_t \leq 0 \ , \quad t \geq 0 \ . \tag{2.11}$$

Thus we can write for any stopping time $\tau$, i.e. which does not anticipate the future (in the sense that at the time of stopping only information independent of future increments of $W$ is usable), by (2.6a)

$$\begin{aligned} E_{x,s}e^{-r\tau}S_\tau &\leq E_{x,s}e^{-r\tau}V(X_\tau, S_\tau) \\ &= E_{x,s}Y_\tau \leq E_{x,s}Y_0 \\ &= V(X_0,S_0) = V(x,s) \ , \end{aligned} \tag{2.12}$$

where we used the fact that $Y$ is a supermartingale (2.9) and (2.11) to obtain the second inequality. If we sup over all such $\tau$ we obtain for all $0 < x \leq s$,

$$V^*(x,s) \leq V(x,s) \ . \tag{2.13}$$

To prove the reverse inequality, let $\tau$ be the $1^{\text{st}}$ $t$ for which

$$X_t = S_t/\alpha \tag{2.14}$$

starting from $X_0 = x$, $S_0 = s$, $x > s/\alpha$. It is clear that $P(\tau < \infty) = 1$ because

$$P\{\tau > T\} = P\left\{\text{for } 0 \le u \le t \le T, \;\; \sigma(W_t - W_u) + \left(\mu - \frac{\sigma^2}{2}\right)(t-u) \ge \log\frac{1}{\alpha}\right\} \tag{2.15}$$

which tends to zero as $T \to \infty$ no matter what the sign of $\mu - \sigma^2/2$ is. The first inequality in (2.12) is thus an equality because of (2.15) and (2.4). The second inequality will also be shown to be an equality if we can prove that $Y_t$, $0 \le t \le \tau$ is a uniformly integrable martingale [6]. We supply a (simpler) direct proof by showing that

$$E \sup_{0 \le t < \infty} Y_t < \infty \;, \tag{2.16}$$

which will directly prove the equality we need.

To prove (2.16) we note that

$$\begin{aligned} Y_t &= e^{-rt}V(X_t, S_t) \\ &\le e^{-rt}V(S_t, S_t) \\ &= Ke^{-rt}S_t \end{aligned} \tag{2.17}$$

since $V(S,S) = KS$ where the constant $K$ is given from (2.4) by

$$K = \frac{1}{\gamma_2 - \gamma_2}(\gamma_2\alpha^{\gamma_1} - \gamma_1\alpha^{\gamma_2}) \;. \tag{2.18}$$

So it is enough to show that $\sup_t \exp(-rt)S_t$ is integrable, i.e.

$$\int_0^\infty dy \; P\left\{\sup_t e^{-rt}S_t > y\right\} < \infty \;. \tag{2.19}$$

By a well-known theorem of Doob [6, 7] for $\alpha > 0$, $\beta > 0$,

$$P\{W_t \le \alpha t + \beta, \; 0 \le t < \infty\} = 1 - e^{-2\alpha\beta} \;. \tag{2.20}$$

If we choose,

$$\alpha = \left(r - \mu + \frac{\sigma^2}{2}\right)/\sigma, \quad \beta = \frac{1}{\sigma}\log\left(\frac{y}{x}\right) \tag{2.21}$$

then from (1.3) for $y > s$, $y > x$

$$P\left\{\sup_t e^{-rt}S_t > y\right\} = P\left\{\sup_t \left[\sup_{0 \le u \le t}\left(\sigma W_u + (\mu - \sigma^2/2)u\right) - \left(\log\frac{y}{x} + rt\right)\right] > 0\right\} \;. \tag{2.22}$$

Now if $W_t \le \alpha t + \beta$ for all $t$, then from (2.21),

$$\sup_{0\le u\le t}\left(\sigma W_u + \left(\mu - \frac{\sigma^2}{2}\right)u\right) \le \sup_{0\le u\le t}\left(\sigma(\alpha u+\beta) + \left(\mu-\frac{\sigma^2}{2}\right)u\right) = \sup_{0\le u\le t}\left(\log\left(\frac{y}{x}\right) + ru\right) \tag{2.23}$$

and since $r > 0$ the right side is growing in $u$ and is maximized at $u = t$. It follows from (2.22) and (2.23) using (2.20), that for $y > s$, $y > x$

$$P\left\{\sup_t e^{-rt}S_t > y\right\} \le e^{-2\alpha\beta} = \left(\frac{y}{x}\right)^{-\left(1+\frac{2(r-\mu)}{\sigma^2}\right)} . \tag{2.24}$$

Thus for $r > \mu$, the integral in (2.19) converges and so (2.16) is proved. Now suppose $\tau$ is the stopping time in (2.14). Since $Y_t$ is a (uniformly integrable) martingale, we have for any fixed $t$,

$$E_{x,s}Y_{t\wedge\tau} = E_{x,s}Y_0 . \tag{2.25}$$

Letting $t \to \infty$, using dominated convergence and (2.16) we see that

$$E_{x,s}Y_\tau = E_{x,s}Y_0 . \tag{2.26}$$

Note that (2.16) also shows that $Y_t$ is a uniformly integrable martingale, which implies (2.26) directly by optional sampling. Thus equality holds also in the second inequality in (2.12) for the choice of $\tau$ in (2.14). But this $\tau$ is included in the sup in (2.1) so that

$$V^*(x,s) \ge V(x,s) \tag{2.27}$$

as well. By (2.13) and (2.27) we have proved

$$V^*(x,s) \equiv V(x,s) \quad \text{for} \quad 0 < x \le s \tag{2.28}$$

as promised.

The proof is unrevealing; how were (2.2)–(2.4) derived? The answer is that we used the "principle of smooth fit." This principle goes back to A. N. Kolmogorov, was discovered in Russia in the '50's, independently later found by Herman Chernoff [8] in the US and also by Lindley in Great Britain. It was used by B. I. Grigelionis and A. N. Shiryaev [4] and others [9], [10], though even now it is not widely appreciated. A new application to Burkholder-Gundy inequalities is in a paper under preparation [12]. It often enables one to obtain (see especially [5], [8], [9], [10]) explicit closed form solutions to optimal stopping or optimal control problems in continuous problems where the discrete versions cannot be solved in explicit form. See [5],

[10] for more details on the technique. In this problem we see that (2.2)–(2.4) were guessed by seeking a $C^2$ function $V(x,s)$ which satisfies $V(x,s) = s$ for $x \geq g(s)$. Note that the continuation region is intuitively guessed to be of this form, i.e. we should not exercise the option if the maximum process, $S_t$, is just about to take an increase. But for $0 \leq x \leq g(s)$ we exercise the option, so $V(x,s) = s$ in this region. The differential equation (2.5) holds in the continuation region, and the principle of smooth fit says, only heuristically of course, that the free boundary $g$ will be determined by $V \in C^2$. This heuristic is only used to guess $V$ as in (2.2)–(2.4); once guessed, the rigorous proof is given in (2.5)–(2.28), in an almost crank-turning way.

Let's look closer at how to guess because it has some new features in this problem. The differential equation (2.5) for $V$ given in (2.4) is obtained by observing that if we elect to continue letting the option run for a small time $h > 0$ then by (2.7), for $g(s) < x < s$

$$V(x,s) = EV(x + x\mu h + \sigma\sqrt{h}\eta, s)e^{-rh} \tag{2.29}$$

and expanding by Ito calculus gives (heuristically) (2.5). The condition of the principle of smooth fit, that $V$ should be in $C^2$, makes Ito's rule applicable, and, at least heuristically, is sufficient to determine the free boundary, $g(s)$. Let's see how it works in the present case.

If $V(x,s)$ satisfies (2.5) in $g(s) < x < s$ then

$$V(x,s) = a(s)x^{\gamma_1} + b(s)x^{\gamma_2} \tag{2.30}$$

where $\gamma_1, \gamma_2$ are as in (2.2). Since $V(x,s)$ must fit smoothly at $x = g(s)$ with $s$ we must have

$$V(g(s),s) = s, \quad V_x(g(s),s) = 0\ . \tag{2.31}$$

Further smoothness conditions, e.g. $V_s(g(s),s) = 1$ now follow automatically and give no further information to determine (guess) the function $g$. Instead, we must obtain a condition along the known (nonfree) boundary $x = s$. This is apparently a novel feature of this problem and is due to the appearance of the process $S_t = \max_{0 \leq u \leq t}(X_u)$. Along this boundary we needed the condition

$$V_s(s,s) \equiv 0 \tag{2.32}$$

in (2.10) at $X_t = S_t$ in order to prove that $Y_t$ is a local martingale there. The conditions (2.31) and (2.32) give a differential equation for $g = g(s)$ by eliminating $a(s)$ and $b(s)$:

$$g' = \frac{\frac{1}{\gamma_1}\left(\frac{s}{g}\right)^{\gamma_1} - \frac{1}{\gamma_2}\left(\frac{s}{g}\right)^{\gamma_2}}{\left(\frac{s}{g}\right)^{\gamma_1+1} - \left(\frac{s}{g}\right)^{\gamma_2+1}}\ . \tag{2.33}$$

It is hard (but possible) to find the general solution to equation (2.33) for $g$, but there is one simple solution to (2.33) — the one we need! At this point one can merely guess at this solution:

$$g(s) = s/\alpha \tag{2.34}$$

with $\alpha$ as in (2.3). One might try to develop a further heuristic that will give the extra boundary condition needed:

$$g(0) = 0 \tag{2.35}$$

but the principle of smooth fit is only a heuristic anyway, so we are content to merely "guess" (2.34). However, we remark that (2.33) can be solved explicitly for other values of $g(0)$. If we make the substitution

$$g(s) = \frac{s}{h(s)} \tag{2.36}$$

then (2.33) separates into

$$\frac{ds}{s} = \frac{h^{\gamma_1 - 1} - h^{\gamma_2 - 1}}{\left(1 - \frac{1}{\gamma_1}\right) h^{\gamma_1} - \left(1 - \frac{1}{\gamma_2}\right) h^{\gamma_2}} dh \tag{2.37}$$

and an explicit integration is possible, although not elementary.

## 3 The Bachelier version

For purposes of comparison we also obtain the pricing formula that the option seller should use to find his break-even point should he believe in the Bachelier rather than the Black-Scholes model of asset fluctuation. We will use primes to denote this model where the price or value of the asset follows the nonexponential evolution:

$$X'_t = x + \mu t + \sigma W_t, \qquad t \geq 0 \tag{3.1}$$

where $\sigma > 0$, $x$, and $\mu$ are given parameters. Again we set analogous to (1.3)

$$S'_t = \max\left(s, \max_{0 \leq u \leq t} X'_u\right) \ . \tag{3.2}$$

The buyer is allowed to exercise his option at any time $t > 0$ and obtains the payoff (analogous to (1.4))

$$S'_t - rt \tag{3.3}$$

where $r > 0$ is the cost of retaining the option for time $t$. Analogous to (1.5) we assume $r > 0$ and $s \geq x$ and

$$r > \mu \ . \tag{3.4}$$

The problem to determine the price of option (3.3) is simpler than that of (1.4) but has apparently not been solved before despite its simplicity, although very similar problems have been discussed in [11]. If we overlooked a prior solution, perhaps it has not been solved with the smooth fit principle, but the elementary solution could have been simply guessed in some other way. Again the full power of smooth fit is perhaps better shown in examples where the free boundary is more difficult to guess than this one.

So we want to determine

$$V^{*'}(x,s) = V^{*'}(x,s,\mu,\sigma,r) = \sup_{\tau} E_{x,s}[S_\tau - r\tau] \tag{3.5}$$

where the sup is taken over all stopping rules $\tau$ with now the important proviso that

$$E\tau < \infty \tag{3.6}$$

to avoid $\infty$-$\infty$ in (3.5).

The answer will be shown to be $V^{*'} = V'$ where we first define $\theta > 0$ by

$$\theta = \frac{\sigma^2}{2\mu} \log \frac{1}{1 - \frac{\mu}{r}} \tag{3.7}$$

and then take

$$V'(x,s) = \begin{cases} s + \dfrac{r}{\mu}(x - s + \theta) - \dfrac{r\sigma^2}{2\mu^2}(1 - e^{-\frac{2\mu}{\sigma^2}(x-s+\theta)}) \ , & s - \theta \le x \le s \\ \\ s \ , & -\infty \le x \le s - \theta \ . \end{cases} \tag{3.8}$$

It is easy to check that $V' \in C^2$ in $-\infty < x \le s$, that

$$\frac{\partial V'}{\partial s}(s,s) = 0 \tag{3.9}$$

and that $V'$ satisfies the O.D.E

$$r = \mu V'_x(x,s) + \frac{1}{2}\sigma^2 V'_{xx}(x,s) \ , \quad x - \theta \le x \le s \ . \tag{3.10}$$

Since $\exp(-y) \ge 1 - y$ it is easy to see that

$$V'(x,s) \ge s \ . \tag{3.11}$$

Thus if we define the process

$$Y'_t = V'(X'_t, S'_t) - rt \ , \quad t \ge 0 \tag{3.12}$$

and note that $X'$ has the Ito differential

$$dX'_t = \mu dt + \sigma dW_t \tag{3.13}$$

we see that

$$dY'_t = \frac{r\sigma}{\mu} dW_t \text{ in } S_t - \theta \le X_t; \quad dY'_t = -r dt \text{ in } X_t \le S_t - \theta \tag{3.14}$$

so that $Y'_t$ is a supermartingale (note (3.9) is needed at $X_t = S_t$). Thus we can write for any stopping rule $\tau$, by (3.11) and (3.12)

$$E_{x,s}(S'_\tau - r\tau) \le E_{x,s} Y'_\tau \tag{3.15}$$

and since $Y'$ is a supermartingale, and $(x_0, s_0) = (x, s)$ we have

$$E_{x,s} Y'_\tau \le E Y'_0 = V'(x, s) \tag{3.16}$$

so that

$$V^{*'}(x, s) \le V'(x, s) \; . \tag{3.17}$$

Now letting $\tau$ be the $1^{\text{st}}$ $t$ for which

$$X_t = S_t - \theta \tag{3.18}$$

gives equality in (3.15) and (3.16) by arguments analogous to those in §2 for $W$ and so $V^{*'}(x, s) = V^{*'}(x, s)$. The choice of $V'$ in (3.8) can be (and was) "derived" or guessed by using the principle of smooth fit in a similar way as $V$ in (2.4).

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Many years ago, L. Shepp learned the concept of "regret" in the context of options from H. B. Sosin and M. B. Goldman, and now he "regrets" that their seeds didn't bear more immediate fruit.

After this paper was written, our attention was brought to several other papers with related methodology and direction, [14a,b,15a,b], which however do not contain any of our results.

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