A Dual Russian Option for Selling Short

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ABSTRACT

We propose a new *call* option where the option *seller pays* the *minimum* price (in *inflated* dollars) that the asset has ever traded at during the time period (which may be indefinitely long) between the *selling* time and the *delivery* time (to be chosen by the *seller*). This option is the dual of the *put* option where the option *buyer receives* the *maximum* price (in *discounted* dollars) that the asset has ever traded at during the time period (which may be indefinitely long) between the *buying* time and the *exercise* time (to be chosen by the *buyer*). Because the settlement payoff is at the minimum (for the call) or the maximum (for the put) there is reduced regret in the sense that it is not necessary for the option holder to worry about missing a good price in the recent past (of course he may regret not holding on longer) since he gets the best price up to the settlement time. We give the exact simple formula for the optimal expected present value (fair price) that can be derived from the option and the (unique) optimal exercise strategy which achieves the optimum value under the assumption that the asset fluctuations follow the Black-Scholes exponential Brownian motion model, widely accepted. The Russian put option was studied earlier.

We show that in both cases, puts or calls, the simpler option where the settlement is based on the *current* price at the time of exercise rather than on the maximum or minimum price to date (i.e. without the "Russian" feature) does not give rise to an interesting stopping problem and there can be no trading on such an option.

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1. Introduction

Suppose the fluctuations in the price of an asset are given by the geometric Brownian motion model

$$X_t = xe^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}, \quad t \geq 0 \tag{1.1}$$

where $x > 0$, $W_0 = 0$ and W_t is a standard Wiener process. The process X, which satisfies the stochastic differential equation $dX = \sigma dW + \mu X dt$, forms the basis for the famous option pricing theory of Black and Scholes [3a, 3b]. The parameters, μ called the drift, and σ the volatility are assumed known.

We solve the following mathematical problem, where $r > 0$ and $y \leq x$ are given, and we want to find a stopping time $\tau \in [0, \infty)$ which need not be a fixed time but can depend on the fluctuations observed to date in any way) to

$$\underset{\tau}{\text{minimize}} \quad Ee^{r\tau}Y_\tau \tag{1.2}$$

where Y is the minimum value, starting at y, for X, i.e.,

$$Y_t = \min\left\{y, \inf_{0 \leq u \leq t} X_u\right\}, \quad t \geq 0\ . \tag{1.3}$$

The motivation for (1.2) is the study of a new financial option which arose first as a dual (call) option of an earlier put option which was studied by us in [1, 2], but to our knowledge is not currently traded in any existing market. This call option which is a short sale operates as follows: the short seller receives a fixed payment at time $t = 0$, which we will calculate, assuming the model (1.1), and incurs an obligation to deliver to the customer (buyer) the asset, stock or bond or other, at some future date, τ, which the seller can choose. Moreover, he can obtain the asset at the best price he could have obtained during, $0 \leq t \leq \tau$, i.e. the value Y_τ, but he must pay off in equivalent dollars at time $t = 0$, i.e. $e^{r\tau}Y_\tau$. Here $r \geq 0$ is a rate of

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inflation *mutually agreed upon* by the short seller and his customer at time $t = 0$. The effect of this option is to provide the short seller, who believes that the asset will decrease in value, a reduced regret vehicle to risk his capital on the venture that the asset will diminish in value, and to obtain a payoff at time 0 commensurate with his risk in taking on the obligation to pay off at some future time in inflated dollars. In practice the short seller could wait forever and apparently avoid his liability, but the customer (stock-broker) would of course require proof at all times that the liability of the short seller can be met — and if the fund of the short seller falls below $e^{rt}Y_t$ at any time t, the option is immediately called, i.e. $\tau = t$.

In this paper the value of the option (i.e. the infimum in (1.2)) will be found exactly, and in particular it will be shown that the minimum in (1.2) is zero when $r < 0$, so there can be no price in this case since the seller has no risk. In (2.2c) we show that even for $r > 0$, if $\mu < \sigma^2/2$ and $r \le r_0 = \frac{1}{2\sigma^2}(\mu - \sigma^2/2)^2$ the minimum in (1.2) is zero so that the short buyer at least will require $\mu > \sigma^2/2$ or $r > r_0$. In all other cases the value is nontrivial and is found in §2.

The option (1.2) is dual to the put option introduced in [1] and dubbed the Russian option, where the mathematical problem was to

$$\underset{\tau}{\text{maximize}} \quad Ee^{-r\tau}S_\tau \tag{1.4}$$

where S is the maximum value, starting at $s \ge x$, for X, i.e.

$$S_t = \max\left\{s, \sup_{0\le u\le t} X_u\right\} . \tag{1.5}$$

Here the buyer of the option pays a certain fair price (namely the maximum in (1.4), found in [1]) and obtains the privilege to obtain $e^{-rt}S_t$ at any time $t = \tau$ of his choosing. In this case the buyer expects X_t to increase and is thus betting that he will obtain a profit. The fair price was calculated in [1] and the result of [1] is stated in Appendix 1 for comparison and convenience.

If we use the Bachelier (1900) linear model of stock price fluctuations, instead of Black-Scholes, i.e.

$$X'_t = x + \sigma W_t + \mu t, \quad t \ge 0 \tag{1.6}$$

and again introduce the running minimum as in (1.3),

$$Y'_t = \min\left\{y, \inf_{0\le u\le t} X'_u\right\} \tag{1.7}$$

where μ, σ, x, y are as before except x, y can be negative, $y \le x$. We solve the problem (in §3) to

$$\underset{\tau}{\text{minimize}} \quad E(Y'_\tau + r\tau) \tag{1.8}$$

where the inflating is now also applied *linearly* (the case of exponential inflation seems to have no simple solution). The simple explicit fair Bachelier value is given in (3.4) below along with the optimal stopping rule τ, (3.11).

Consider the simpler option where the short seller uses not the minimum price Y_t but the actual current price X_t at the time he delivers the asset (i.e. completes the transaction). We prove in the Appendix that this option is not interesting in the sense that the optimal stopping time is either $\tau = 0$ or $\tau = \infty$, and the reward is either zero or infinite. We prove that a similar observation is true also for the dual put option studied earlier in [1]. Thus the "Russian" feature of *reduced risk* is the critical aspect to make the options feasible.

2. The optimal pricing formula for the Russian call option

Let $x, y, \mu, \sigma, r, X_t, Y_t$ be as in (1.1) and (1.3). Let us first calculate when the value

$$V^*(x,y) = V^*(x,y,\mu,\sigma,r) = \inf_\tau \; E_{x,y} e^{r\tau} Y_\tau \;, \tag{2.1}$$

with the inf taken over all stopping rules, degenerates to zero.

Lemma: *If any of the following hold then* $V^* = 0$

$$r < 0 \tag{2.2a}$$

$$r = 0, \quad \mu \le \sigma^2/2 \tag{2.2b}$$

$$0 < r \le \frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2, \quad \mu < \frac{\sigma^2}{2} \tag{2.2c}$$

Proof. We set $\tau \equiv T \to \infty$ in case (2.2a,b) and since $Y_t \to 0$ in these cases (since Y_t decreases) and e^{rt} remains bounded we are done. Note that $\tau = \infty$ is, strictly speaking, excluded as a stopping time. The case (2.2c) is a little harder, but again we show that the fixed times $\tau_T \equiv T$ achieve 0 in the limit as $T \to \infty$. We must show that under (2.2c)

$$\lim_{T\to\infty} e^{rT} E \inf_{0\le t\le T} \left(e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t} \right) = 0 \;. \tag{2.3}$$

By the well-known formula, for $b > 0$, with Φ the standard normal cumulative,

$$P\{W_t \le at + b,\ 0 \le t \le T\} = \Phi\left(a\sqrt{T} + \frac{b}{\sqrt{T}}\right) - e^{-2ab}\Phi\left(a\sqrt{T} - \frac{b}{\sqrt{T}}\right) \tag{2.4}$$

we write (2.3), after some calculation, as

$$\begin{aligned}&\lim_{T\to\infty} e^{rT}\int_0^\infty e^{-x} d_x P\left\{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t \ge -x,\ 0 \le t \le T\right\} \\ &= \lim_{T\to\infty} e^{rT}\left[e^{\mu T}\left(1 + \frac{\sigma^2}{2\mu}\right)\Phi\left(-\left(\frac{\sigma}{2} + \frac{\mu}{\sigma}\right)\sqrt{T}\right) + \left(1 - \frac{\sigma^2}{2\mu}\right)\Phi\left(-\left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right)\sqrt{T}\right)\right] .\end{aligned} \tag{2.5}$$

Since the argument of the Φ's is negative because $\mu < \sigma^2/2$, we see that the term inside the limit is

$$o\left(e^{rT}\left[e^{\mu T} e^{-\left(\frac{\sigma}{2} + \frac{\mu}{\sigma}\right)^2 \frac{T}{2}} + e^{-\left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right)^2 \frac{T}{2}}\right]\right) \to 0 \tag{2.6}$$

since $r + \mu - \left(\frac{\sigma}{2} + \frac{\mu}{\sigma}\right)^2 \frac{1}{2} \le 0$ and $r - \left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right)^2 \frac{1}{2} \le 0$ and the lemma is proved.

In the remaining cases of μ, σ, r, the value of V^* will be nontrivial and will be proved to be the values given by V below.

For $r = 0$, $\mu > \sigma^2/2$, and throughout $y \le x$ of course,

$$V(x, y) = y\left(1 - \left(\frac{y}{x}\right)^{\frac{2\mu}{\sigma^2} - 1} \frac{\sigma^2}{2\mu}\right) , \tag{2.7}$$

and $\tau_{opt} = \infty$ (asymptotically $\tau \equiv T \to \infty$ achieves the minimum).

For $0 < r < \frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2$, $\mu > \frac{\sigma^2}{2}$

$$V(x, y) = \frac{y}{\gamma_2 - \gamma_1}\left(\gamma_2\left(\frac{x}{\theta y}\right)^{\gamma_1} - \gamma_1\left(\frac{x}{\theta y}\right)^{\gamma_2}\right) , \quad \theta = \left(\frac{1 - \frac{1}{\gamma_2}}{1 - \frac{1}{\gamma_1}}\right)^{\frac{1}{\gamma_2 - \gamma_1}} \tag{2.8}$$

where γ_1 and γ_2 are the roots ($\gamma_1 < \gamma_2 < 0$) of $\frac{\sigma^2}{2}\gamma^2 - \left(\frac{\sigma^2}{2} - \mu\right)\gamma + r = 0$,

$$\gamma_{1,2} = \frac{\frac{\sigma^2}{2} - \mu \pm \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 - 2r\sigma^2}}{\sigma^2} \tag{2.9}$$

N. B. γ_1 and γ_2 are negative, $\gamma_1 < \gamma_2 < 0$, and $\theta > 1$ in (2.8).

For $r = \frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2$, $\mu > \frac{\sigma^2}{2}$ (a limiting case of (2.8), $\gamma_1 = \gamma_2 = \gamma$)

$$V(x, y) = y\left(\left(\frac{x}{\theta y}\right)^\gamma - \gamma\left(\frac{x}{\theta y}\right)^\gamma \log\left(\frac{x}{\theta y}\right)\right) , \quad \theta = e^{\frac{1}{\gamma(\gamma - 1)}} . \tag{2.10}$$

N. B. The quadratic equation for γ_1 and γ_2 has equal roots $\gamma = \frac{1}{\sigma^2}\left(\frac{\sigma^2}{2} - \mu\right) < 0$ and $\theta > 1$.

For $r > \frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2$ both roots of the quadratic are complex conjugates, $\gamma, \overline{\gamma}$

$$V(x,y) = \frac{y}{\gamma - \overline{\gamma}}\left(\gamma\left(\frac{x}{\theta y}\right)^{\overline{\gamma}} - \overline{\gamma}\left(\frac{x}{\theta y}\right)^{\gamma}\right) \quad , \quad \theta = \left(\frac{1-\frac{1}{\gamma}}{1-\frac{1}{\overline{\gamma}}}\right)^{\frac{1}{\gamma-\overline{\gamma}}} \tag{2.11}$$

where $\overline{\gamma}$ is the conjugate (replace i by $-i$) of

$$\gamma = \frac{\frac{\sigma^2}{2} - \mu + i\sqrt{2r\sigma^2 - \left(\mu - \frac{\sigma^2}{2}\right)^2}}{\sigma^2} \ . \tag{2.12}$$

N. B. γ is in the upper half-plane and $1 - 1/\gamma = re^{i\phi}$ is in the upper half-plane so that with $\beta = \mathrm{Im}(\gamma) > 0$, $\theta = \exp(\phi/\beta) > 1$, as always, and V in (2.11) is real (since its equal to its conjugate).

Theorem: *$V^*(x,y) = V(x,y)$ in each of the cases* (2.7), (2.8), (2.10), *and* (2.11) *and in each case* (2.8), (2.10), (2.11), *$V(x,y) = y$ for $x \geq \theta y$. The optimal stopping rule τ_{opt} is the first time t that $X_t = \theta Y_t$ in each case.*

Proof. Each of the proofs are similar except for details and are similar to the proof in [1] in the dual case (1.4), except for (2.7), which is rather simpler. Since $Y_t \downarrow$, when $r = 0$, clearly the short seller should wait forever and buy the asset for delivery at the overall minimum which can be no greater than the value $Y_0 = y$. But then clearly

$$V^*(x,y) = EY_\infty \ . \tag{2.13}$$

One could find this directly in this case, but it is simpler to proceed more generally as follows.

Verify in each of the cases (2.7), (2.8), (2.10), and (2.11) that $V \in C_2$ in $0 < y \leq x$, and that for $0 < y \leq x \leq \theta y$ where θ is given in each case ($\theta = \infty$ in (2.7))

$$\frac{1}{2}\sigma^2 x^2 V_{xx}(x,y) + \mu x V_x(x,y) + rV(x,y) = 0 \tag{2.14}$$

$$V(x,y) \leq y \ ; \quad V(x,y) = y \ \text{ for } \ x \geq \theta y \tag{2.15}$$

$$V_y(y,y) \triangleq \frac{\partial V}{\partial y}(x,y)\Big|_{x=y} = 0 \ . \tag{2.16}$$

Since $V(x,y) = y$ for $x \geq \theta y$, the inequality in (2.15) holds for all x, y. Now X_t has Ito differential, from (1.1)

$$dX_t = X_t(\mu dt + \sigma dW_t) \tag{2.17}$$

and so the process

$$Z_t = e^{rt} V(X_t, Y_t) \tag{2.18}$$

is a submartingale, i.e. in the region $0 < \theta Y_t \leq X_t$

$$dZ_t = d(e^{rt}\theta Y_t) = \theta Y_t r e^{rt} dt \geq 0 \tag{2.19}$$

since Y_t is constant when $X_t > Y_t$, while in the region $0 < Y_t \leq X_t \leq \theta Y_t$ since Y_t grows only when $X_t = Y_t$ and $V_y(y,y) = 0$ in (2.16),

$$\begin{aligned} dZ_t &= e^{rt}\left[\frac{1}{2}V_{xx}(X_t,Y_t)(dX_t)^2 + V_x(X_t,Y_t)dX_t + rV(X_t,Y_t)dt\right] \\ &= e^{rt}V_x(X_t,Y_t)X_t\sigma dW_t \end{aligned} \tag{2.20}$$

using (2.14), and so Z_t in (2.18) is a positive local martingale in (2.20) and hence again a submartingale and

$$E_{x,y} dZ_t \geq 0\ , \quad t \geq 0\ . \tag{2.21}$$

Thus we can write for any stopping time τ, i.e. any stopping rule which does not anticipate the future (in the sense that at the time of stopping only information independent of future increments of W is usable), by (2.15),

$$\begin{aligned} E_{x,y} e^{r\tau} Y_\tau &= E_{x,y} e^{r\tau} V(X_\tau, Y_\tau) \\ &= E_{x,y} Z_\tau \ \geq \ E_{x,y} Z_0 \\ &= V(X_0, Y_0) \ = \ V(x,y) \end{aligned} \tag{2.22}$$

where we used the fact that Z is a submartingale, (2.21), to obtain the second inequality. If we take the inf over all such τ, we obtain that for all $0 < y \leq x$,

$$V^*(x,y) \geq V(x,y)\ . \tag{2.23}$$

To prove the reverse inequality, let τ be the 1st time t for which

$$\theta Y_t = X_t \tag{2.24}$$

in each of (2.8), (2.10), and (2.11); the case (2.7) has $\theta = \tau = \infty$, starting from $X_0 = x$, $Y_0 = y \leq x < \theta y$. It is clear that $P(\tau < \infty) = 1$ because

$$P\{\tau > T\} = P\left\{\text{for } 0 \leq u \leq t \leq T,\ \ \sigma(W_t - W_u) + \left(\mu + \frac{\sigma^2}{2}\right)(t-u) \leq \log\frac{\theta y}{x}\right\} \tag{2.25}$$

which tends to zero as $T \to \infty$ no matter what the sign of $\mu - \frac{\sigma^2}{2}$ is. The first inequality in (2.22) is thus an equality for this τ on account of the fact that at $\tau = t$ in (2.24), $V(X_\tau, Y_\tau) = Y_\tau$ by (2.15). The second inequality in (2.22) is also an equality since when $y \le x < y\theta$, we have for every fixed T since Z_t is a local martingale,

$$E_{x,y} Z_{\tau \wedge T} = E_{x,y} Z_0 \ . \tag{2.26}$$

Now let $T \to \infty$ and use Fatou's lemma,

$$E_{x,y} Z_\tau \le E_{x,y} Z_0 \tag{2.27}$$

so the reverse inequality in (2.22) holds, i.e. τ achieves $V(x, y)$. This means that

$$V^*(x, y) \le V(x, y) \tag{2.28}$$

and with (2.23) this completes the proof of the theorem.

As in [1], the method of derivation (guessing) at (2.7), (2.8), (2.10), and (2.11) was the principle of smooth fit — which goes back to A. N. Kolmogorov in Russia and partially explains why the call option is called the Russian call option (cf. [1]).

3. The Bachelier version of the Russian call option

For purposes of comparison we also obtain the pricing formula that the option customer and seller should use to compute the fair price or break-even point should one or the other prefer the Bachelier rather than the Black-Scholes model of asset fluctuation. We will use primes to denote this model where the price or value of the asset follows the nonexponential evolution:

$$X'_t = x + \mu t + \sigma W_t, \quad t \ge 0 \tag{3.1}$$

where $\sigma > 0$, $-\infty < x < \infty$, and μ are given parameters, assumed known. Again we set, analogous to (1.3) for $y \le x$

$$Y'_t = \min \left(y, \ \inf_{0 \le u \le t} \ X'_u \right) \ . \tag{3.2}$$

The short seller is allowed to provide delivery at any time $\tau = t > 0$ and must pay $Y'_t + rt$ where here we use inflated dollars based on a linear scale (an exponential scaling would be more natural, but the math problem becomes too hard, apparently).

The problem is now to determine the price of the Russian call option with linear scaling under the Bachelier model, i.e. to find

$$V^{*'}(x,y) = V^{*'}(x,y,\mu,\sigma,r) = \inf_{\tau} E_{x,y}[Y'_t + r\tau] \tag{3.3}$$

where the inf is taken over all stopping times τ. If the reader wishes to compare with [1] he will notice that the condition $E\tau < \infty$ assumed in [1, §3] is not needed here because the infimum imposes regularity intrinsically.

The answer will be shown to be

$$V'(x,y) = \begin{cases} y + \dfrac{r\sigma^2}{2\mu^2}\left(\dfrac{2\mu}{\sigma^2}(y+\gamma-x) - \left(e^{\frac{2\mu}{\sigma^2}(y+\gamma-x)} - 1\right)\right) \ ; & y \le x \le y+\gamma \\ y \ ; \quad y+\gamma \le x \end{cases} \tag{3.4}$$

where

$$\gamma = \frac{\sigma^2}{2\mu}\log\left(1 + \frac{\mu}{r}\right) \tag{3.5}$$

provided that $\mu > -r$. If instead $\mu \le -r$ then the option seller can obtain an infinite reward — indeed in this case $\liminf(Y'_t + rt) = -\infty$ as $t \to \infty$.

The proof is exactly analogous to those of §2. We verify that $V' \in C^2$ in $-\infty < y \le x < \infty$ and

$$\frac{1}{2}\sigma^2 V'_{xx}(x,y) + \mu V'_x(x,y) + r = 0 \quad \text{in} \quad -\infty < y \le x \le y+\gamma < \infty \tag{3.6}$$

$$V'(x,y) \le y \ ; \quad V(x,y) = y \ \text{ for } \ x \ge y+\gamma \tag{3.7}$$

$$V'_y(y,y) \triangleq \frac{\partial V}{\partial y}(x,y)\Big|_{x=y} = 0 \tag{3.8}$$

Thus if we define the process

$$Z'_t = V'(X'_t, Y'_t) + rt, \quad t \ge 0 \tag{3.9}$$

and note that X' has the Ito differential

$$dX'_t = \mu dt + \sigma dW_t \tag{3.10}$$

we see that

$$dZ'_t = \frac{r\sigma}{\mu}dW_t \ \text{ in } \ Y_t \le X_t \le Y_t + \gamma \ ; \ \ dZ'_t = rdt \ \text{ in } \ X_t \ge Y_t + \gamma \tag{3.11}$$

so that Z'_t is a submartingale (note (3.8) is needed at $X_t = Y_t$). Thus for any stopping time τ, by (3.7)

$$E_{x,y}(Y'_\tau + r\tau) \geq E_{x,y}Z'_\tau \tag{3.12}$$

and since Z' is a submartingale and $(X'_0, Y'_0) = (x, y)$,

$$E_{x,y}Z'_\tau \geq E_{x,y}Z'_0 = V'(x, y) \tag{3.13}$$

so that

$$V^{*'}(x, y) \geq V'(x, y)\ . \tag{3.14}$$

Now let τ be the 1st t for which $X_t = Y_t + \gamma$ if $y \leq x < y + \gamma$, and note that if $r > -\mu$ then $P(\tau < \infty) = 1$ and a somewhat clumsy but straightforward argument can be given to show that $EZ'_\tau = EZ'_0$ to complete the proof that

$$V^{*'}(x, y) = V'(x, y)\ . \tag{3.15}$$

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The basic idea of this paper, i.e. that of dualizing the put option in [1], is due to J. Darrell Duffie and J. Michael Harrison, to whom I am indebted and grateful.

Appendix

For completeness and to call attention to the dual results in [1] for the put option, let X_t and S_t be as in (1.1), (1.5), let r be an agreed upon interest rate. The fair for the put option is

$$V^*_{put}(x,s) = \sup_\tau E_{x,s} e^{-r\tau} S_\tau \ . \tag{A1}$$

As shown in [1], this is infinite unless $r > \mu$ and $r > 0$. If $r > \mu$ and $r > 0$ then $V^*_{put}(x,s) \equiv V_{put}(x,s)$ for $x \le s$, where

$$V_{put}(x,s) = \begin{cases} \dfrac{s}{\gamma_2 - \gamma_1}\left(\gamma_2\left(\dfrac{\alpha x}{s}\right)^{\gamma_1} - \gamma_1\left(\dfrac{\alpha x}{s}\right)^{\gamma_2}\right) \ , & \dfrac{s}{\alpha} \le x \le s \\ s \ , & 0 < x \le \dfrac{s}{\alpha} \end{cases} \tag{A2}$$

where $0 < \alpha < 1$ is given by

$$\alpha = \left(\frac{1 - \frac{1}{\gamma_1}}{1 - \frac{1}{\gamma_2}}\right)^{\frac{1}{\gamma_2 - \gamma_1}} \tag{A3}$$

and γ_1 and γ_2 are the two roots, $\gamma_1 < 0 < 1 < \gamma_2$ of the quadratic equation

$$\frac{1}{2}\sigma^2\gamma^2 + \gamma\left(\mu - \frac{\sigma^2}{2}\right) - r = 0 \tag{A4}$$

$$\gamma_{1,2} = \frac{\frac{\sigma^2}{2} - \mu \pm \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} \ . \tag{A5}$$

The optimal stopping time τ is the first t for which

$$X_t = S_t/\alpha \ . \tag{A6}$$

The similarity to the results given here is apparent.

We show here that without the reduced risk or "Russian" feature, the problem loses interest. Thus suppose X_t is as in (1.1) and we consider the problem to determine

$$V_p(x) = \sup_\tau E X_\tau e^{-r\tau} \tag{A7}$$

where the current value, X_τ, of the asset is used at the time the put is exercised rather than S_τ as in (A1).

Theorem A1. *If $r \geq \mu$ then $V_p(x) = x$; if $r < \mu$ then $V_p(x) = \infty$.*

Proof. Let $Y_t = X_t e^{-rt}$. Then

$$dY_t = ((\mu - r)dt + \sigma dW_t)e^{-rt}X_t \tag{A8}$$

so that if $r \geq \mu$ then $EdY_t \leq 0$ and $EY_\tau \leq EY_0 = x$ for any τ, so $V_p(x) \leq x$ and taking $\tau = 0$ show $V_p(x) = x$. If $r < \mu$ then the fixed time rule $\tau \equiv t$ gives

$$V_p(x) > EX_t e^{-rt} = e^{(\mu - r)t} \to \infty \quad \text{as} \quad t \to \infty \tag{A9}$$

so that $V_p(x) = \infty$ in this case. This proves Theorem A1.

Similarly, in the case of the call option, let

$$V_c(x) = \inf_\tau E_x X_\tau e^{r\tau} \tag{A10}$$

if the current price is used at settlement time instead of the minimum as in (1.3).

Theorem A2. *If $r \geq -\mu$ then $V_c(x) = x$; if $r < -\mu$ then $V_c(x) = 0$.*

Proof. Let $Z_t = X_t e^{rt}$. Then

$$dZ_t((r + \mu)dt + \sigma dW_t)e^{rt}X_t \tag{A11}$$

so then $r \geq -\mu$ means that $EdZ_t \geq 0$ and so $EZ_\tau \geq x$ for any τ, so $V_c(x) \geq x$ and taking $\tau = 0$ shows that $V_c(x) = x$. For $r < -\mu$, the fixed time rule $\tau \equiv t$ gives

$$V_c(x) \leq EX_t e^{rt} = e^{(r+\mu)t} \to 0 \quad \text{as} \quad t \to \infty \tag{A12}$$

so that $V_c(x) = 0$ in this case. This proves Theorem A2.

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