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Testing Simple Polygons[1]

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ABSTRACT

We consider the problem of verifying a simple polygon in the plane using test points. A *test point* is a geometric probe that takes as input a point in Euclidean space, and returns $+$ if the point is inside the object being probed or $-$ if it is outside. We give a procedure for verifying an n-sided, non-degenerate, simple polygon using $5n$ test points. This testing strategy works even if the test polygon has $n+1$ vertices. We also give algorithms using $O(n)$ test points for simple polygons which may be degenerate and for test polygons that may have upto $n+2$ vertices. All of these algorithms work for polygons with holes. We also give an extension of the basic testing algorithm to d dimensions.

1 Introduction

Geometric probing [CY87, Ski88, LR89, BS91, Kar91] is the subarea of computational geometry that investigates how to determine an object or some property of an object using a measuring device called a *probe*. Geometric probing problems are specified by several different parameters [Ski88, Ski89]. One parameter that must be specified is the property of the geometric object that is to be determined using probes. In this paper we look at the problem of verifying a geometric object. For verification problems we are given a description of a target object, including its location and orientation, and we must use probes to verify or *test* that a given object correctly matches the description.

Another parameter that is used to specify a geometric probing problem is the type of probe used. In this paper we use point probes or *test points*. Point probes measure whether a single point is inside or outside the object being probed. That is, a point probe takes as input a point in Euclidean space, and it returns $+$ (*positive*) if the point is inside the object being probed or $-$ (*negative*) if it is outside. This type of probe was developed independently by Romanik [RS90a, RS90b] and Mitchell [Mit90]. Here we use the model defined by Romanik, which is defined formally in Section 2 below. This model was used by Romanik and Salzberg [RS92] for obtaining results on testing orthogonal shapes. Note that in this model exact verification cannot be done since we assume infinite precision and we assume that a point probe does not return information about whether it is on the boundary of an object. Therefore, a testing algorithm can only test an object to within a given error bound.

In this paper we give an explicit procedure for verifying a simple n-sided polygon in the plane by using at most $5n$ test points, provided that the target polygon has no three collinear vertices and the test polygon has at most $n+1$ vertices. We also give strategies using $O(n)$ test points for the case where the target polygon has three or more collinear vertices and the case where the test polygon has $n+2$ vertices. If the test polygon can have three vertices more than the target polygon, then verification with any finite number of test points is impossible. Previous work by Romanik [Rom92] using this model provided a testing scheme using $7n$ points for polygons with no three collinear vertices.

An alternate probing model is that of the finger probe: a *finger probe* takes as input a directed line originating from infinity, and it returns the first point of intersection between the line and the object being probed, if one exists, or it returns ∞ if the line completely misses the object. See [Ski88], [CY87], [RS90a, RS90b] and [Ski89].

Section 2 of this paper gives definitions. Section 3 gives results for simple polygons where both the target and test polygon have n vertices. First a result is given for the case where the target polygon has no three collinear vertices, and then the result is generalized to handle collinearities. Section 4 generalizes the results of section 3 by considering testing strategies for polygons that may have one or two more vertices than the target polygon. Section 5 extends the first result from Section 3 to polyhedra in E^d, where no two facets are coplanar. Section 6 summarizes the results and discusses future work.

2 Testing With Point Probes

In this section we give definitions that describe our model of testing with point probes that are general enough to be applied to any set of geometric objects in the plane (and easily generalizes to higher dimensions). The objects in this paper will be simple polygons (polyhedra, in higher dimensions).

An *object* q is a measurable subset $q \subset E^2$ of the plane; in particular, it is a Borel set. An *object class* is a set Q of objects. Given a *test object* $r \in Q$ and a *target object* $q \in Q$, r is *consistent* with q on some finite set of test points t if it contains the same subset of t as q, i.e. $t \cap r = t \cap q$. The *error* of r, with respect to q, is given by $A(q \triangle r)$, where $A(p)$ denotes the area of p and $q \triangle r$ denotes the symmetric difference of the sets. Let S denote the set of all finite sets of points in E^2, let I denote the open interval of rationals $(0, 1)$.

Definition. A computable function $T: Q \times I \rightarrow S$ is a *testing algorithm for Q with test set size k* if there exists a constant k dependent only on Q such that for all $\epsilon \in I$ and for all $q \in Q$, there exists a $t \in S$ such that $T(q, \epsilon) = t$ and $|t| \leq k$ and for all $r \in Q$, if r is consistent with q on t, then $A(q \triangle r) \leq \epsilon$. $T(q, \epsilon)$ is called a *test set for q with respect to the class Q*. For each $t_i \in T(q, \epsilon)$, if $t_i \in q$ then t_i is a *positive test point*; otherwise, t_i is a *negative test point.*

Thus given a target object $q \in Q$ and $\epsilon \in I$, T produces a test set for q such that any test object that is consistent with q on this set has error no more than ϵ. If such a T and k exist, then Q is *k-testable.* Note that in general a testing algorithm may produce test sets whose sizes are functions of both ϵ and the complexity of the target object, but the testing algorithms we develop in this paper produce constant size test sets.

3 Testing n-Sided Simple Polygons

In this section we first develop a testing algorithm with test set size 6n (Theorem 1) and then improve it to achieve size 5n (Corollary 2) subject to the following two assumptions:

Assumption 1 *The target polygon is a simple polygon that has no two collinear edges.*

Assumption 2 *The target polygon and the test polygon both have n vertices.*

Theorem 1 *The class of simple polygons in E^2 with n vertices and no collinear edges is $6n$-testable.*

Proof: Our goal is to use 6 test points to verify each edge of the polygon. The 6 points are divided into 3 pairs of positive/negative points, one pair near each end of the edge, and the third pair "somewhere in the middle of the edge" (see Figure 1). More precisely, each pair is represented by the line segment connecting the two points. Assume that these line segments have some length δ_1, and are placed along an edge. (The orientation of the line segments with respect to the edge is unimportant in the current case. Later, when we remove Assumptions 1 and 2, we must choose this orientation carefully.) One segment is placed at a

distance at most some length δ_2 from each of the vertices at the ends of the edge. The third pair of points is placed between them, at a position chosen so that we do not create any "unnecessary" collinearities. Think of each pair of test points as being a single point on the boundary of the target polygon (i.e., $\delta_1 = 0$). By Assumption 1, we can place these points such that any line in the plane can stab at most three such points, and if it does stab three, these must be points on one of the target polygon's edges. Now, as we increase δ_1 from zero, creating pairs of test points, we can insure, for small enough δ_1's, that any line in the plane will separate at most three test pairs, and if a line does separate three pairs, these pairs are generated by one edge of the target polygon.

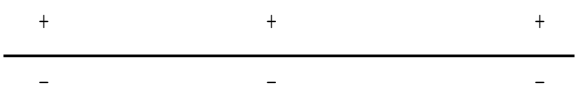

Figure 1: Six points verify an edge of a simple polygon

We use a simple counting argument to show that the edges of a test polygon consistent with the target polygon separate the test pairs in the same circular ordering as does the target polygon (under Assumptions 1 and 2). Any n-sided simple polygon consistent on the test points must separate all $3n$ pairs of points with its n edges. By the placement of the pairs of test points, no edge can separate more than two pairs of points except by separating three pairs along an edge. Therefore, in order to separate all $3n$ pairs with n edges, each edge must separate three pairs along some edge of the target polygon.

Finally, we choose parameters δ_1 and δ_2 based on the specific target polygon q and a specific choice of ϵ to guarantee that any n-sided simple polygon r whose edges separate the $3n$ test pairs in the same circular ordering as the target polygon q satisfies $A(q \triangle r) \leq \epsilon$. Clearly, $A(q \triangle r)$ is a continuous function of the parameters δ_i, and decreases to zero as δ_1 and δ_2 approach zero. Thus for any choice of ϵ one can choose $\delta_1, \delta_2 > 0$ that are small enough. □

We can reduce to $5n$ the number of test points by "reusing" either one positive or one negative test point at each vertex, depending on whether the interior angle of the polygon is convex or reflex. Picture moving the two test pairs at the ends of edges that share a vertex closer to that vertex, until two of the test points coincide, and then rotating the two segments until they are collinear. (See Figure 2.) As before, we are careful not to create any new degeneracies. This can be achieved since no three vertices are collinear, and thus no line can stab three vertices. Furthermore, one can define small disks centered at each vertex, such that no line can stab more than two such disks. Since no line can separate both pairs at a vertex, if a line separates three pairs, these pairs must include the "middle" pair of an edge and one pair near each of the vertices at the ends of the edge. Therefore, the line must be "close to" one of the edges of the target polygon.

Corollary 2 *The class of simple polygons in E^2 with n vertices satisfying Assumption 1 is $5n$-testable.*

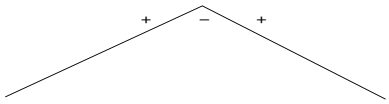

Figure 2: "Reusing" a test point at a vertex

Since the proof of Theorem 1 relies only on the general position assumption, it also applies to simple polygons with holes[1].

Corollary 3 *The class of simple polygons with holes in E^2 with n vertices satisfying Assumption 1 is $5n$-testable.*

This proof does not hold for polygons for which two or more edges may be collinear, because a single edge could now separate more than three pairs of test points. As Figure 3 shows, there are polygons that satisfy Assumption 2 but not Assumption 1, where the edges of the test polygon separate different pairs of points than the target. However, we note that these two polygons are very similar by our definition, as the area of the symmetric difference can be made arbitrarily small by shrinking δ_1. Thus, these polygons do not provide a counterexample to the testability of general n-sided polygons. Note that if a polygon has collinear vertices, but no collinear edges, our testing method of Corollaries 2 and 3 applies, since we can always place the testing pairs so as not to cause any unnecessary degeneracies.

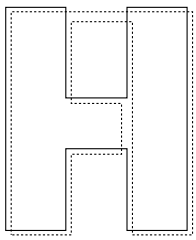

Figure 3: Polygons whose edges separate different pairs of points

If we wish to relax our requirement that edges be in general position, then we can replace Assumption 1 by the following assumption and use $6n$ test points:

Assumption 3 *There exists a bound, $\delta > 0$, such that for each test polygon, the minimum distance between a vertex and an edge of which it is not an endpoint is at least δ.*

We state without proof the following theorem.

Theorem 4 *The class of simple polygons in E^2 satisfying Assumptions 2 and 3 is $6n$-testable.*

[1]The authors would like to thank Jit Bose for suggesting the problem of testing simple polygons with holes.

Is Assumption 3 necessary? To date we have been unable to identify a target polygon that is not testable with the above scheme, even if Assumption 3 is removed, but we have no proof that the testing scheme is guaranteed to work! Consequently, we eliminate Assumption 3 at the expense of adding more (but still a linear number of) test points. No testing scheme that uses only a linear number of test points can avoid having thin strips by which the target and test polygons may differ, as in Figure 3; that is, no testing scheme can guarantee that clockwise traversals of the test and target polygons separate the test pairs in the same cyclical order. However, our next testing scheme, which builds upon the $6n$-size testing scheme of Theorems 1 and 4, does guarantee that the two polygons have a small symmetric difference.

The revisions to the testing scheme of Theorem 1 focus on collinear edges. Let $e_1 = (v', v)$ and $e_2 = (w, w')$ be collinear edges of the target polygon such that their endpoints are in order v', v, w, w' along the line containing both edges. If no other edge of the target polygon lies along the line segment between v and w, these edges are *consecutive*. If a clockwise traversal of the boundary of the polygon moves from v' to v and from w to w', or from w' to w and from v to v', then these edges are *aligned*. If e_1 and e_2 are consecutive and aligned, then (v, w) will be called a *phantom edge* (depicted by a dashed line in our figures).

Augment the previous testing scheme with three pairs of test points along each such phantom edge, called *phantom edge test pairs* or *phantom pairs*, placed collinear with the test pairs defined by real edges (*real pairs*) and so as to avoid any unnecessary degeneracies. Real pairs have one point of each sign. The phantom pairs along one phantom edge are all of the same sign (all positive or all negative). See Figure 4. If an edge of the test polygon separates pairs of test points generated by two or more collinear edges of the target polygon, then this edge will also erroneously separate the phantom pairs, necessitating an additional edge to reseparate them.

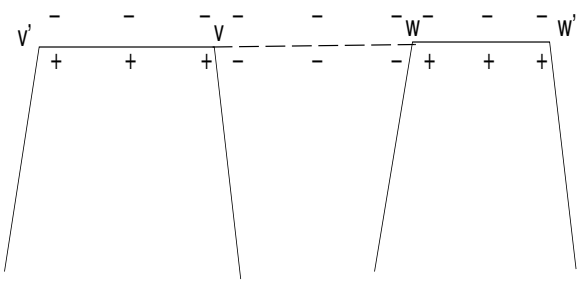

Figure 4: Testing degenerate polygons satisfying Assumption 2.

To formalize this argument, define the *value* of an edge of the test polygon, in a clockwise traversal of the polygon, as the sum of the values of the real and phantom pairs it separates, as follows. If the directed edge separates a real pair, with the positive test point on its right side, the value is +1, whereas if the negative test point is on the right, the value is -1. For a phantom pair, we consider the positive side of the phantom edge generating the pair to be the same as the positive sides of the real edges bounding the phantom edge. The value of a

phantom pair is -1 if the pair is separated with the positive side on the right of the separating edge, and +1 if it is on the left. See Figure 5, where the edges of the test polygon are denoted as arrows, and the edges of the target polygon by solid lines.

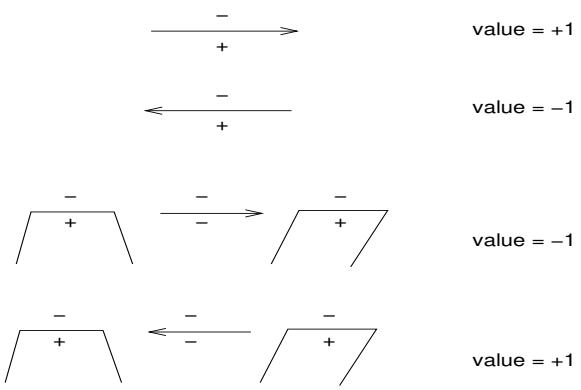

Figure 5: Assignment of value

Lemma 5 *The value of each edge of the test polygon is at most 3.*

Proof: Assume by contradiction that there exists an edge that has value 4 or more. A value of 4 can only be obtained by separating real pairs along consecutive, aligned edges or by separating phantom pairs along aligned phantom edges. However, an edge that separates real pairs on consecutive, aligned edges will also separate the pairs on the phantom edge between these edges, and these pairs will yield the opposite sign. Similarly, an edge that separates pairs from more than one phantom edge will separate real pairs giving the opposite sign. □

Theorem 6 *The class of simple polygons in E^2 with n vertices can be tested with $6n + 6l$ test points, where l is the number of phantom edges (pairs of consecutive, aligned edges) of the target polygon.*

Proof: We use the testing scheme described above. Any test polygon that is consistent with the target polygon on the test points must have value exactly $3n$, since every such polygon must separate phantom pairs an equal number of times in each direction, and real pairs exactly once more with the positive test points on its right. The argument is now reminiscent of that presented in Theorem 1. Since each edge of the test polygon has value at most 3 (Lemma 5), and the number of edges is n, each edge must have value exactly 3. This implies that the three pairs generated by a real or phantom edge of the target polygon are separated by the same edge(s) of the test polygon. As a result, all edges of the test polygon are "close to" edges of the target polygon, and the only place in which the target and test polygons can differ is in the "strips" between the two points of a test pair, as in the example

of Figure 3. However, the area of the strips can be made arbitrarily small by decreasing δ_1. Finally, we note that $l \leq n$, so the number of test points remains linear in n. □

The same testing strategy can be used for simple polygons with holes. The edges of a hole are traversed in a counter-clockwise order, so two consecutive collinear edges of the target polygon, $e_1 = (v', v)$ and $e_2 = (w, w')$, where e_2 is an edge of a hole, are *aligned* if a clockwise traversal of the boundary of the polygon moves from v' to v and a counter-clockwise traversal of the boundaries of the holes moves from w to w', or visa versa. As before, a phantom edge occurs between every pair of consecutive, aligned edges. Also, the value of an edge of the test polygon is determined by making a clockwise traversal of the boundary of the polygon and a counter-clockwise traversal of the boundaries of the holes, with the value of a real pair being determined as before. By traversing the holes in counter-clockwise order, the arguments of Theorem 6 remain valid, and the following corollary results.

Corollary 7 *The class of simple polygons with holes in E^2 with n vertices can be tested with $6n + 6l$ test points, where l is the number of phantom edges of the target polygon.*

4 Testing Simple Polygons with more than n sides

In this section, we consider the question of whether an accurate linear-size testing scheme can be devised when Assumption 2 is relaxed and the test polygon may have more sides than the target polygon. We will assume that the vertices of the target polygon are in general position (Assumption 1). Without Assumption 3, no finite-sized testing scheme is possible: create a test polygon by adding one edge at a convex-hull vertex of the target polygon, forming a long and very skinny "needle" (see Figure 6); the needle can be made arbitrarily long, and thus the area of symmetric difference between the test and target polygons can be made arbitrarily large. Even with Assumption 3, a test polygon with three more edges than the target polygon can fool any finite-sized testing scheme: create a test polygon by creating a long "needle" somewhere in the middle of one of the externally visible edges of the target polygon, avoiding all test points; this needle replaces one edge with four edges; the area of the symmetric difference between the two polygons can be made arbitrarily large by increasing the length of the needle. Using Assumption 3, however, we do achieve positive results: a testing scheme of size $5n$ when the number of additional edges is at most one; and a testing scheme of size $7n$ when the number of additional edges is at most two.

We will be making use of the following simple claim:

Claim 8 *Given a line segment f and a point p at distance at most d from f, then any line segment g that contains p but does not cross f must have an endpoint at distance at most d from f, or else f must have an endpoint at distance at most d from g.*

Proof (of Claim): If f and g are parallel, then since g contains a point p at distance d from f, either f or g will have an endpoint at distance d from the other. If f and g are not parallel, then the proof is an easy case analysis, based on the intersection point of the

two lines containing segments f and g, which can lie on exactly one of the segments or on neither segment. □

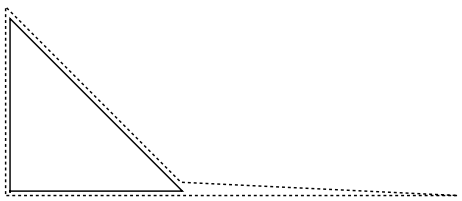

Figure 6: Polygons with $n+1$ sides cannot be tested

Theorem 9 *The class of simple polygons in E^2 with n or $n+1$ vertices, satisfying Assumptions 1 and 3, can be tested with $5n$ test points.*

Proof: As in Corollary 2, we choose δ_1 and δ_2 small enough to ensure that the distance from a vertex to any point on either of the two test-segments near it is less than $\frac{\delta}{2}$. The proof proceeds as in Theorem 1: there are $3n$ pairs to be separated, and no one edge of the test polygon can separate more than 3 pairs; each test pair must be separated by exactly one edge, since two edges separating a pair would cause one of the test points to have an incorrect label, and three or more edges separating a pair would, by Claim 8, cause some vertex to be within δ of a nonadjacent edge, violating Assumption 3. We say that an edge of the target polygon *corresponds to* an edge of the test polygon if the two edges separate the same three pairs of test points. If the test polygon has $n+1$ edges, then there are three cases to consider:

1. All n edges of the target polygon have corresponding edges in the test polygon, and so one edge $f = (v_1, v_2)$ of the test polygon separates no pairs. Let f connect two edges f_1 and f_2 of the test polygon. If the corresponding edges e_1 and e_2 of the target polygon do not share a vertex, then the two edges of the target polygon incident on e_1 have corresponding test polygon edges f_3 and f_4, both of which must share an endpoint with f_1 by Claim 8 and Assumption 3. But f and f_1 also share a vertex, providing a contradiction. If e_1 and e_2 do share a vertex v, then f_1 and f_2 both separate a pair of points near the vertex v and pass at most distance $\frac{\delta}{2}$ from vertex v. Thus either v_1 is "close" to the edge f_2 or v_2 to f_1, violating the combination of Claim 8 with Assumption 3.

2. The test polygon r has $n-1$ edges which separate 3 pairs each, and hence have corresponding edges in the target polygon. Of the remaining two edges of the test polygon, one edge f_1 separates 1 pair, the other f_2 separates 2 pairs, all of which are separated by the single target edge e. Replace f_1 and f_2 by a single edge f separating all three pairs and extend the adjacent edges to form a new polygon r'. By Theorem 1 we know that for every value of $\epsilon_1 > 0$ we can choose the parameters δ_1 and δ_2 so

that $A(q \triangle r') \leq \epsilon_1$, where q is the target polygon. To show that for any $\epsilon_2 > 0$ we can choose the parameters small enough so that $A(r \triangle r') \leq \epsilon_2$, we observe first that as δ_1 is decreased to 0, each test pair becomes a single point on the boundary of the polygon, and $A(r \triangle r') = 0$. As the test points within each pair are moved continuously apart from one another, the maximum potential difference in slope between f_2 and f grows continuously. The limits on the slope of f_2 (determined by δ_1) and on the placement of the vertex v_f shared by f_2 and f_1, (dictated by Assumption 3) prevent f_1 from forming a long needle with either of its adjacent edges when δ_1 is increased from zero, and so the area $A(r \triangle r')$ also grows continuously. Choosing $\epsilon_1 = \epsilon_2 = \epsilon/2$, and the parameters δ_i accordingly, completes the argument.

3. There are $n - 2$ edges of the test polygon that separate 3 pairs each, and hence have corresponding edges in the target polygon. Each of the remaining three edges of the test polygon f_1, f_2, and f_3 separates 2 pairs. Since each of the edges f_i separates two pairs of points, it must be that the three edges separate six pairs that are separated by two edges e_1 and e_2 of the target polygon that share a vertex v. Specifically, f_1 (f_2) separates two pairs of test points generated by edge e_1 (e_2) – one middle pair and one pair near the vertex other than v. f_3 must separate the two test pairs near the vertex v. Any other (non-equivalent) configuration would result in the test polygon being non-simple, or a violation of Assumption 3. However, by our placement of test points, no line can separate the two pairs of test points near one vertex, so this case is impossible.

□

In order to test polygons that may have two more sides than the target polygon, our testing scheme must be modified. Consider a test polygon that looks like the target except that one of the externally visible edges of the target polygon is replaced by three edges in the test polygon, forming a "zig-zag" – see Figure 7. The test polygon is consistent with the target polygon, no matter how small the parameters δ_1 and δ_2 are chosen, but the area of the symmetric difference can be made arbitrarily large compared to the area of the target polygon.

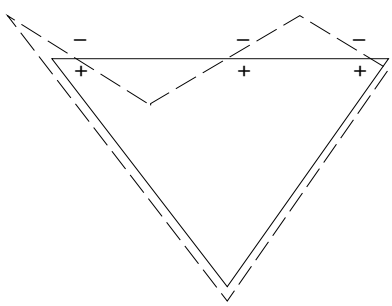

Figure 7: $5n$ points do not suffice for testing a polygon with $n + 2$ sides

To overcome this difficulty, add one pair of test points near each end of each edge of the target polygon, at distance δ_3 from the vertex, such that $\delta_3 > \delta_2$, every point on the test-segment is within $\frac{\delta}{2}$ of the vertex, and unnecessary degeneracies are avoided. Eliminate the middle pair of test points. The net gain in test points is $2n$. See Figure 8.

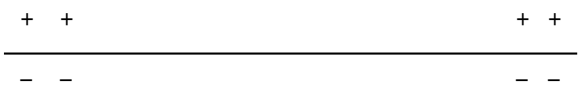

Figure 8: 8 points verify an edge of a polygon with $n + 2$ sides

Claim 10 *The four pairs of test points near a vertex of the target polygon must be separated by exactly two edges of a successful test polygon.*

Proof (of Claim): By our placement of test points, no line can separate all four pairs of test points near one vertex. If three or more edges separate the test pairs near a vertex, then there will be at least two edges that do not share an endpoint passing within $\frac{\delta}{2}$ of the vertex, and by Claim 8 an endpoint of one of these edges will be within δ of the other edge, which violates Assumption 3. Therefore, exactly two edges must be used to separate the four pairs. □

Claim 11 *The four pairs of test points generated by a single edge of the target polygon can be separated by at most two different edges of a successful test polygon.*

Proof (of Claim): If four pairs of test points generated by an edge of the target polygon are separated by three different edges of the test polygon, then the two test pairs near one endpoint of the target edge are separated by two different test edges. By Claim 10 these two edges must also separate the two test pairs near the vertex on the adjacent target edge. The test edge separating the pair closest to the vertex cannot also separate the pair closest to the vertex on the adjacent edge because these two pairs are collinear. If the two edges separate the pairs on the adjacent edge in the opposite way, then they will cross, contradicting simplicity. □

As in the proof of Theorem 9, we say that an edge of the target polygon *corresponds to* an edge of the test polygon if the two edges separate the same four pairs of test points.

Claim 12 *At least $n - 2$ edges of the target polygon must have corresponding edges in a successful test polygon.*

Proof (of Claim): If only $n - 3$ edges of the target polygon have corresponding edges in the test polygon, then the test polygon (if it has $n + 2$ edges) must use five edges to separate the remaining 12 test pairs. Therefore, some edge f of the test polygon must separate three test pairs, a feat possible only if these three pairs lie along one edge e of the target

polygon. Consequently, f separates only one of the test pairs near one endpoint of e, so at least two more edges must be used to separate the remaining three pairs near that vertex, contradicting Claim 10. □

Theorem 13 *The class of simple polygons in E^2 satisfying Assumptions 1 and 3 with n, $n+1$, or $n+2$ vertices can be tested with $7n$ test points.*

Proof: If the target polygon has n vertices, then there are $4n$ test pairs to be separated, and no one edge of the test polygon can separate more than four pairs. Also, each test pair must be separated by exactly one edge, since two edges separating a pair would cause one of the test points to have an incorrect label, and three or more edges separating a pair would, by Claim 8, cause some vertex to be within δ of a nonadjacent edge, violating Assumption 3. The proof mimics that of Theorem 9. We assume that the test polygon has $n+2$ edges and consider the following three cases:

1. All n edges of the target polygon have corresponding edges in the test polygon, and so two edges f_1 and f_2 of the test polygon separate no pairs, where f_1 and f_2 may or may not be adjacent in the test polygon. In either case, consideration of the edges of the test polygon adjacent to them will show, by Claim 8, that there is a vertex that is "too close" to an edge of which it is not an endpoint, contradicting Assumption 3 as in the first case of Theorem 9.

2. There are $n-1$ edges of the test polygon that separate four pairs each, and hence have corresponding edges in the target polygon. The test pairs of the remaining edge of the target polygon are separated by two of the remaining three edges of the test polygon, by Claim 11. If the three edges of the test polygon that do not correspond to edges of the target polygon are not consecutive along the boundary of the test polygon, then one of them must connect two edges of the test polygon that do correspond to edges of the target polygon, and as in Case (1) of Theorem 9, we get a contradiction to Assumption 3. If the three edges are consecutive, then the edge that separates no pairs cannot be extreme or the argument of Case (1) in Theorem 9 would produce a contradiction. Furthermore, if one of the three edges separates a single test pair, we get a contradiction to Claim 10 since three edges will be used to separate the four pairs near a vertex. The remaining possibility is that two of the three edges separate two test pairs and the third connects the two and separates no test pairs. An edge that separates two test pairs that are generated by a single edge of the target polygon must be very close in slope to that target edge, by our choice of the parameters. Thus, as in Case (2) of Theorem 9, the three edges can be replaced by a single edge, and the resulting polygon is very similar to the test polygon.

3. There are $n-2$ edges of the test polygon that separate four pairs each, and hence have corresponding edges in the target polygon. The test pairs of each of the remaining edges e_1 and e_2 of the target polygon are separated by exactly two edges of the test

polygon, by Claim 11. Each edge of the test polygon separating test pairs along e_1 or e_2 must separate exactly two pairs, since if a test polygon edge separated three test pairs, then there would be at least three edges separating the four pairs near a vertex, violating Claim 10. Therefore, arguments similar to those of Case (2) in Theorem 9 will complete the proof.

□

It is tempting to think that by possibly adding another test pair (or some constant number of test pairs) per edge we could test polygons with $n + 3$ vertices. However, as we noted before, this is false.

5 Higher Dimensions/Extensions

Consider now the problem of testing polyhedra in E^d. We can extend our two-dimensional result as follows:

Theorem 14 *The class of polyhedra in E^d having n facets, no two of which are coplanar, can be tested with $2(d+1)n$ test points.*

Proof: (Sketch) As in two dimensions, we use pairs of test points that we distribute on facets of the target polyhedron. In each pair of test points, the separation between the points is δ_1, which is extremely small. One point in each pair is a positive point (being just inside the target), and one is a negative point (being just outside the target).

We distribute $d+1$ pairs on each facet of the target, taking care to make certain that no set of $d+1$ pairs can be separated by a hyperplane, *except* in the case that the $d+1$ pairs are all on a common facet. (That is, we create no accidental degeneracies – in general, any d pairs can be separated by a (unique) hyperplane, but no $d+1$ pairs can be separated by a single hyperplane.)

As in the two-dimensional case, any n-faceted test polyhedron that is consistent with the test points must be efficient in separating the $(d+1)n$ test pairs; namely, each facet of the test polyhedron must separate $(d+1)$ test pairs. This follows from the fact that each facet can separate no more than $d+1$ pairs. But this implies that each facet of the test polyhedron must be nearly coincident with some facet of the target polyhedron, separating exactly those test pairs that correspond to the target facet.

We claim that this implies that the volume of the symmetric difference between the test and target polyhedra can be made arbitrarily small by choosing δ_1 sufficiently small. To see this, consider a box B that is large enough to contain the target polyhedron. Consider one facet, f, of the target, and let $W_f = B \cap \bigcup_{h \in C_f} h$, where C_f denotes the set of all hyperplanes that separate the $(d+1)$ test pairs of f. As δ_1 goes to 0, the volume of W_f goes to 0; thus, we can, for any given $\epsilon > 0$, pick δ_1 so that the volume of W_f is less than ϵ/n, for every facet f. But the volume of the symmetric difference between the target and test polyhedra

is bounded above by $\sum_f vol(W_f)$, so we can pick δ_1 in such a way that this volume is less than ϵ. □

Remarks: (1). Note that the theorem did not assume that the target polyhedron was simply connected — e.g., it may have holes.

(2). The nondegeneracy assumption in the above theorem can likely be removed, if we add some additional test pairs as we did in the two-dimensional case.

Another extension of our results is to the case of curved boundaries. If the target is assumed to have piecewise-algebraic boundary with constant degree k, then the argument in the above theorem generalizes: Place enough test pairs $(d+k)$ in *general position* on each curved facet so that the only way that a test object can separate all pairs is for it to nearly coincide with the target facet.

6 Conclusions/Future Work

In this paper we have examined the problem of verifying simple polygons in the plane using test points. We have given a testing scheme using $5n$ test points for the case where the vertices of the n-sided target polygon are in general position, and the test polygon has at most one more vertex than the target. We have modified this scheme to one that uses $6n+6l$ test points for target polygons with l consecutive, aligned collinear edges. We have also given a testing scheme using $7n$ test points for target polygons with n vertices in general position where the test polygon may contain two more vertices than the target.

All of our results extend to simple polygons with holes. We have also extended our results for two dimensions to higher dimensions under a general position assumption. Extending the testing scheme to higher dimensions in the face of degeneracies remains an open question.

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