1. Introduction

Let $\mathbf{M}$ be a matroid on a finite set. Let $\mathfrak{M}(\mathbf{M})$ denote the set of oriented matroids whose underlying matroid is $\mathbf{M}$. We define an equivalence relation on $\mathfrak{M}(\mathbf{M})$ in terms of "reorientations" of oriented matroids. The object of this paper is to characterize the set $\mathcal{R}(\mathbf{M})$ of reorientation classes of oriented matroids on $\mathbf{M}$.

In the case that an oriented matroid $\tilde{\mathbf{M}}$ is represented by a configuration of vectors in $\mathbf{R}^k$, reorienting $\tilde{\mathbf{M}}$ corresponds to replacing some of the vectors by their negatives. Thus the reorientation class of $\tilde{\mathbf{M}}$ corresponds to the configuration of lines through the origin generated by the vectors; that is, to a configuration of points in $\mathbf{R}P^{k-1}$.

Our first characterization of $\mathcal{R}(\mathbf{M})$, guided by this example, is combinatorial. In section 2 we define a "projective orientation" of $\mathbf{M}$ in terms of the combinatorics of its circuits and cocircuits. We prove (Theorem 1) that there is a one-to-one correspondence between $\mathcal{R}(\mathbf{M})$ and the set of projective orientations of $\mathbf{M}$.

In the third section we characterize $\mathcal{R}(\mathbf{M})$ algebraically, using a certain Abelian group associated to $\mathbf{M}$, called its "inner Tutte group." This group, denoted by $\mathbb{T}^{(0)}_{\mathbf{M}}$, was defined in [2] as the kernel of a certain homomorphism from a larger Abelian group, called the "Tutte group" of the matroid $\mathbf{M}$, to a free Abelian group. For the Tutte group several presentations in terms of generators and relations were proposed in [2], but no such presentation was known for $\mathbb{T}^{(0)}_{\mathbf{M}}$. We define two Abelian groups $\mathbb{T}^{(1)}_{\mathbf{M}}$ and $\mathbb{T}^{(2)}_{\mathbf{M}}$ by generators and relations, and prove (Theorems 3,4) that both of them are canonically isomorphic to $\mathbb{T}^{(0)}_{\mathbf{M}}$.

Inner Tutte groups of matroids were used in [10] to describe the projective equivalence classes of matroids with coefficients. The notion of an oriented matroid is a special case of the notion of a matroid with coefficients, and the reorientation class is just another name for the projective equivalence class in this case. It was shown in [10] that $\mathcal{R}(\mathbf{M})$ is in one-to-one correspondence with the set of those homomorphisms $\mathbb{T}^{(0)}_{\mathbf{M}} \to \{+1, -1\}$ which satisfy certain conditions. Using this we derive another combinatorial description of $\mathcal{R}(\mathbf{M})$ from our presentation of the inner Tutte group by generators and relations as $\mathbb{T}^{(2)}_{\mathbf{M}}$. (See Theorem 2 and Corollaries 1 and 2 for precise statements).

2. Projective orientations

In this paper we shall consider only finite matroids, oriented matroids etc. First we recall some basic facts and notations (see, e.g., [1], [7] and [8]).

We use the definition of a matroid $\mathbf{M}$ as a pair $(E, \mathfrak{C})$, where E is the ground set and $\mathfrak{C}$ is the set of circuits of $\mathbf{M}$. We denote by $\mathfrak{C}^*$ the set of cocircuits of $\mathbf{M}$, so the matroid dual to $\mathbf{M}$ is $\mathbf{M}^* = (E, \mathfrak{C}^*)$. The rank of matroid $\mathbf{M}$ we denote by $rk(\mathbf{M})$, the rank of $A \subseteq E$ in $\mathbf{M}$ by $r_{\mathbf{M}}(A)$. The subscript $\mathbf{M}$ in $r_{\mathbf{M}}$ will often be omitted.

We shall often use the elimination axiom in its strong form: if $C_1, C_2 \in \mathfrak{C}$, $a \in C_1 \cap C_2$ and $b \in C_1 \setminus C_2$, then there is $C_3 \in \mathfrak{C}$ with $b \in C_3 \subseteq C_1 \cup C_2 \setminus \{a\}$. We shall also use the orthogonality of $\mathfrak{C}$ and $\mathfrak{C}^*$ — that is, for $C \in \mathfrak{C}$ and $D \in \mathfrak{C}^*$ we have $|C \cap D| \neq 1$ — and that for $C \in \mathfrak{C}$ and $x, y \in C$ with $x \neq y$ there is $D \in \mathfrak{C}^*$ with $D \cap C = \{x, y\}$.

We denote by $\mathbf{M}/A$ the matroid obtained from $\mathbf{M}$ by contraction through $A \subseteq E$, and by $\mathbf{M} \setminus A$ the matroid obtained by deletion of $A \subseteq E$. We shall also write $\mathbf{M} \cdot A$ instead of $\mathbf{M} \setminus (E \setminus A)$. Following Tutte, we shall use the word *flat* for any set of the form $F = C_1 \cup \cdots \cup C_k$ with $C_i \in \mathfrak{C}$; so $E \setminus F$ is a closed set for $\mathbf{M}^*$. We shall call the flat F connected if $\mathbf{M} \cdot F$ is a connected matroid. If $\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_d = F$ is a maximal chain of flats contained in F, then d depends only on F and is called the *dimension* of F. We denote it by $\dim(F) = \dim_{\mathbf{M}}(F)$. We denote by $\langle A \rangle$ the maximal flat $F \subseteq A$ for $A \subseteq E$; so $E \setminus \langle A \rangle$ is the closure of $E \setminus A$ in $\mathbf{M}^*$. Set $\dim(A) = \dim(\langle A \rangle)$; then $\dim_{\mathbf{M}}(A) = rk(\mathbf{M}^*) - r_{\mathbf{M}^*}(E \setminus A) - 1$.

The flats of dimension 1 will be called lines; those of dimension 2, planes. We shall also say "points" instead of "circuits" in this context. Thus "the point C lies on the line L" just means $C \subset L$ and "the lines L_1 and L_2 intersect in the point C" means $\langle L_1 \cap L_2 \rangle = C$.

The subset $\mathcal{C} \subseteq \mathfrak{C}$ is called a *linear subclass* if for any two distinct points $C_1, C_2 \in \mathcal{C}$, lying on the same line, all points on this line belong to $\mathcal{C}$. The sequence of points $(C_1, \ldots, C_k)$ is called a *path off* $\mathcal{C}$, if any two subsequent points are distinct points on a connected line and no point of this sequence belongs to $\mathcal{C}$. The fundamental theorem about linear subclasses [7, Theorem 4.34] says that if $\mathbf{M}$ is connected, then for any linear subclass $\mathcal{C}$ and any $C, C_1 \in \mathfrak{C} \setminus \mathcal{C}$ there is a path $(C_1, \ldots, C_k = C)$ off $\mathcal{C}$.

The most important (and, in fact, general) example of a linear subclass is the following. Let $\mathbf{M} = (E, \mathfrak{C})$ be a matroid, $a \in E$ with $\{a\} \notin \mathfrak{C}$ and $\mathbf{M}' = \mathbf{M}/\{a\}$. Then for each circuit $C' \in \mathfrak{C}_{\mathbf{M}'}$ there is a unique circuit $C \in \mathfrak{C}$ with $C \setminus \{a\} = C'$. We denote this circuit C by $\theta C'$. The circuits $C' \in \mathfrak{C}_{\mathbf{M}'}$ with $a \notin \theta C'$ form a linear subclass in $\mathbf{M}'$ [7, Proposition 7.412].

¿From [7, Theorem 4.371] we know that if $\mathbf{M}$ is connected, then for any two elements of E there is a circuit containing both of them. We have no reference for the following fact, so we put it here with the proof.

Proposition. *If $\mathbf{M} = (E, \mathfrak{C})$ is a connected matroid, then for any three elements of E there is either a circuit or a cocircuit which contains all of them.*

Proof. Let $\mathbf{M}$ be a minor-minimal matroid, for which it is not so, and let $x, y, z \in E$ be contained neither in the same circuit, nor in the same cocircuit of $\mathbf{M}$. As $\mathbf{M}$ is connected, there is a circuit containing both x and y, so if $\dim(E) > 1$, then there is a connected flat F, containing both x and y, with $\dim(F) = \dim(E) - 2$ [7, Theorem 4.25]. By [7, Theorem 4.26] there are two connected flats F_1 and F_2 with $\dim(F_i) = \dim(E) - 1$, $F \subseteq F_1 \cap F_2$ and $F_1 \cup F_2 = E$. So either F_1 or F_2 contains x, y and z; hence $\mathbf{M}$ is not minimal. So $\dim(E) = 1$ (if $\dim(E) = 0$, then $\{x, y, z\} \subseteq E \in \mathfrak{C}$).

The same arguments applied to $\mathbf{M}^*$ show that $rk(\mathbf{M}) = 2$. So $|E| = 4$ and $E = \{x, y, z, t\}$. If t is parallel to one of the other elements, say to x, then it is not parallel to either y or z (because $\mathbf{M}$ is connected of rank 2), so $\{y, z, t\} \in \mathfrak{C}$ and $\{x, t\} \in \mathfrak{C}$, thus $\{x, y, z\} \in \mathfrak{C}$. If t is not parallel to any of the other elements of E, then $\{x, y, z\} \in \mathfrak{C}^*$, for $\{t\}$ is closed subset of E of corank 1. This contradicts the choice of $\mathbf{M}$. $\square$

Let $\mathbf{M} = (E, \mathfrak{C})$ be a matroid. A *circuit signature* $\mathcal{O}$ on $\mathbf{M}$ is a set of functions $\mathcal{O}C : C \to \{1, -1\}$, one for each $C \in \mathfrak{C}$. Analogously, a *cocircuit signature* $\mathcal{O}^*$ is a

set of functions $\mathcal{O}^*D : D \to \{1,-1\}$, one for each $D \in \mathfrak{C}^*$. We say that circuit and cocircuit signatures are *orthogonal* (denoted $\mathcal{O} \perp \mathcal{O}^*$), if for any $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$ with nonempty intersection $\{\mathcal{O}C(x)\mathcal{O}^*D(x) \mid x \in C \cap D\} = \{1,-1\}$.

The circuit signatures $\mathcal{O}_1$ and $\mathcal{O}_2$ are called *equivalent* (denoted $\mathcal{O}_1 \sim \mathcal{O}_2$) if, for any $C \in \mathfrak{C}$, either $\mathcal{O}_1C(x) = \mathcal{O}_2C(x)$ for all $x \in C$, or $\mathcal{O}_1C(x) = -\mathcal{O}_2C(x)$ for all $x \in C$. We also consider the same equivalence relation for cocircuit signatures. It is easy to see that if $\mathcal{O}_1 \perp \mathcal{O}_1^*$ and $\mathcal{O}_1 \sim \mathcal{O}_2$, $\mathcal{O}_1^* \sim \mathcal{O}_2^*$, then $\mathcal{O}_2 \perp \mathcal{O}_2^*$. The pair $(E, \overline{\mathcal{O}})$, where $\overline{\mathcal{O}}$ is an equivalence class of a circuit signature $\mathcal{O}$ on a matroid $\mathbf{M} = (E, \mathfrak{C})$, is called an *oriented matroid*, if there exists a cocircuit signature $\mathcal{O}^*$, such that $\mathcal{O} \perp \mathcal{O}^*$. In this case $(E, \overline{\mathcal{O}^*})$ is also an oriented matroid, which is called *dual* to $(E, \overline{\mathcal{O}})$. It is easy to see that $\overline{\mathcal{O}^*}$ is determined uniquely by $\overline{\mathcal{O}}$.

Let now $\mathcal{O}$ be a circuit signature on $\mathbf{M} = (E, \mathfrak{C})$, and for some $A \subseteq E$ let us consider another circuit signature $\mathcal{O}_A$, such that

$$\mathcal{O}_AC(x) = \begin{cases} \mathcal{O}C(x), & \text{for } x \notin A \\ -\mathcal{O}C(x), & \text{for } x \in A. \end{cases}$$

We call $\mathcal{O}_A$ the *reorientation* of $\mathcal{O}$. It is clear that if $\mathcal{O} \perp \mathcal{O}^*$, then $\mathcal{O}_A \perp \mathcal{O}_A^*$, so if $(E, \overline{\mathcal{O}})$ is an oriented matroid, then $(E, \overline{\mathcal{O}_A})$ is an oriented matroid, whose dual is $(E, \overline{\mathcal{O}_A^*})$. Oriented matroids, which can be obtained from each other by reorientation, form a *reorientation class*.

Now again let $\mathcal{O}$ and $\mathcal{O}^*$ be arbitrary circuit and cocircuit signatures on a given matroid $\mathbf{M}$. For $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$, $x, y \in C \cap D$ we set

$$\text{(2-1)} \qquad \begin{pmatrix} C & D \\ x & y \end{pmatrix} = \frac{\mathcal{O}C(x)\mathcal{O}^*D(x)}{\mathcal{O}C(y)\mathcal{O}^*D(y)}.$$

The following properties immediately follow from this definition:

$$\text{(PO1)} \qquad \begin{pmatrix} C & D \\ x & x \end{pmatrix} = 1;$$

$$\text{(PO2)} \qquad \begin{pmatrix} C & D \\ x & y \end{pmatrix} \begin{pmatrix} C & D \\ y & z \end{pmatrix} \begin{pmatrix} C & D \\ z & x \end{pmatrix} = 1;$$

$$\text{(PO3)} \qquad \begin{pmatrix} C_1 & D_1 \\ x & y \end{pmatrix} \begin{pmatrix} C_2 & D_2 \\ x & y \end{pmatrix} = \begin{pmatrix} C_1 & D_2 \\ x & y \end{pmatrix} \begin{pmatrix} C_2 & D_1 \\ x & y \end{pmatrix}.$$

Each time, when we write $\begin{pmatrix} C & D \\ x & y \end{pmatrix}$, we mean that $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$, $x, y \in C \cap D$. We also have

$$\text{(PO1}') \qquad \begin{pmatrix} C & D \\ x & y \end{pmatrix} \begin{pmatrix} C & D \\ y & x \end{pmatrix} = 1,$$

but we do not give it a separate number, for it follows easily from (PO1) and (PO2).

If $\mathcal{O} \perp \mathcal{O}^*$– so $(E, \overline{\mathcal{O}})$ is an oriented matroid–then we also have

$$\text{(PO4)} \qquad \left\{ \begin{pmatrix} C & D \\ x & y \end{pmatrix} \;\middle|\; x, y \in C \cap D \right\} \ni -1 \qquad \text{for } C \in \mathfrak{C},\ D \in \mathfrak{C}^* \text{ with } C \cap D \neq \emptyset.$$

As a special case of (PO4) we have, using (PO1), (PO1′),

$$\text{(PO4}') \qquad \begin{pmatrix} C & D \\ x & y \end{pmatrix} = -1 \qquad \text{for } C \in \mathfrak{C},\ D \in \mathfrak{C}^* \text{ with } C \cap D = \{x, y\}.$$

It is easy to see that the values of $\begin{pmatrix} C & D \\ x & y \end{pmatrix}$ depend only on the equivalence classes of $\mathcal{O}$ and $\mathcal{O}^*$. Moreover, if we change $\mathcal{O}$ to $\mathcal{O}_A$ and $\mathcal{O}^*$ to $\mathcal{O}_A^*$ for the same $A \subseteq E$, these values will not change. So $\begin{pmatrix} C & D \\ x & y \end{pmatrix}$ depend only on the reorientation class of the oriented matroid $(E, \overline{\mathcal{O}})$.

Definition. Let $\mathbf{M} = (E, \mathfrak{C})$ be a matroid with $\mathfrak{C}^*$ its set of cocircuits. A *projective orientation* of $\mathbf{M}$ is a function $\mathbf{P}$, defined on the set $\{(C, D, x, y) \in \mathfrak{C} \times \mathfrak{C}^* \times E \times E \mid x, y \in C \cap D\}$, whose values, denoted by $\begin{pmatrix} C & D \\ x & y \end{pmatrix}$, belong to $\{1, -1\}$ and satisfy conditions (PO1)—(PO4) for all possible arguments.

Remark. If we have a projective orientation $\mathbf{P}$ of $\mathbf{M}$, then $\begin{pmatrix} D & C \\ x & y \end{pmatrix}_{\mathbf{M}^*} = \begin{pmatrix} C & D \\ x & y \end{pmatrix}$ defines a projective orientation $\mathbf{P}^*$ of $\mathbf{M}^*$.

Now we shall consider an important example.

Example 1. Let $\mathbf{M} = \mathbf{U}_2(4)$ be the uniform matroid of rank 2 on the four-element set $E = \{1, 2, 3, 4\}$. So $\mathfrak{C} = \{C_i \mid i = 1, \ldots, 4\}$, where $C_i = E \setminus \{i\}$. Then $\mathfrak{C}^* = \mathfrak{C}$, but the set C_i as an element of $\mathfrak{C}^*$ will be denoted by D_i. We shall describe all possible projective orientations of $\mathbf{M}$.

Figure 1 shows three configurations of four vectors in $\mathbf{R}^2$. In each case the underlying matroid is $\mathbf{M}$. Each configuration determines an oriented matroid, and hence a projective orientation $\mathbf{P}$ of $\mathbf{M}$. We shall determine these three projective orientations of $\mathbf{M}$. It will be shown later (Lemma 2.1) that there are no others.

FIGURE 1. THE THREE PROJECTIVE CONFIGURATIONS OF $\mathbf{U}_2(4)$.

We shall concentrate on case (a), since the others can be obtained from (a) by permutations of E. First we indicate how to construct circuit and cocircuit signatures for the given configuration. Regarding $\mathcal{O}C_4$, for example, since $v(3)$ is a positive linear combination of $v(1)$ and $v(2)$, we may define $\mathcal{O}C_4$ by $\mathcal{O}C_4(1) = \mathcal{O}C_4(2) = 1$, $\mathcal{O}C_4(3) = -1$. Let $L(i)$ denote the line through the origin and $v(i)$, for $i = 1,2,3,4$. Then $v(1)$ is on the opposite side of $L(4)$ from $v(2)$ and $v(3)$; so we may define $\mathcal{O}^*D_4$ by $\mathcal{O}^*D_4(2) = \mathcal{O}^*D_4(3) = 1$, $\mathcal{O}^*D_4(1) = -1$. Now we can compute

$$\begin{pmatrix} C_4 & D_4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} C_4 & D_4 \\ 2 & 3 \end{pmatrix} = -1; \qquad \begin{pmatrix} C_4 & D_4 \\ 1 & 3 \end{pmatrix} = 1.$$

The end result (for case (a) is:

$$\begin{pmatrix} C_1 & D_1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} C_2 & D_2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} C_3 & D_3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} C_4 & D_4 \\ 1 & 3 \end{pmatrix} = 1;$$

$$\text{all other } \begin{pmatrix} C_i & D_j \\ k & l \end{pmatrix} = -1.$$

In case (b) we obtain:

$$\begin{pmatrix} C_1 & D_1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} C_2 & D_2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} C_3 & D_3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} C_4 & D_4 \\ 2 & 3 \end{pmatrix} = 1;$$

$$\text{all other } \begin{pmatrix} C_i & D_j \\ k & l \end{pmatrix} = -1;$$

while in case (c) we have:

$$\begin{pmatrix} C_1 & D_1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} C_2 & D_2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} C_3 & D_3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} C_4 & D_4 \\ 1 & 2 \end{pmatrix} = 1;$$

$$\text{all other } \begin{pmatrix} C_i & D_j \\ k & l \end{pmatrix} = -1.$$

The same projective orientation $\mathbf{P}$ of $\mathbf{M}$ is obtained from any reorientation of the oriented matroid represented by the configuration of Figure 1(a). It follows that $\mathbf{P}$ depends only on the configuration of the lines $L(i)$ for $i = 1,2,3,4$. Now four distinct lines through the origin in $\mathbf{R}^2$ have just one *projective* invariant: their cross-ratio $\tilde{\phi} = \tilde{\phi}(L(1), L(2), L(3), L(4)) \in \mathbf{R} \setminus \{\vec{0}\}$. The corresponding *combinatorial* invariant is a triple

$$\vec{\phi} = (\phi_{12|34}, \phi_{13|24}, \phi_{14|23}),$$

where

$$\phi_{ij|kl} = \tilde{\phi}(L(i), L(j), L(k), L(l)) \Big/ |\tilde{\phi}(L(i), L(j), L(k), L(l))| = \pm 1.$$

(The symmetries of $\phi_{ij|kl}$ are generated by $\phi_{ij|kl} = \phi_{ji|kl} = \phi_{kl|ij}$.) There is a constraint on the possible values of the $\phi_{ij|kl}$: exactly one of them must equal -1.

In the configurations of Figure 1, we have $\phi_{13|24} = -1$ in (a), $\phi_{14|23} = -1$ in (b), and $\phi_{12|34} = -1$ in (c).

We can express $\phi_{ij|kl}$ in terms of $\mathbf{P}$:

$$\begin{aligned}\phi_{ij|kl} &= \frac{\mathcal{O}^*D_i(k)\mathcal{O}^*D_j(l)}{\mathcal{O}^*D_i(l)\mathcal{O}^*D_j(k)} \\ &= \begin{pmatrix} C_i & D_i \\ k & l \end{pmatrix} \Big/ \begin{pmatrix} C_i & D_j \\ k & l \end{pmatrix} \\ &= -\begin{pmatrix} C_i & D_i \\ k & l \end{pmatrix} \qquad \text{by (PO4}'\text{).}\end{aligned}$$

(This equals $\begin{pmatrix} C_j & D_j \\ k & l \end{pmatrix}$ by (PO3) and (PO4′).) □

This example shows that the data $\{\begin{pmatrix} C & D \\ x & y \end{pmatrix}\}$ of a projective orientation of a given matroid $\mathbf{M}$ are redundant. We shall return to this point at the end of section 3, when we discuss generalised cross-ratios.

Theorem 1. *For any projective orientation of a matroid $\mathbf{M} = (E, \mathfrak{C})$ there exist orthogonal circuit and cocircuit signatures $\mathcal{O}$ and $\mathcal{O}^*$ on $\mathbf{M}$ such that (2-1) holds, and the reorientation class of the oriented matroid $(E, \overline{\mathcal{O}})$ is uniquely determined by the given projective orientation.*

Proof. The proof takes up the rest of this section. First we shall prove the uniqueness. It follows immediately from [6, Theorem 6.1], but it is also quite easy to prove it directly.

It is clear that it is enough to consider the case when $\mathbf{M}$ is connected. Let $(\mathcal{O}_1, \mathcal{O}_1^*)$ and $(\mathcal{O}_2, \mathcal{O}_2^*)$ be two pairs of circuit and cocircuit signatures, giving the same projective orientation of $\mathbf{M}$. Let us fix an arbitrary linear ordering of E and write the elements of circuits and cocircuits of $\mathbf{M}$ in increasing order to compare them lexicographically. Using equivalence and reorientation, we can change $(\mathcal{O}_2, \mathcal{O}_2^*)$, without changing the reorientation class of $(E, \mathcal{O}_2)$, in such a way that $\mathcal{O}_1C(x) = \mathcal{O}_2C(x)$ for x being the minimal element of $C \in \mathfrak{C}$ or C being the minimal circuit containing x, and $\mathcal{O}_1^*D(x) = \mathcal{O}_2^*D(x)$ for x being the minimal element of $D \in \mathfrak{C}^*$. Let $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$, $x, y \in C \cap D$ be such that $\mathcal{O}_1C(x) = \mathcal{O}_2C(x)$ and $\mathcal{O}_1^*D(x) = \mathcal{O}_2^*D(x)$; then from $\mathcal{O}_1C(y) = \mathcal{O}_2C(y)$ follows $\mathcal{O}_1^*D(y) = \mathcal{O}_2^*D(y)$ and vice versa. It is also easy to see that given $C, C' \in \mathfrak{C}$, $x, y \in C \cap C'$ such that $\mathcal{O}_1C(x) = \mathcal{O}_2C(x)$ and $\mathcal{O}_1C'(x) = \mathcal{O}_2C'(x)$, then from $\mathcal{O}_1C(y) = \mathcal{O}_2C(y)$ follows $\mathcal{O}_1C'(y) = \mathcal{O}_2C'(y)$ (because we have $\dfrac{\mathcal{O}_iC(y)\mathcal{O}_iC'(x)}{\mathcal{O}_iC(x)\mathcal{O}_iC'(y)} = \begin{pmatrix} C & D \\ y & x \end{pmatrix} \Big/ \begin{pmatrix} C' & D \\ y & x \end{pmatrix}$ for any $D \ni x, y$), and the same for cocircuits.

We use now induction on x to prove that for any $x \in E$, $C \ni x$, $D \ni x$ we have $\mathcal{O}_1C(x) = \mathcal{O}_2C(x)$ and $\mathcal{O}_1^*D(x) = \mathcal{O}_2^*D(x)$. If x is the minimal element x_0 of E, then it is obvious. Now let it be true for all $x < y$. We have $\mathcal{O}_1C_0(y) = \mathcal{O}_2C_0(y)$, where C_0 is the first circuit containing y. Now $x_0 \in C_0$, so $\mathcal{O}_1^*D(y) = \mathcal{O}_2^*D(y)$ for all $D \ni x_0, y$; hence $\mathcal{O}_1C(y) = \mathcal{O}_2C(y)$ for all $C \ni x_0, y$. For any $x < y$ we have either a circuit or a cocircuit containing x_0, x and y. It follows that for any circuit

C having x as its minimal element, $\mathcal{O}_1 C(y) = \mathcal{O}_2 C(y)$; and for any cocircuit D having x as its minimal element, $\mathcal{O}_1^* D(y) = \mathcal{O}_2^* D(y)$. If y is the minimal element of a circuit C or a cocircuit D, then $\mathcal{O}_1 C(y) = \mathcal{O}_2 C(y)$ or $\mathcal{O}_1^* D(y) = \mathcal{O}_2^* D(y)$ by the choice of $(\mathcal{O}_2, \mathcal{O}_2^*)$. The uniqueness part of the theorem follows.

Lemma 2.1. *The theorem is true for* $\mathbf{M} = \mathbf{U}_2(4)$. *Moreover* $\mathbf{M}$ *has exactly three projective orientations in this case.*

Proof. $C_i \cap D_j = \{k, l\}$ for $i \neq j$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, so by (PO4′) we have

$$\begin{pmatrix} C_i & D_j \\ k & l \end{pmatrix} = -1. \tag{2-2}$$

Now we shall forget (PO4) for some time and deduce what we can from (PO1)—(PO3) and (2-2). First from (PO3) and (2-2) we have for $\{i, j, k, l\} = \{1, 2, 3, 4\}$

$$\begin{pmatrix} C_i & D_i \\ k & l \end{pmatrix} \begin{pmatrix} C_j & D_j \\ k & l \end{pmatrix} = 1, \tag{2-3}$$

thus from (PO1′) and (2-3)

$$\begin{pmatrix} C_i & D_i \\ k & l \end{pmatrix} = \begin{pmatrix} C_j & D_j \\ l & k \end{pmatrix}. \tag{2-4}$$

Applying (2-4) to (PO2) we get

$$\begin{pmatrix} C_i & D_i \\ j & k \end{pmatrix} \begin{pmatrix} C_j & D_j \\ k & i \end{pmatrix} \begin{pmatrix} C_k & D_k \\ i & j \end{pmatrix} = 1. \tag{2-5}$$

Multiplying it by

$$\begin{pmatrix} C_i & D_i \\ k & l \end{pmatrix} \begin{pmatrix} C_k & D_k \\ l & i \end{pmatrix} \begin{pmatrix} C_l & D_l \\ i & k \end{pmatrix} = 1$$

we get with the help of (PO1)—(PO3) and (2-4)

$$\begin{pmatrix} C_i & D_i \\ j & l \end{pmatrix}^2 \begin{pmatrix} C_j & D_j \\ k & i \end{pmatrix}^2 = 1.$$

By (PO1) this is the same as

$$\begin{pmatrix} C_i & D_i \\ j & l \end{pmatrix}^2 = \begin{pmatrix} C_j & D_j \\ i & k \end{pmatrix}^2. \tag{2-6}$$

This is not surprising, since (by definition of a projective orientation) all the values are either 1 or -1, but we have not yet needed to use this part of the definition, and we shall proceed for some time without using it.

We set

$$\begin{pmatrix} C_i & D_i \\ j & l \end{pmatrix} = \gamma_{jl,ik} \begin{pmatrix} C_j & D_j \\ k & i \end{pmatrix}, \tag{2-7}$$

and by (2-6) we have $\gamma_{jl,ik}^2 = 1$. It follows that $\gamma_{jl,ik} = \gamma_{ik,jl}$ and, using (PO1′) and (2-4), we have also $\gamma_{jl,ik} = \gamma_{jl,ki}$. Now we rewrite (2-5) as

$$\begin{pmatrix} C_i & D_i \\ j & l \end{pmatrix} \begin{pmatrix} C_j & D_j \\ l & i \end{pmatrix} \begin{pmatrix} C_l & D_l \\ i & j \end{pmatrix} = 1,$$

and apply (2-7) to the first factor, then (PO2) and (PO1′) to get

$$\begin{pmatrix} C_j & D_j \\ l & k \end{pmatrix} = \gamma_{jl,ik} \begin{pmatrix} C_l & D_l \\ j & i \end{pmatrix}.$$

and thus $\gamma_{jl,ik} = \gamma_{lk,ji}$. This implies that all the γ's are equal, and for each permutation (i,j,k,l) of $(1,2,3,4)$ we have

$$\begin{pmatrix} C_i & D_i \\ j & l \end{pmatrix} = \gamma \begin{pmatrix} C_j & D_j \\ k & i \end{pmatrix} \tag{2-8}$$

for some fixed γ with $\gamma^2 = 1$.

Now let us set

$$f = \begin{pmatrix} C_1 & D_1 \\ 3 & 4 \end{pmatrix}, \qquad g = \begin{pmatrix} C_1 & D_1 \\ 2 & 4 \end{pmatrix}. \tag{2-9}$$

¿From (2-4) and (2-8) we have

$$\begin{pmatrix} C_2 & D_2 \\ 4 & 3 \end{pmatrix} = f, \qquad \begin{pmatrix} C_2 & D_2 \\ 1 & 3 \end{pmatrix} = \gamma g; \tag{2-10}$$

$$\begin{pmatrix} C_3 & D_3 \\ 1 & 2 \end{pmatrix} = \gamma f, \qquad \begin{pmatrix} C_3 & D_3 \\ 4 & 2 \end{pmatrix} = g; \tag{2-11}$$

$$\begin{pmatrix} C_4 & D_4 \\ 2 & 1 \end{pmatrix} = \gamma f, \qquad \begin{pmatrix} C_4 & D_4 \\ 3 & 1 \end{pmatrix} = \gamma g. \tag{2-12}$$

Now at last we use (PO4). From (2-9)—(2-12) we have that

(1) at least one of f and g equals -1,
(2) at least one of f and γg equals -1,
(3) at least one of γf and g equals -1,
(4) at least one of γf and γg equals -1.

It follows that $\gamma = 1$, thus

$$\begin{pmatrix} C_i & D_i \\ j & l \end{pmatrix} = \begin{pmatrix} C_j & D_j \\ k & i \end{pmatrix} \qquad \text{for } \{i,j,k,l\} = \{1,2,3,4\}. \tag{2-13}$$

Now it is easy to construct circuit and cocircuit signatures on $\mathbf{M}$. Let us set $\mathcal{O}C_i(5-i) = \mathcal{O}^*D_i(5-i) = 1$ for $i = 1,\ldots,4$; $\mathcal{O}C_1(2) = \mathcal{O}C_1(3) = \mathcal{O}^*D_4(2) = 1$

(all this can be achieved in any case by equivalences and reorientations). Using (2-2) and (2-9)—(2-12) with $\gamma = 1$ we easily fill in the rest of the tables:

$E \backslash \mathfrak{C}$	C_1	C_2	C_3	C_4
1		1	-1	1
2	1		1	f
3	1	1		$-g$
4	1	$-f$	$-g$	

$E \backslash \mathfrak{C}^*$	D_1	D_2	D_3	D_4
1		g	$-f$	1
2	g		1	1
3	f	1		-1
4	1	-1	-1	

It is easy to see that (2-1) is fulfilled. So for $\mathbf{M} = \mathbf{U}_2(4)$ the theorem is true. Moreover only the following values for (f, g) are permitted: $(-1, 1)$, $(-1, -1)$ and (1,-1). Thus $\mathbf{M}$ has exactly these three projective orientations. By (2-9) they are represented by the configurations of lines in $\mathbf{R}^2$ shown in Figure 1 (a), (b) and (c) respectively. □

For the rest of this section let $\mathbf{M} = (E, \mathfrak{C})$ be a matroid with fixed projective orientation.

Lemma 2.2. *Let A and B be subsets of E with $A \cap B = \emptyset$. Then we can correctly define a projective orientation of the matroid $\mathbf{M}' = (\mathbf{M} \setminus A)/B$ by the formula*

$$\begin{pmatrix} C' & D' \\ x & y \end{pmatrix}_{\mathbf{M}'} = \begin{pmatrix} C & D \\ x & y \end{pmatrix},$$

where $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$, $C \cap A = D \cap B = \emptyset$, $C' = C \setminus D \in \mathfrak{C}_{\mathbf{M}'}$, $D' = D \setminus A \in \mathfrak{C}^*_{\mathbf{M}'}$ and $x, y \in C \cap D$.

Proof. The only thing we have to verify is the correctness of the above definition. So let us assume that there is another circuit $C_1 \neq C$ with $C_1 \cap A = \emptyset$ and $C_1 \setminus B = C'$. As $\mathbf{M}'$ is a matroid, there exists $D_1' \in \mathfrak{C}^*_{\mathbf{M}'}$ such that $D_1' \cap C' = \{a, b\}$, and therefore there exists $D_1 \in \mathfrak{C}^*$ with $D_1 \cap B = \emptyset$ and $D_1 \cap C_1 = D_1 \cap C = \{x, y\}$. Now we use (PO3) and (PO4′) to get $\begin{pmatrix} C_1 & D \\ x & y \end{pmatrix} = \begin{pmatrix} C & D \\ x & y \end{pmatrix}$. In the same way we can prove that nothing depends on the choice of $D \in \mathfrak{C}^*$ with $D \cap B = \emptyset$ and $D \setminus A = D'$. □

Lemma 2.3. *Under the conditions of* Lemma 2.1 *let us have $\mathbf{M}' \cong \mathbf{U}_2(4)$, $E \setminus (A \cup B) = \{x_1, x_2, x_3, x_4\}$, $C_1, C_2 \in \mathfrak{C}$, $D_1, D_2 \in \mathfrak{C}^*$, $C_i \cap A = D_i \cap B = \emptyset$, $C_i \setminus B = D_i \setminus A = \{x_1, x_2, x_3, x_4\} \setminus \{x_i\}$ $(i = 1, 2)$. Then $\begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} C_2 & D_2 \\ x_1 & x_4 \end{pmatrix}$.*

Proof. This follows easily from Lemmas 2.1 and 2.2. □

Lemma 2.4. *Let $C_1, C_2, C_3 \in \mathfrak{C}$, $D_1, D_2 \in \mathfrak{C}^*$ and $x_1, x_2, x_3 \in D_1 \cap D_2$ with $x_i \in C_j$ for $i \neq j$. Then*

$$\text{(2-14)} \qquad \begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix} \begin{pmatrix} C_2 & D_1 \\ x_3 & x_1 \end{pmatrix} \begin{pmatrix} C_3 & D_1 \\ x_1 & x_2 \end{pmatrix} = \begin{pmatrix} C_1 & D_2 \\ x_2 & x_3 \end{pmatrix} \begin{pmatrix} C_2 & D_2 \\ x_3 & x_1 \end{pmatrix} \begin{pmatrix} C_3 & D_2 \\ x_1 & x_2 \end{pmatrix}.$$

Proof. First we notice that the fulfillment of (2-14) does not depend on the choice of C_i (by PO3)). So if (2-14) is true, then we say that D_1 and D_2 *match* at $\{x_1, x_2, x_3\}$.

Let $\mathbf{M}$ be a matroid with the minimal number of elements for which there exist $D_1, D_2 \in \mathfrak{C}^*$ and $x_1, x_2, x_3 \in D_1 \cap D_2$ with D_1 not matching D_2 at $\{x_1, x_2, x_3\}$. In this case the elements x_1, x_2 and x_3 are pairwise distinct, and there is no $C \in \mathfrak{C}$ with $x_1, x_2, x_3 \in C$ (or we could take all $C_i = C$ with both sides of (2-14) equal to 1). As $\mathbf{M}$ is minimal we have $D_1 \cup D_2 = E$ and $D_1 \cap D_2 = \{x_1, x_2, x_3\}$. We choose C_i in such a way that $|C_i \cap D_2| = 2$, thus $C_i \subset D_1$.

We shall show now that each element of D_1 is in exactly two of the C_i, and there are no other circuits inside D_1. Let $x \in D_1 \setminus D_2$, $x \neq C_1 \cup C_2$. We consider two cases:

(1) $|D_1| = 4$. In this case $C_1 = \{x_2, x_3\}$, $C_2 = \{x_1, x_3\}$. Using elimination we obtain $C_3 = \{x_1, x_2\}$. By (PO4′) all the factors in both sides of (2-14) equal -1; thus D_1 and D_2 match at $\{x_1, x_2, x_3\}$.

(2) $|D_1| > 4$. If $D_2 \in \mathfrak{C}^*_{\mathbf{M}\setminus\{x\}}$, then $D_1 \setminus \{x\}$ and D_2 do not match at $\{x_1, x_2, x_3\}$ as cocircuits of $\mathbf{M} \setminus \{x\}$. Hence $D_2 \notin \mathfrak{C}^*_{\mathbf{M}\setminus\{x\}}$, i.e. there is $D_3 \in \mathfrak{C}^*$ with $x \in D_3 \subset D_2 \cup \{x\}$. If $x_3 \notin D_3$, then we eliminate some element $y \in D_3 \setminus D_1 \neq \emptyset$ from $D_2 \cup D_3$ to get a cocircuit $D_4 \ni x_3$ with $x \in D_4 \subset D_2 \cup \{x\}$. So we may assume that $x_3 \in D_3$. From orthogonality to C_1 and C_2 we have also $x_2 \in D_3$ and $x_1 \in D_3$. From the minimality of $\mathbf{M}$ it follows that D_3 matches both D_1 and D_2 at $\{x_1, x_2, x_3\}$ (because $C_3 \cup C_1 \neq E$ and $C_3 \cup C_2 \neq E$). Hence D_1 and D_2 match at $\{x_1, x_2, x_3\}$.

In either case we get a contradiction to the choice of $\mathbf{M}$. This shows that each element of D_1 is in at least two of the C_i.

Now let $x \in C_1 \cap C_2 \cap C_3$. Eliminating x from $C_2 \cup C_3$ we get $C_4, C_5 \in \mathfrak{C}$ with $x_2 \in C_4$, $x_3 \in C_5$. If none of them contains both x_2 and x_3, then both of them contain x_1 (by orthogonality to D_2). Thus we can apply elimination again to get a circuit $C_6 \ni x_2, x_3$. Hence in any case we can replace C_1 by a circuit not containing x. If we could do the same for C_2 we would obtain the previous situation, which was found to be impossible. So each element of D_1 is in exactly two of the C_i. Thus they are determined uniquely and there are no other circuits inside D_1 (or we could use elimination to change C_i).

Now let $x_4 \in D_1 \setminus D_2$. Without loss of generality we may assume that $x_4 \in C_2 \cap C_3$. Then we have a cocircuit $\{x_1, x_4\}$ of $\mathbf{M} \cdot D_1$. Thus there is a cocircuit $D_4 \subset D_2 \cup \{x_4\}$ with $D_4 \cap D_1 = \{x_1, x_4\}$. Let $y \in D_4 \setminus D_1$. We eliminate y from $D_4 \cup D_2$ to get a cocircuit D_5 with $x_2 \in D_5 \subseteq (D_4 \cup D_2) \setminus \{y\}$. From $D_5 \not\subseteq D_2$ we have $x_4 \in D_5$ and from orthogonality to C_1 we have $x_3 \in D_5$. If $x_1 \notin D_5$, then we have

$$\begin{pmatrix} C_1 & D_5 \\ x_2 & x_3 \end{pmatrix} \begin{pmatrix} C_2 & D_5 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} C_3 & D_5 \\ x_4 & x_2 \end{pmatrix} = -1,$$

but

$$\begin{pmatrix} C_2 & D_4 \\ x_1 & x_4 \end{pmatrix} = \begin{pmatrix} C_3 & D_4 \\ x_1 & x_4 \end{pmatrix} = -1,$$

so by (PO3)

$$\begin{pmatrix} C_2 & D_1 \\ x_1 & x_4 \end{pmatrix} = \begin{pmatrix} C_3 & D_1 \\ x_1 & x_4 \end{pmatrix},$$

hence

$$\begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix}\begin{pmatrix} C_2 & D_1 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} C_3 & D_1 \\ x_4 & x_2 \end{pmatrix} = \begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix}\begin{pmatrix} C_2 & D_1 \\ x_3 & x_1 \end{pmatrix}\begin{pmatrix} C_2 & D_1 \\ x_1 & x_4 \end{pmatrix}$$
$$\cdot\begin{pmatrix} C_3 & D_1 \\ x_4 & x_1 \end{pmatrix}\begin{pmatrix} C_3 & D_1 \\ x_1 & x_2 \end{pmatrix} = \begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix}\begin{pmatrix} C_2 & D_1 \\ x_3 & x_1 \end{pmatrix}\begin{pmatrix} C_3 & D_1 \\ x_1 & x_2 \end{pmatrix}$$
$$\neq \begin{pmatrix} C_1 & D_2 \\ x_2 & x_3 \end{pmatrix}\begin{pmatrix} C_2 & D_2 \\ x_3 & x_1 \end{pmatrix}\begin{pmatrix} C_3 & D_2 \\ x_1 & x_2 \end{pmatrix} = -1.$$

So D_5 and D_1 do not match at $\{x_4, x_2, x_3\}$. But that contradicts the minimality of $\mathbf{M}$, for $D_5 \cup D_1 \neq E$. Hence $x_1 \in D_5$.

Again from the minimality of $\mathbf{M}$ we have that D_5 and D_1 match at $\{x_1, x_2, x_3\}$; thus D_5 and D_2 do not match at $\{x_1, x_2, x_3\}$ and we must have $D_5 \cup D_2 = E$. It follows that $|D_1| = 4$.

The same arguments applied to D_2 show that $|D_2| = 4$ and hence $|E| = 5$.

Now we have $E = \{x_1, x_2, x_3, x_4, x_5\}$, $D_1 = \{x_1, x_2, x_3, x_4\}$ $D_2 = \{x_1, x_2, x_3, x_5\}$, $D_4 = \{x_1, x_4, x_5\}$, $C_1 = \{x_2, x_3\}$, $C_2 = \{x_1, x_3, x_4\}$ and $C_3 = \{x_1, x_2, x_4\}$. Using elimination and orthogonality we get $D_3 = \{x_2, x_3, x_4, x_5\}$, $C_4 = \{x_1, x_4, x_5\}$, $C_5 = \{x_2, x_4, x_5\}$, $C_6 = \{x_3, x_4, x_5\}$, $C_7 = \{x_1, x_3, x_5\}$ and $C_8 = \{x_1, x_2, x_5\}$. It is easy to see that $\mathbf{M} \setminus \{x_2\} \cong \mathbf{M} \setminus \{x_3\} \cong \mathbf{U}_2(4)$. Using Lemma 2.2 we have

$$\begin{pmatrix} C_2 & D_1 \\ x_3 & x_1 \end{pmatrix} = \begin{pmatrix} C_4 & D_4 \\ x_5 & x_4 \end{pmatrix} = \begin{pmatrix} C_3 & D_1 \\ x_2 & x_1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix}\begin{pmatrix} C_2 & D_1 \\ x_3 & x_1 \end{pmatrix}\begin{pmatrix} C_3 & D_1 \\ x_1 & x_3 \end{pmatrix} = -1 = \begin{pmatrix} C_1 & D_2 \\ x_2 & x_3 \end{pmatrix}\begin{pmatrix} C_2 & D_2 \\ x_3 & x_1 \end{pmatrix}\begin{pmatrix} C_3 & D_2 \\ x_1 & x_2 \end{pmatrix}.$$

Lemma 2.4 follows. □

Lemma 2.5. *If $\{a, b\} \in \mathfrak{C}$ and the statement of the theorem is true for $\mathbf{M}' = \mathbf{M} \setminus \{a\}$, then it is also true for $\mathbf{M}$.*

Proof. If $a = b$, then a is a loop in $\mathbf{M}$ and $\mathbf{M}$ is a direct product of $\mathbf{M}'$ and the zero-rank matroid on the set $\{a\}$. The statement of the theorem is obviously true for the zero-rank matroid, so if it is true for $\mathbf{M}'$,then it is true for $\mathbf{M}$.

Let $a \neq b$. For $C \in \mathfrak{C}_{\mathbf{M}'}$ we set

$$C_a = \begin{cases} C, & \text{for } b \notin C \\ C \setminus \{b\} \cup \{a\}, & \text{for } b \in C. \end{cases}$$

It is easy to see that $\mathfrak{C} = \mathfrak{C}_{\mathbf{M}'} \cup \{C_a \mid C \in \mathfrak{C}_{\mathbf{M}'}\} \cup \{\{a, b\}\}$. Each cocircuit of $\mathbf{M}$ either contains both a and b or neither of them. Hence the map $D \mapsto D \setminus \{a\}$ is a bijection of $\mathfrak{C}$ onto $\mathfrak{C}_{\mathbf{M}'}$.

Let $C \in \mathfrak{C}_{\mathbf{M}'}$, $b \in C$, $D \in \mathfrak{C}^*$ and $x, y \in (C \cap D) \setminus \{b\}$ be given, with $x \neq y$. Then there is a cocircuit D_1 of $\mathbf{M}'$ such that $D_1 \cap C = \{x, y\}$, thus $D_1 \in \mathfrak{C}^*$. This implies that $D_1 \cap C_a = \{x, y\}$. Hence from

$$\begin{pmatrix} C & D \\ x & y \end{pmatrix}\begin{pmatrix} C_a & D_1 \\ x & y \end{pmatrix} = \begin{pmatrix} C_a & D \\ x & y \end{pmatrix}\begin{pmatrix} C & D_1 \\ x & y \end{pmatrix}$$

we have, using (PO4′),

$$\begin{pmatrix} C & D \\ x & y \end{pmatrix} = \begin{pmatrix} C_a & D \\ x & y \end{pmatrix}. \tag{2-15}$$

Now let $C \in \mathfrak{C}_{\mathbf{M}'}$, $D \in \mathfrak{C}^*$ and $b, x \in C \cap D$ be given, with $x \neq b$. So $a \in D$ also. Choose $D_1 \in \mathfrak{C}^*$ with $D_1 \cap C = \{b, x\}$. According to Lemma 2.4

$$\begin{pmatrix} \{a,b\} & D \\ a & b \end{pmatrix} \begin{pmatrix} C & D \\ b & x \end{pmatrix} \begin{pmatrix} C_a & D \\ x & a \end{pmatrix} = \begin{pmatrix} \{a,b\} & D_1 \\ a & b \end{pmatrix} \begin{pmatrix} C & D_1 \\ b & x \end{pmatrix} \begin{pmatrix} C_a & D_1 \\ x & a \end{pmatrix}.$$

Applying (PO4′) and (PO1′), we obtain

$$\begin{pmatrix} C & D \\ b & x \end{pmatrix} = \begin{pmatrix} C_a & D \\ b & x \end{pmatrix} \tag{2-16}$$

Let us fix now circuit and cocircuit signatures on $\mathbf{M}'$, consistent with the given projective orientation. Leaving the signatures of the elements of $\mathfrak{C}_{\mathbf{M}'}$ the same, we set

$$\begin{gathered} \mathcal{O}\{a,b\}(a) = 1, \quad \mathcal{O}\{a,b\}(b) = -1; \\ \mathcal{O}C_a(x) = \mathcal{O}C(x), \quad \mathcal{O}C_a(a) = \mathcal{O}C(b), \qquad \text{for } b, x \in C \in \mathfrak{C}_{\mathbf{M}'}; \\ \mathcal{O}^*D(x) = \mathcal{O}^*(D \setminus \{a\})(x), \qquad \text{for } D \in \mathfrak{C}^*,\ x \in D \in \mathfrak{C}^*; \\ \mathcal{O}^*D(a) = \mathcal{O}^*D(b), \qquad \text{for } a \in D \in \mathfrak{C}^*. \end{gathered}$$

Using (2-15), (2-16) we see that these signatures are consistent with the projective orientation. □

Lemma 2.6. *Let* $\mathbf{M}$ *be connected, without parallel elements, and such that* $rk(\mathbf{M}^*) > 2$, *and let us assume that for each proper minor of* $\mathbf{M}$ *the statement of the theorem is true. Then it is also true for* $\mathbf{M}$.

Proof. First we choose $a \in E$ such that $\mathbf{M}' = \mathbf{M}/\{a\}$ is connected (which is possible by [7, Theorem 4.25]). Then we fix circuit and cocircuit signatures on $\mathbf{M}'$, consistent with the given projective orientation. That is, we fix $\mathcal{O}C(x)$ and $\mathcal{O}^*D(x)$ for $x \neq a$ and for those $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$ such that $C \setminus \{a\} \in \mathfrak{C}_{\mathbf{M}'}$, $D \subseteq E \setminus \{a\}$.

Let $C \in \mathfrak{C}$, $C \setminus \{a\} \notin \mathfrak{C}_{\mathbf{M}'}$; thus $a \notin C$. Let us choose some $x \in C$, and for any $y \in C$ set $\mathcal{O}C(y) = \begin{pmatrix} C & D \\ y & x \end{pmatrix} \mathcal{O}^*D(x)/\mathcal{O}^*D(y)$, where $D \in \mathfrak{C}_{\mathbf{M}'}$ with $y, x \in D$ (such D exists because $\mathbf{M}'$ is connected). We show that this definition is correct: For $y = x$ it just means $\mathcal{O}C(x) = 1$. For $y \neq x$ we can find a circuit $C' \ni y, x$ with $C' \setminus \{a\} \in \mathfrak{C}_{\mathbf{M}'}$; thus for any $D' \in \mathfrak{C}^*_{\mathbf{M}'}$ with $y, x \in D'$, from

$$\begin{pmatrix} C & D \\ y & x \end{pmatrix} \begin{pmatrix} C' & D' \\ y & x \end{pmatrix} = \begin{pmatrix} C' & D \\ y & x \end{pmatrix} \begin{pmatrix} C & D' \\ y & x \end{pmatrix}$$

we have, using (2-1) for $\begin{pmatrix} C' & D \\ y & x \end{pmatrix}$ and $\begin{pmatrix} C' & D' \\ y & x \end{pmatrix}$,

$$\begin{pmatrix} C & D \\ y & x \end{pmatrix} \mathcal{O}^*D(x)/\mathcal{O}^*D(y) = \begin{pmatrix} C & D' \\ y & x \end{pmatrix} \mathcal{O}^*D'(x)/\mathcal{O}^*D'(y).$$

Now consider $y, z \in C$. We want to show that

$$\begin{pmatrix} C & D \\ y & z \end{pmatrix} = \frac{\mathcal{O}C(y)\mathcal{O}^*D(y)}{\mathcal{O}C(z)\mathcal{O}^*D(z)} \tag{2-17}$$

for $D \in \mathfrak{C}^*_{\mathbf{M}'}$ with $y, z \in D$. As above, it does not matter for which D to prove it. So if there exists a cocircuit of $\mathbf{M}'$, containing x, y and z, then (2-17) follows from (PO2). Otherwise there is a circuit C' with $\{x, y, z\} \subseteq C \setminus \{a\} \in \mathfrak{C}_{\mathbf{M}'}$. Let now D' be a cocircuit of $\mathbf{M}'$, containing x and y, and D'' be a cocircuit of $\mathbf{M}'$, containing x and z. By Lemma 2.3, applied to $\mathbf{M}^*$, we have

$$\begin{pmatrix} C & D \\ y & z \end{pmatrix} \begin{pmatrix} C & D'' \\ z & x \end{pmatrix} \begin{pmatrix} C & D' \\ x & y \end{pmatrix} = \begin{pmatrix} C' & D \\ y & z \end{pmatrix} \begin{pmatrix} C' & D'' \\ z & x \end{pmatrix} \begin{pmatrix} C' & D' \\ x & y \end{pmatrix}$$

Using (2-1) for all factors but $\begin{pmatrix} C & D \\ y & z \end{pmatrix}$ we get (2-17).

In the same way we define $\mathcal{O}^*D(y)$ for $D \in \mathfrak{C}^*$ with $y, a \in D$, $y \neq a$ and verify that for all signature values already fixed the equality (2-1) is true.

So only $\mathcal{O}C(a)$ and $\mathcal{O}^*D(a)$ for $a \in C \in \mathfrak{C}$ and $a \in D \in \mathfrak{C}^*$ are still undefined. First we choose $D_0 \in \mathfrak{C}^*$ with $a \in D_0$ and set $\mathcal{O}^*D_0(a) = 1$. Then for each $C \in \mathfrak{C}$ with $a \in C$ we set $\mathcal{O}C(a) = \begin{pmatrix} C & D_0 \\ a & x \end{pmatrix} \mathcal{O}C(x)\mathcal{O}^*D_0(x)$, where $x \in (C \cap D_0) \setminus \{a\}$ (that this is correct follows from (PO2)). Next we choose $C_0 \in \mathfrak{C}$ with $a \in C_0$ and for each $D \in \mathfrak{C}^*$ with $a \in D$ we set $\mathcal{O}^*D(a) = \begin{pmatrix} C_0 & D \\ a & y \end{pmatrix} \mathcal{O}C_0(y)\mathcal{O}^*D(y)/\mathcal{O}C_0(a)$ for some $y \in (C_0 \cap D) \setminus \{a\}$ (again the correctness is obvious). All we need is to show that

$$\begin{pmatrix} C & D \\ a & x \end{pmatrix} = \frac{\mathcal{O}C(a)\mathcal{O}^*D(a)}{\mathcal{O}C(x)\mathcal{O}^*D(x)} \tag{2-18}$$

for all $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$ with $a \in C \cap D$ and $x \in C \cap D$. If $C = C_0$, then it follows from the definition of $\mathcal{O}D(a)$. So it is enough to show that our definition does not depend on the choice of C_0. From the fundamental theorem about linear subclasses we know that for any $C \in \mathfrak{C}$ with $a \in C$ there is a path $(C_0, C_1, \ldots, C_k = C)$, where $a \in C_i$, $\dim(C_i \cup C_{i-1}) = 1$ and $(\mathbf{M} \cdot (C_i \cup C_{i-1}))/\{a\}$ is connected $(i = 1, \ldots, k)$. Hence it is enough to consider the case $\dim(C_0 \cup C) = 1$. But in this case (2-18) follows from the statement of the theorem for $\mathbf{M} \cdot (C_0 \cup C)$, which is a proper minor of $\mathbf{M}$. □

Now it is easy to complete the proof of the theorem.

Let $\mathbf{M}$ be a matroid with the minimal number of elements, for which the statement of the theorem is false. Then $\mathbf{M}$ is connected, has no parallel elements (by Lemma 2.5), and $rk(\mathbf{M}^*) \leq 2$ (by Lemma 2.6). As the statement of the theorem is essentially the same for dual matroids, we may replace $\mathbf{M}$ by $\mathbf{M}^*$. Now $rk(\mathbf{M}) \leq 2$ and there are no coparallel elements in $\mathbf{M}$. Hence $\mathbf{M} \cong \mathbf{U}_2(4)$; but for $\mathbf{U}_2(4)$ the statement of the theorem is true (by Lemma 2.1). Therefore it is true for any $\mathbf{M}$. □

3. Presentations of the inner Tutte group of a matroid by generators and relations

Theorem 1, and especially its proof, are naturally connected with the theory of Tutte groups of matroids, developed in [2] and subsequent papers by the same authors. We recall now some definitions and theorems from this theory in a slightly modified form. The main difference is that we use circuits instead of hyperplanes (complements of cocircuits). Thus everything we state here is dual to the results of [2] etc.

The extended Tutte group $\mathbb{T}^{\mathfrak{C}}_{\mathbf{M}}$ of a matroid $\mathbf{M}$ is defined to be the abelian group with generators ε and $C(x)$ for $x \in C \in \mathfrak{C}$ and relations

$$\text{(TG0)} \qquad \varepsilon^2 = 1,$$

$$\text{(TG1)} \qquad \frac{C_1(x_2)C_2(x_3)C_3(x_1)}{C_1(x_3)C_2(x_1)C_3(x_2)} = \varepsilon \qquad \text{for } C_i \in \mathfrak{C},\ L = C_1 \cup C_2 \cup C_3 = C_i \cup C_j$$

$$(i \neq j), \quad x_i \in L \setminus C_i \ \text{ and } \dim(L) = 1.$$

(This is just a short way to say that $\mathbb{T}^{\mathfrak{C}}_{\mathbf{M}}$ is defined as the quotient group of the free abelian group with basis consisting of the stated generators, by the subgroup generated by all the quotients of the left-hand side of a relation by its right-hand side.)

The group $\mathbb{T}^{\mathfrak{C},\mathfrak{C}^*}_{\mathbf{M}}$ is defined to be the abelian group with generators ε, $C(x)$ for $x \in C \in \mathfrak{C}$ and $D(x)$ for $x \in D \in \mathfrak{C}^*$, and relations (TG0) and

$$\text{(TG1}') \qquad C(x)D(x) = \varepsilon C(y)D(y) \qquad \text{for } C \in \mathfrak{C},\ D \in \mathfrak{C}^* \text{ with } \{x,y\} = C \cap D.$$

The Tutte group $\mathbb{T}_{\mathbf{M}} \cong \mathbb{T}^{\mathfrak{H},\mathfrak{C}}_{\mathbf{M}}$, as defined in [2], is isomorphic to the subgroup of $\mathbb{T}^{\mathfrak{C},\mathfrak{C}^*}_{\mathbf{M}}$ generated by ε, $C(x)C(y)^{-1}$ for $x, y \in C \in \mathfrak{C}$, and $D(x)D(y)^{-1}$ for $x, y \in D \in \mathfrak{C}^*$. For simplicity we denote this subgroup by $\mathbb{T}_{\mathbf{M}}$.

It follows from [2, Theorem 1.3, Theorem 1.4] that there is an injective homomorphism $\mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \to \mathbb{T}^{\mathfrak{C},\mathfrak{C}^*}_{\mathbf{M}}$ with $\varepsilon \mapsto \varepsilon$ and $C(x) \mapsto C(x)$ for $x \in C \in \mathfrak{C}$. So we may view $\mathbb{T}^{\mathfrak{C}}_{\mathbf{M}}$ as a subgroup of $\mathbb{T}^{\mathfrak{C},\mathfrak{C}^*}_{\mathbf{M}}$. Let us consider the homomorphism $\mathbb{T}^{\mathfrak{C},\mathfrak{C}^*}_{\mathbf{M}} \to \mathbb{Z}^E \times \mathbb{Z}^{\mathfrak{C}} \times \mathbb{Z}^{\mathfrak{C}^*}$, given by $\varepsilon \mapsto 0$, $C(x) \mapsto (\delta_{\bullet,x}, \delta_{\bullet,C}, 0)$ and $D(x) \mapsto (-\delta_{\bullet,x}, 0, \delta_{\bullet,D})$. The kernel of this homomorphism is called the inner Tutte group and denoted by $\mathbb{T}^{(0)}_{\mathbf{M}}$. It is easy to see that $\mathbb{T}^{(0)}_{\mathbf{M}} \subseteq \mathbb{T}_{\mathbf{M}} \subseteq \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \subseteq \mathbb{T}^{\mathfrak{C},\mathfrak{C}^*}_{\mathbf{M}}$. Moreover $\mathbb{T}_{\mathbf{M}} \cong \mathbb{T}^{(0)}_{\mathbf{M}} \times \mathbb{Z}^{|E|-c}$, where c is the number of connected components of $\mathbf{M}$, $\mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \cong \mathbb{T}_{\mathbf{M}} \times \mathbb{Z}^{\mathfrak{C}}$ and $\mathbb{T}^{\mathfrak{C},\mathfrak{C}^*}_{\mathbf{M}} \cong \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \times \mathbb{Z}^{\mathfrak{C}^*}$ (cf. [2, Theorem 1.5]). Thus the structure of any of these groups is known as soon as we know the structure of $\mathbb{T}^{(0)}_{\mathbf{M}}$.

According to [9, Proposition 2.9(i)] the inner Tutte group is generated by ε and all products of the form $C_1(x)C_1(y)^{-1}C_2(y)C_2(x)^{-1}$ for $C_1, C_2 \in \mathfrak{C}$, $x, y \in C_1 \cap C_2$, and $\dim(C_1 \cup C_2) = 1$.

Theorem 2. *Let* $\mathbf{M}$ *be a fixed matroid. Then there is a one-to-one correspondence between the set* $\mathcal{R}(\mathbf{M})$ *of reorientation classes of oriented matroids on* $\mathbf{M}$ *and the set of homomorphisms* $\chi : \mathbb{T}_{\mathbf{M}}^{(0)} \to \{+1, -1\}$ *such that*

(OR1) $$\chi(\varepsilon) = -1$$

and
(OR2)
$$+1 \in \left\{ \chi\left(\frac{C_1(x_3)C_2(x_4)}{C_1(x_4)C_2(x_3)}\right), \chi\left(\frac{C_4(x_3)C_2(x_1)}{C_4(x_1)C_2(x_3)}\right) \right\} \qquad \text{for } x_1, x_2, x_3, x_4 \in L \subseteq E$$

with $\dim L = 1$ *and* $C_i = \langle L \setminus \{x_i\} \rangle$ *being pairwise distinct circuits.*

All the circuit signatures from the reorientation class corresponding to χ *may be obtained as follows: let* $\tilde{\chi} : \mathbb{T}_{\mathbf{M}}^{\mathfrak{C}} \to \{+1, -1\}$ *be any homomorphism such that* $\tilde{\chi}|_{\mathbb{T}_{\mathbf{M}}^{(0)}} = \chi$*; then the circuit signature corresponding to* $\tilde{\chi}$ *is defined by* $\mathcal{O}C(x) = \tilde{\chi}(C(x))$ *for any* $C \in \mathfrak{C}$ *and* $x \in C$*.*

Proof. This is Theorem 4.4 from [3], which is a reformulation of Theorem 6.1 from [2]. □

Corollary 1. *Theorem 2 remains true if* (OR2) *is replaced by*
(OR2′)
$$-1 \in \left\{ \chi\left(\frac{C(x)D(x)}{C(y)D(y)}\right) \;\middle|\; x \in C \cap D \right\} \qquad \text{for any } C \in \mathfrak{C}, D \in \mathfrak{C}^*, y \in C \cap D.$$

Proof. Let χ be a homomorphism which satisfies (OR1) and (OR2′), and let x_1, x_2, x_3, x_4 and L be as in (OR2). Then $\{x_1, x_3, x_4\}$ is a cocircuit of $\mathbf{M} \cdot L$; so there exists a cocircuit D of $\mathbf{M}$ with $D \cap L = \{x_1, x_3, x_4\}$. By (TG1′), $D(x_4)D(x_3)^{-1} = \varepsilon C_1(x_4)^{-1}C_1(x_3)$ and $D(x_1)D(x_3)^{-1} = \varepsilon C_4(x_1)^{-1}C_4(x_3)$. So

$$-1 \in \left\{ 1, -\chi\left(\frac{C_1(x_3)C_2(x_4)}{C_1(x_4)C_2(x_3)}\right), -\chi\left(\frac{C_4(x_3)C_2(x_1)}{C_4(x_1)C_2(x_3)}\right) \right\};$$

hence

$$+1 \in \left\{ \chi\left(\frac{C_1(x_3)C_2(x_4)}{C_1(x_4)C_2(x_3)}\right), \chi\left(\frac{C_4(x_3)C_2(x_1)}{C_4(x_1)C_2(x_3)}\right) \right\}.$$

Conversely, let χ be a homomorphism satisfying (OR1) and (OR2). By Theorem 2 it corresponds to a reorientation class. We pick a circuit signature $\mathcal{O}$ from it, and a cocircuit signature $\mathcal{O}^*$ orthogonal to $\mathcal{O}$. It is easy to see that for any $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$, and $x, y \in C \cap D$ we must have

$$\frac{\mathcal{O}C(x)\mathcal{O}^*D(x)}{\mathcal{O}C(y)\mathcal{O}^*D(y)} = \chi\left(\frac{C(x)D(x)}{C(y)D(y)}\right).$$

Hence (OR2′) is also true, since $\mathcal{O}^*$ is orthogonal to $\mathcal{O}$. □

In [3] those elements of the inner Tutte group which can be expressed as products of the form $C_1(x)C_1(y)^{-1}C_2(y)C_2(x)^{-1}$ for $C_1, C_2 \in \mathfrak{C}$, $x, y \in C_1 \cap C_2$, and $\dim(C_1 \cup$

$C_2) = 1$, were called "generalized cross ratios", and some relations for them were established. We shall return to this point at the end of this section, but first we shall describe a presentation of the inner Tutte group with a broader set of generators.

Let $\mathbb{T}^{(1)}_{\mathbf{M}}$ be the abelian group with generators ε and $\begin{pmatrix} C & D \\ x & y \end{pmatrix}$ for $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$, $x, y \in C \cap D$, and relations: (TG0);

(ITG1) $$\begin{pmatrix} C & D \\ x & x \end{pmatrix} = 1;$$

(ITG2) $$\begin{pmatrix} C & D \\ x & y \end{pmatrix}\begin{pmatrix} C & D \\ y & z \end{pmatrix}\begin{pmatrix} C & D \\ z & x \end{pmatrix} = 1;$$

(ITG3) $$\begin{pmatrix} C_1 & D_1 \\ x & y \end{pmatrix}\begin{pmatrix} C_2 & D_2 \\ x & y \end{pmatrix} = \begin{pmatrix} C_1 & D_2 \\ x & y \end{pmatrix}\begin{pmatrix} C_2 & D_1 \\ x & y \end{pmatrix};$$

(ITG4) $$\begin{pmatrix} C & D \\ x & y \end{pmatrix} = \varepsilon, \qquad \text{if } C \cap D = \{x, y\};$$

(ITG5)
$$\begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} C_2 & D_2 \\ x_1 & x_4 \end{pmatrix}, \qquad \text{if } \dim(C_1 \cup C_2) = 1,\ C_1 \cap D_1 = \{x_2, x_3, x_4\},$$

$C_2 \cap D_2 = \{x_1, x_3, x_4\}$ and $C_1 \cap D_2 = C_2 \cap D_1 = \{x_3, x_4\}$.

Here we use $\begin{pmatrix} C & D \\ x & y \end{pmatrix}$ in a different sense than in §2—not as functions, having values in $\{1, -1\}$, but as abstract generators of a group. Nonetheless, note that (ITG1)–(ITG3) are the same as (PO1)–(PO3), and (ITG4) is (PO4′) with ε instead of -1.

Lemma 3.1. *Assume (TG0) and (ITG1)–(ITG4). Then (ITG5) is equivalent to the following condition, which corresponds to Lemma 2.3.*

(ITG5′) $$\begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} C_2 & D_2 \\ x_1 & x_4 \end{pmatrix}, \qquad \text{if } E' = \{x_1, x_2, x_3, x_4\} = E \setminus (A \cup B),$$

$(\mathbf{M} \setminus A)/B \cong \mathbf{U}_2(4)$, $C_i \cap A = D_i \cap B = \emptyset$ *and* $E' \setminus C_i = E' \setminus D_i = \{i\}$ *for* $i = 1, 2, 3, 4$.

Proof. Let C_1, C_2, D_1, D_2 be as in (ITG5) and $E' = \{x_1, x_2, x_3, x_4\}$. Then, applying the elimination axiom, we obtain a circuit $C_3 \subseteq C_1 \cup C_2 \setminus \{x_3\}$ which contains x_1, hence also x_4 (by orthogonality to D_2), and hence also x_2 (by orthogonality to D_1). In the same way we obtain $C_4 \in \mathfrak{C}$ and $D_3, D_4 \in \mathfrak{C}^*$ with the property $C_i \cap D_j = E' \setminus \{x_i, x_j\}$ for $i, j \in \{1, 2, 3, 4\}$. Thus $(\mathbf{M} \setminus A)/B \cong \mathbf{U}_2(4)$, where $A = E \setminus (C_1 \cup C_2)$ and $B = (C_1 \cup C_2) \setminus E'$. Hence (ITG5) follows from (ITG5′).

Vice versa, under the conditions of (ITG5′) there exist circuits $C_1', C_2' \subseteq E \setminus A$ with $C_i' \cap E' = C_i \cap E' \quad (i = 1,2)$ and $\dim(C_1' \cup C_2') = 1$ (this follows from [5, Theorem 4.41]). Now

$$\begin{pmatrix} C_1' & D_1 \\ x_2 & x_3 \end{pmatrix} \begin{pmatrix} C_1 & D_4 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} C_1' & D_4 \\ x_2 & x_3 \end{pmatrix} \begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix}$$

by (ITG3), thus

$$\begin{pmatrix} C_1' & D_1 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} C_1 & D_1 \\ x_2 & x_3 \end{pmatrix}$$

by (ITG4). Analogously,

$$\begin{pmatrix} C_2' & D_2 \\ x_1 & x_4 \end{pmatrix} = \begin{pmatrix} C_2 & D_2 \\ x_1 & x_4 \end{pmatrix}.$$

Hence (ITG5′) follows from (ITG3)–(ITG5). □

Theorem 3. *Let* $\mathbf{M}$ *be a fixed matroid. Then there is a homomorphism* $\tau : \mathbb{T}_{\mathbf{M}}^{(1)} \to \mathbb{T}_{\mathbf{M}}^{\mathfrak{C},\mathfrak{C}^*}$, *sending* ε *to* ε *and* $\begin{pmatrix} C & D \\ x & y \end{pmatrix}$ *to* $C(x)D(x)C(y)^{-1}D(y)^{-1}$, *which induces an isomorphism* $\mathbb{T}_{\mathbf{M}}^{(1)} \cong \mathbb{T}_{\mathbf{M}}^{(0)}$.

Proof. To prove the existence of τ we must only show that the identities (ITG1)–(ITG5) hold for $\tau \begin{pmatrix} C & D \\ x & y \end{pmatrix} = C(x)D(x)C(y)^{-1}D(y)^{-1}$. This is evident for (ITG1)–(ITG4). We already know that under the conditions of (ITG5), $\mathbf{M} \cdot (C_1 \cup C_2)/((C_1 \cup C_2) \setminus E') \cong \mathbf{U}_2(4)$. By (TG1′)

$$\text{(3-1)} \qquad \frac{C(x)}{C(y)} = \varepsilon \frac{D(y)}{D(x)} \qquad \text{for } C \in \mathfrak{C},\ D \in \mathfrak{C}^* \text{ with } C \cap D = \{x, y\},$$

and by (TG1)

$$\text{(3-2)} \qquad \frac{C_i(x_j)C_j(x_k)C_k(x_i)}{C_i(x_k)C_j(x_i)C_k(x_j)} = \varepsilon \qquad \text{for } \{i,j,k,l\} = E'$$

Using (3-1) and (3-2) twice, we get

$$\begin{aligned} \frac{C_1(x_2)D_1(x_2)}{C_1(x_3)D_1(x_3)} &= \varepsilon \frac{C_1(x_2)C_4(x_3)}{C_1(x_3)C_4(x_2)} = \frac{C_2(x_1)C_3(x_2)C_4(x_3)}{C_3(x_1)C_2(x_3)C_4(x_2)} \\ &= \varepsilon \frac{C_2(x_1)C_3(x_4)}{C_3(x_1)C_2(x_4)} = \frac{C_2(x_1)D_2(x_1)}{C_2(x_4)D_2(x_4)}, \end{aligned}$$

which is (ITG5) for $\tau \begin{pmatrix} C & D \\ x & y \end{pmatrix}$.

It is obvious that $\tau \mathbb{T}_{\mathbf{M}}^{(1)} \subseteq \mathbb{T}_{\mathbf{M}}^{(0)}$. So we must only prove the following:

(1) $\mathbb{T}_{\mathbf{M}}^{(0)} \subseteq \tau \mathbb{T}_{\mathbf{M}}^{(1)}$;
(2) $\ker \tau = \{1\}$.

The proof of the first statement can be obtained by translating the uniqueness part of the proof of Theorem 1 into the language of Tutte groups; but it also follows, as already mentioned, from [9, Proposition 2.9(i)], because $C_1(x)C_1(y)^{-1}C_2(y)C_2(x)^{-1} = \varepsilon\tau\begin{pmatrix} C_1 & D_2 \\ x & y \end{pmatrix}$ for $C_1, C_2 \in \mathfrak{C}$, $D_2 \in \mathfrak{C}^*$, $x, y \in C_1 \cap C_2$ and $C_2 \cap D_2 = \{x, y\}$.

To prove the second statement it is enough to construct a homomorphism $\pi : \mathbb{T}_{\mathbf{M}}^{\mathfrak{C},\mathfrak{C}^*} \to \mathbb{T}_{\mathbf{M}}^{(1)}$ with $\pi \circ \tau = \mathrm{id}$. This can be done by rewriting the rest of the proof of Theorem 1, replacing -1 by ε, $\mathcal{O}C(x)$ by $\pi C(x)$, $\mathcal{O}^*D(x)$ by $\pi D(x)$ and the statements like "any given projective orientation of $\mathbf{M}$ determines canonically the projective orientation of its minor $\mathbf{M}' = (\mathbf{M} \setminus A)/B$" by "there is a canonical homomorphism $\mathbb{T}_{\mathbf{M}'}^{(1)} \to \mathbb{T}_{\mathbf{M}}^{(1)}$ with $\varepsilon \mapsto \varepsilon$ for $\mathbf{M}' = (\mathbf{M} \setminus A)/B$". The details are left to the reader. □

Now Theorem 1 may be considered as a combinatorial corollary of Theorem 3 and Corollary 1.

Remark. The inner Tutte group was used in [10] to describe the projective equivalence classes of matroids with coefficients. We shall not go into details, but those who are acquainted with these notions can see that such classes correspond to homomorphisms $\varkappa : \mathbb{T}_{\mathbf{M}}^{(1)} \to K^*$, satisfying the properties:

(1) $\varkappa(\varepsilon) = \varepsilon_K$;

(2) $\sum\limits_{x \in C \cap D} \varkappa\begin{pmatrix} C & D \\ x & y \end{pmatrix} \in K_0$ for any $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$ and $y \in C \cap D$;

where $\mathbf{M}$ is an ordinary matroid and $(K; +; \cdot; \varepsilon_K; K_0)$ is a fuzzy ring.

The consideration of $\mathbf{U}_2(4)$ in §2 shows that we can use even a larger group than $\mathbb{T}_{\mathbf{M}}^{(1)}$, defined only by relations (ITG1)–(ITG3), if the fuzzy ring satisfies the following property:

if $\gamma, f, g \in K^*, \gamma^2 = 1$ and $(1+f+g), (1+\gamma f+g), (1+f+\gamma g), (1+\gamma f+\gamma g) \in K_0$, then $\gamma = 1$.

This property holds for all fuzzy integral domains (because it follows from (FR6‴) in [5]), but it is not true in general even for perfect fuzzy rings (for example for $\mathbb{F}_7//(\mathbb{F}_7^*)^2$).

In the proof of the next theorem we shall need Tutte's homotopy theory [7, Ch. 6]. Here is a brief summary of it.

Let $\mathcal{C}$ be a linear subclass in a matroid $\mathbf{M} = (E, \mathfrak{C})$. We call a path $(C_1, \ldots, C_k)$ off $\mathcal{C}$ *re-entrant* if $C_1 = C_k$. In [7, § 6.1] the following four kinds of elementary re-entrant paths were described:

(1) Any path of the form (C_1, C_2, C_1) is called an elementary re-entrant path of the first kind.

(2) Any path of the form (C_1, C_2, C_3, C_1), where $\dim(C_1 \cup C_2 \cup C_3) \leq 2$, is called an elementary re-entrant path of the second kind.

(3) Let P be a plane of $\mathbf{M}$ and $P \supset C_\alpha, C_\beta \in \mathcal{C}$ such that any connected line in P contains either C_α or C_β. Then any path on P of the form

$(C_1, C_2, C_3, C_4, C_1)$ with $C_\alpha = \langle (C_1 \cup C_2) \cap (C_3 \cup C_4) \rangle$ and $C_\beta = \langle (C_1 \cup C_4) \cap (C_2 \cup C_3) \rangle$ is called an elementary re-entrant path of the third kind.

(4) Let J be a 3-flat of $\mathbf{M}$ with 3 points X_1, X_2 and X_3 such that $X_i \cup X_j$ is a disconnected line when $i \neq j$. Let there be just six connected planes P_i in J, two containing each of these lines: $X_i \cup X_j \subset P_k, P_{k+3}$ for $\{i,j,k\} = \{1,2,3\}$. We have in J also lines $L_{ij,\alpha\beta} = \langle P_{i+3\alpha} \cap P_{j+3\beta} \rangle$ for $1 \leq i < j \leq 3$, $\alpha, \beta \in \{0,1\}$, and points $Y_{\alpha\beta\gamma} = \langle P_{1+3\alpha} \cap P_{2+3\beta} \cap P_{3+3\gamma} \rangle$ for $\alpha, \beta, \gamma \in \{0,1\}$. There are no other lines or points on J. Let $Y_{\alpha\beta\gamma} \in \mathcal{C}$ iff $\alpha+\beta+\gamma \equiv 1 \pmod 2$. Then the path $(C_1, C_2, C_3, C_4, C_1)$, where $C_1 = X_1$, $C_2 = Y_{000}$, $C_3 = X_2$ and $C_4 = Y_{011}$, is called an elementary re-entrant path of the fourth kind.

If $P = (C_1, \ldots, C_k)$ and $Q = (C_k, \ldots, C_m)$ are paths off $\mathcal{C}$, then we can define their product $PQ = (C_1, \ldots, C_m)$ which is also a path off $\mathcal{C}$. Now if we have two paths PR and PQR off $\mathcal{C}$, where Q is an elementary re-entrant path, then we call either of them an elementary deformation of the other. Two given paths P and Q are called homotopic if one can be derived from the other by a finite sequence of elementary deformations.

Tutte's Homotopy Theorem. [7, Theorem 6.41] *Any re-entrant path off $\mathcal{C}$ is homotopic to a path consisting of one point.*

We define now $\mathbb{T}^{(2)}_{\mathbf{M}}$ as the abelian group with generators ε and $\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$, where $C_i \in \mathfrak{C}$ for $i = 1,2,3,4$, $L = C_1 \cup C_2 \cup C_3 \cup C_4 = C_k \cup C_l$ for $k = 1,2$, $l = 3,4$, $\dim(L) = 1$, and relations: (TG0);

(CR1) $$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_3 \end{bmatrix} = 1;$$

(CR2) $$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \\ C_4 & C_5 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \\ C_5 & C_3 \end{bmatrix} = 1;$$

(CR3) $$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = \begin{bmatrix} C_3 & C_4 \\ C_1 & C_2 \end{bmatrix};$$

(CR4) $$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} C_1 & C_4 \\ C_2 & C_3 \end{bmatrix} \begin{bmatrix} C_1 & C_3 \\ C_4 & C_2 \end{bmatrix} = \varepsilon;$$

(CR5) $\varepsilon = 1$ for $\mathbf{M}$ having the Fano-matroid or its dual as a minor;

(CR6) $$\begin{bmatrix} C_1 & C_2 \\ C_6 & C_9 \end{bmatrix} \begin{bmatrix} C_2 & C_3 \\ C_4 & C_7 \end{bmatrix} \begin{bmatrix} C_3 & C_1 \\ C_5 & C_8 \end{bmatrix} = 1$$ for $C_1, \ldots, C_9 \in \mathfrak{C}$ such that:

(1) $\dim(L_i) = 1$ for $L_i = C_j \cup C_k$, where $\{i,j,k\} = \{1,2,3\}$,
(2) $\dim(P) = 2$, where $P = C_1 \cup C_2 \cup C_3$,
(3) $C_{i+3}, C_{i+6} \subset L_i$ for $i = 1,2,3$,
(4) $\dim(L) = \dim(L') = 1$, where $L = C_4 \cup C_5 \cup C_6$ and $L' = C_7 \cup C_8 \cup C_9$,
(5) $\{C_1, C_2, C_3\} \cap \{C_4, \ldots, C_9\} = \emptyset$.

Theorem 4. *There is a homomorphism* $\lambda : \mathbb{T}^{(2)}_{\mathbf{M}} \to \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}}$ *with* $\lambda(\varepsilon) = \varepsilon$, $\lambda \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = C_1(x_3)C_1(x_4)^{-1}C_2(x_4)C_2(x_3)^{-1}$, *where* $x_3 \in L \setminus C_3$, $x_4 \in L \setminus C_4$ *with* $L = C_1 \cup C_2 \cup C_3 \cup C_4$; *and* λ *induces an isomorphism* $\mathbb{T}^{(2)}_{\mathbf{M}} \cong \mathbb{T}^{(0)}_{\mathbf{M}}$.

Proof. The proof includes Lemmas 3.2–3.5. The correctness of the definition of $\lambda \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ and the relations (CR1)–(CR5) for the images of the generators of $\mathbb{T}^{(2)}_{\mathbf{M}}$ were established in [3, Lemma 2.4, Proposition 2.5]. To prove that under the conditions of (CR6)

$$\lambda \begin{bmatrix} C_1 & C_2 \\ C_6 & C_9 \end{bmatrix} \cdot \lambda \begin{bmatrix} C_2 & C_3 \\ C_4 & C_7 \end{bmatrix} \cdot \lambda \begin{bmatrix} C_3 & C_1 \\ C_5 & C_8 \end{bmatrix} = 1 \tag{3-3}$$

we choose $a \in P \setminus L$ and $b \in P \setminus L'$. If $L = L_3$, then $C_1, C_2 \subset L$, but $C_1 \cup C_5 = L_2$ (as $C_5 \neq C_1$), thus $L_2 = L$ and $C_3 \subset L$, hence $L = P$, which is impossible. Therefore $\dim\langle L \cap L_3\rangle < 1$, thus $\langle L \cap L_3\rangle = C_6$, and by $C_1 \neq C_6$ we get $C_1 \not\subset L$, hence $\dim(C_1 \cup L) > 1$, i.e. $C_1 \cup L = P$ and $a \in C_1$. In the same way we prove that $a \in C_2, C_3$ and $b \in C_1, C_2, C_3$. Hence the left-hand side of (3-3) can be rewritten as

$$\frac{C_1(a)C_2(b)}{C_1(b)C_2(a)} \cdot \frac{C_2(a)C_3(b)}{C_2(b)C_3(a)} \cdot \frac{C_3(a)C_1(b)}{C_3(b)C_1(a)} = 1.$$

So the homomorphism λ is well defined. $\mathbb{T}^{(0)}_{\mathbf{M}} = \lambda\mathbb{T}^{(2)}_{\mathbf{M}}$ by [9, Proposition 2.9(i)], [3, Theorem 2.7(i)]. We must only prove that $\ker\lambda = \{1\}$. As in the proof of Theorem 3 we do it by constructing a homomorphism $\rho : \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \to \mathbb{T}^{(2)}_{\mathbf{M}}$ with $\rho \circ \lambda = \mathrm{id}$.

Let us choose $a \in E$ and for $\mathbf{M}' = \mathbf{M}/\{a\}$ denote by λ' the corresponding homomorphism $\mathbb{T}^{(2)}_{\mathbf{M}'} \to \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}'}$. We recall that for $C' \in \mathfrak{C}_{\mathbf{M}'}$ there is a unique circuit $C = \theta C' \in \mathfrak{C}$ with $C \setminus \{a\} = C'$, and

$$C = \begin{cases} C', & \text{for } C' \in \mathfrak{C} \\ C' \cup \{a\}, & \text{for } C' \notin \mathfrak{C}. \end{cases}$$

According to [7, Theorems 4.42–4.44] θ can be uniquely extended to a map of the set of flats of $\mathbf{M}'$ to the set of flats of $\mathbf{M}$, preserving dimensions and inclusion relations. We shall denote this map also by θ, and for any flat F' of $\mathbf{M}'$ we shall write F instead of $\theta F'$.

Lemma 3.2. *There exists a canonical homomorphism* $\xi : \mathbb{T}^{(2)}_{\mathbf{M}'} \to \mathbb{T}^{(2)}_{\mathbf{M}}$, *given by* $\xi(\varepsilon) = \varepsilon$ *and* $\xi \begin{bmatrix} C'_1 & C'_2 \\ C'_3 & C'_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$.

Proof. This follows immediately from the fact that θ preserves dimension and inclusion relations of flats. □

According to [2, Proposition 4.1] there is also a canonical homomorphism $\zeta : \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}'} \to \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}}$, given by $\zeta(\varepsilon) = \varepsilon$ and $\zeta(C'(x)) = C(x)$. It is easy to see that the

diagram

$$\begin{array}{ccc} \mathbb{T}^{(2)}_{\mathbf{M}'} & \xrightarrow{\xi} & \mathbb{T}^{(2)}_{\mathbf{M}} \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}'} & \xrightarrow{\zeta} & \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \end{array} \tag{3-4}$$

is commutative.

Let us assume that there is a homomorphism $\rho' : \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}'} \to \mathbb{T}^{(2)}_{\mathbf{M}'}$ with $\rho' \circ \lambda' = \mathrm{id}$. We want to construct $\rho : \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \to \mathbb{T}^{(2)}_{\mathbf{M}}$ in such a way that

$$\begin{array}{ccc} \mathbb{T}^{(2)}_{\mathbf{M}'} & \xrightarrow{\xi} & \mathbb{T}^{(2)}_{\mathbf{M}} \\ {\scriptstyle\rho'}\uparrow & & \uparrow{\scriptstyle\rho} \\ \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}'} & \xrightarrow{\zeta} & \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \end{array} \tag{3-5}$$

is also a commutative diagram.

It is easy to see that if a is a loop in $\mathbf{M}$, then ξ is an isomorphism, $\lambda = \zeta \circ \lambda' \circ \xi^{-1}$, ζ is injective and $\mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \cong \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}'} \times \mathbb{Z}$. Let $\psi : \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \to \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}'}$ be the projection on the first factor. Then $\rho = \xi \circ \rho' \circ \psi$ makes (3-5) commutative and $\rho \circ \lambda = \mathrm{id}$.

In the sequel we assume that a is not a loop in $\mathbf{M}$.

We set $\rho(\varepsilon) = \varepsilon$ and $\rho C(x) = \xi \rho' C'(x)$ for $x \in C' \in \mathfrak{C}_{\mathbf{M}'}$ and $c = \theta C'$. Now let $C'_1, C'_2 \in \mathfrak{C}_{\mathbf{M}'}$ with $a \in C'_1 \cap C'_2$, and let $L' = C'_1 \cup C'_2$ be a connected line in $\mathbf{M}$. We set

$$t(C_1, C_2) = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \cdot \xi \rho' \left(\frac{C'_2(x)}{C'_1(x)} \right), \tag{3-6}$$

where $x \in C'_1 \cap C'_2$, $C_3 = \langle L \setminus \{x\} \rangle$, $C_4 = \langle L \setminus \{a\} \rangle$. To prove the correctness of this definition we observe that

$$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \cdot \xi \rho' \left(\frac{C'_2(x)}{C'_1(x)} \right) \cdot \begin{bmatrix} C_1 & C_2 \\ C_4 & C_5 \end{bmatrix} \cdot \xi \rho' \left(\frac{C'_1(y)}{C'_2(y)} \right)$$
$$= \begin{bmatrix} C_1 & C_2 \\ C_3 & C_5 \end{bmatrix} \cdot \xi \rho' \left(\frac{C'_2(x) C'_1(y)}{C'_1(x) C'_2(y)} \right) = \xi \begin{bmatrix} C'_1 & C'_2 \\ C'_3 & C'_5 \end{bmatrix} \cdot \xi \begin{bmatrix} C'_1 & C'_2 \\ C'_5 & C'_3 \end{bmatrix} = 1,$$

where $y \in C'_1 \cap C'_2$, $C'_5 = \langle L' \setminus \{y\} \rangle_{\mathbf{M}'}$, $C'_3 = \langle L' \setminus \{x\} \rangle_{\mathbf{M}'}$ and $C_5 = \theta C'_5 = \langle L \setminus \{y\} \rangle$.

Lemma 3.3. *Let $\mathcal{C}$ be the linear subclass in $\mathbf{M}'$, consisting of all $C' \in \mathfrak{C}_{\mathbf{M}'}$ with $a \notin \theta C'$. Then $t(C_1, C_2) \cdot \ldots \cdot t(C_{k-1}, C_k) \cdot t(C_k, C_1) = 1$ for any re-entrant path $(C'_1, \ldots, C'_k, C'_1$ off $\mathcal{C}$ with $C_i = \theta C'_i$ $(i = 1, \ldots, k)$.*

Proof. By Tutte's homotopy theorem it is enough to consider only elementary re-entrant paths.

Let (C'_1, C'_2, C'_1) be an elementary re-entrant path of the first kind. Then, by the definition of $t(C_1, C_2)$, $t(C_1, C_2) \cdot t(C_2, C_1) = 1$

Let (C'_1, C'_2, C'_3, C'_1) be an elementary re-entrant path of the second kind and let $F' = C'_1 \cup C'_2 \cup C'_3$. If $\dim_{\mathbf{M}'}(F') = 1$, then there are $x_i \in F' \setminus C'_i$ $(i = 1,2,3)$ and $x_i \in C'_j \cap C'_k$ for $\{i,j,k\} = \{1,2,3\}$. Let $C_4 = \langle F \setminus \{a\}\rangle$. We have

$$t(C_1,C_2)\cdot t(C_2,C_3)\cdot t(C_3,C_1) = \begin{bmatrix} C_1 & C_2 \\ C_4 & C_3 \end{bmatrix} \cdot \xi\rho'\left(\frac{C'_2(x_3)}{C'_1(x_3)}\right)$$
$$\cdot \begin{bmatrix} C_2 & C_3 \\ C_4 & C_1 \end{bmatrix} \cdot \xi\rho'\left(\frac{C'_3(x_1)}{C'_2(x_1)}\right) \cdot \begin{bmatrix} C_3 & C_1 \\ C_4 & C_2 \end{bmatrix} \cdot \xi\rho'\left(\frac{C'_1(x_2)}{C'_3(x_2)}\right) = \varepsilon \cdot \xi\rho'(\varepsilon) = 1.$$

If $\dim_{\mathbf{M}'}(F') = 2$, then F' is a connected plane. Let $L'_i = C'_j \cup C'_k$ for $\{i,j,k\} = \{1,2,3\}$. We know from [7, Theorem 4.171] that any two distinct lines on F' have just one common point, and from [7. Theorem 4.26] that any point on F' lies in the intersection of two distinct connected lines on F'.

Let us first consider the case when there is a line L' on F' with $C'_i \not\subset L'$ for $i = 1,2,3$ (we call it case (A)). Then for any $x \in F' \setminus L'$ we have $x \in C'_1 \cap C'_2 \cap C'_3$ (because $L' \cup C'_i = F'$ for $i = 1,2,3$), and $C'_{i+3} = \langle L'_i \setminus \{x\}\rangle_{\mathbf{M}'} = \langle L'_i \cap L'\rangle_{\mathbf{M}'}$ are distinct from C'_1, C'_2, C'_3. Let $C_{i+6} = \langle L_i \setminus \{a\}\rangle$ for $i = 1,2,3$. We have

$$t(C_1,C_2)\cdot t(C_2,C_3)\cdot t(C_3,C_1) = \begin{bmatrix} C_1 & C_2 \\ C_9 & C_6 \end{bmatrix} \cdot \xi\rho'\left(\frac{C'_2(x)}{C'_1(x)}\right)$$
$$\cdot \begin{bmatrix} C_2 & C_3 \\ C_7 & C_4 \end{bmatrix} \cdot \xi\rho'\left(\frac{C'_3(x)}{C'_2(x)}\right) \cdot \begin{bmatrix} C_3 & C_1 \\ C_8 & C_5 \end{bmatrix} \cdot \xi\rho'\left(\frac{C'_1(x)}{C'_3(x)}\right) = 1 \cdot \xi\rho'(1) = 1$$

by (CR6), because $L_0 = C_7 \cup C_8 \cup C_9 = \langle F \setminus \{a\}\rangle$ is a line on F.

Let us assume now that there is no such line L' on F'. As L'_i is connected, there is a point $C'_{i+3} \subset L'_i$ with $C'_{i+3} \neq C'_j, C'_k$ for $\{i,j,k\} = \{1,2,3\}$, and the only other line containing C'_{i+3} is $L'_{i+3} = C'_i \cup C'_{i+3}$, hence L'_{i+3} is connected. If $C'_4 \notin \mathcal{C}$, then

$$t(C_1,C_2)\cdot t(C_2,C_3)\cdot t(C_3,C_1) = t(C_1,C_2)\cdot t(C_2,C_4)\cdot t(C_4,C_3)\cdot t(C_3,C_1)$$

(because C'_2, C'_3 and C'_4 are on the same connected line L'_1)

$$= \big(t(C_1,C_2)\cdot t(C_2,C_4)\cdot t(C_4,C_1)\big)\cdot\big(t(C_1,C_4)\cdot t(C_4,C_3)\cdot t(C_3,C_1)\big)$$

(because C'_1 and C'_4 are on the same connected line L'_4).

Both factors in the last expression are of the type considered in case (A), with the role of L' played by L'_6 for the first one and by L'_5 for the second one. Hence we have again $t(C_1,C_2)\cdot t(C_2,C_3)\cdot t(C_3,C_1) = 1$.

The same arguments work if any of C'_{i+3} $(i = 1,2,3)$ can be chosen outside $\mathcal{C}$. Otherwise let $C'_{i+6} = \big\langle L'_{j+3} \cap L'_{k+3}\big\rangle$ for $\{i,j,k\} = \{1,2,3\}$. As $C'_i \notin \mathcal{C}$ and $C'_{i+3} \in \mathcal{C}$ for $i = 1,2,3$, we have $C'_{i+6} \notin \mathcal{C}$ for $i = 1,2,3$. Hence if $C'_7 \neq C'_8$ we may use the arguments of the same kind as above to get

$$t(C_1,C_2)\cdot t(C_2,C_3)\cdot t(C_3,C_1) = \big(t(C_1,C_2)\cdot t(C_2,C_9)\cdot t(C_9,C_1)\big)$$
$$\cdot\big(t(C_1,C_8)\cdot t(C_8,C_3)\cdot t(C_3,C_1)\big)\cdot\big(t(C_8,C_9)\cdot t(C_9,C_7)\cdot t(C_7,C_8)\big)$$
$$\cdot\big(t(C_7,C_2)\cdot t(C_2,C_3)\cdot t(C_3,C_7)\big) = 1,$$

because each of the four factors equals 1 by the case (A) with the role of L' played by L'_6, L'_5, L'_1 and L'_4 respectively.

Finally if $C'_7 = C'_8 (= C'_9)$, then we choose $x_i \in F' \setminus L'_i$ $(i = 1, \ldots, 6)$ and see that $(\mathbf{M} \cdot F)/(F' \setminus \{x_1, \ldots, x_6\})$ is isomorphic to the dual of the Fano-matroid; thus $\varepsilon = 1$ in $\mathbb{T}^{(2)}_{\mathbf{M}}$. Hence

$$t(C_1, C_2) \cdot t(C_2, C_3) \cdot t(C_3, C_1) = \xi\rho' \left(\frac{C'_1(x_5) C'_2(x_6) C'_3(x_4)}{C'_1(x_6) C'_2(x_4) C'_3(x_5)} \right)$$

(using (TG1) for $x_4 \in L'_3 \setminus C'_1$, $x_5 \in L'_3 \setminus C'_2$ and $x_6 \in L'_3 \setminus C'_3$)

$$= \xi\rho' \left(\varepsilon \frac{C'_6(x_5) C'_3(x_4)}{C'_6(x_4) C'_3(x_5)} \right)$$

(using (TG1) for $x_1 \in L'_6 \setminus C'_3$, $x_3 \in L'_6 \setminus C'_6$ and $x_5 \in L'_6 \setminus C'_7$)

$$= \xi\rho' \left(\frac{C'_6(x_1) C'_7(x_3) C'_3(x_4)}{C'_7(x_1) C'_3(x_3) C'_6(x_4)} \right)$$

(using (TG1) for $x_1 \in L'_6 \setminus C'_3$, $x_3 \in L'_6 \setminus C'_6$ and $x_4 \in L'_6 \setminus C'_7$)

$$= \xi\rho'(\varepsilon) = 1.$$

Let $(C'_1, C'_2, C'_3, C'_4, C'_5 = C'_1)$ be an elementary re-entrant path of the third kind and let $P' = C'_1 \cup C'_2 \cup C'_3 \cup C'_4$, $L'_i = C'_i \cup C'_{i+1}$ and $x_i \in P' \setminus L'_i$ for $i = 1, 2, 3, 4$, $C'_\alpha = \langle L'_1 \cap L'_3 \rangle$ and $C'_\beta = \langle L'_2 \cap L'_4 \rangle$. Then

$$t(C_1, C_2) \cdot t(C_2, C_3) \cdot t(C_3, C_4) \cdot t(C_4, C_1) = \xi\rho' \left(\frac{C'_1(x_2) C'_2(x_3) C'_3(x_4) C'_4(x_1)}{C'_1(x_3) C'_2(x_4) C'_3(x_1) C'_4(x_2)} \right).$$

But $x_2 \in L'_1 \setminus C'_2$ (because $L'_1 \cup L'_2 = P'$, $C'_2 \subset L'_2$ and $x_2 \in P' \setminus L'_2$), $x_3 \in L'_1 \setminus C'_\alpha$ (because $L'_1 \cup L'_3 = P'$, $C'_\alpha \subset L'_3$ and $x_3 \in P' \setminus L'_3$) and $x_4 \in L'_1 \setminus C'_1$ (because $L'_1 \cup L'_4 = P'$, $C'_1 \subset L'_4$ and $x_4 \in P' \setminus L'_4$), hence by (TG1)

$$\frac{C'_1(x_2) C'_2(x_3)}{C'_1(x_3) C'_2(x_4)} = \varepsilon \frac{C'_\alpha(x_2)}{C'_\alpha(x_4)}.$$

Analogously,

$$\frac{C'_3(x_4) C'_4(x_1)}{C'_3(x_1) C'_4(x_2)} = \varepsilon \frac{C'_\alpha(x_4)}{C'_\alpha(x_2)}.$$

Therefore $t(C_1, C_2) \cdot t(C_2, C_3) \cdot t(C_3, C_4) \cdot t(C_4, C_1) = 1$.

Let $(C'_1, C'_2, C'_3, C'_4, C'_5 = C'_1)$ be an elementary re-entrant path of the fourth kind and let $J' = C'_1 \cup C'_2 \cup C'_3 \cup C'_4$, We adopt for the flats on J' the notations from the definition of such path (with the prime mark added). Let $x_i \in J' \setminus P'_i$ for

$i = 1, \ldots, 6$. Then $(\mathbf{M} \cdot J)/(J' \setminus \{x_1, \ldots, x_6\})$ is isomorphic to the Fano-matroid (cf. [9, Proposition 4.6]). Hence $\varepsilon = 1$ in $\mathbb{T}^{(2)}_{\mathbf{M}}$ and

$$t(C_1, C_2) \cdot t(C_2, C_3) \cdot t(C_3, C_4) \cdot t(C_4, C_1) = \xi\rho' \left(\frac{Y'_{000}(x_4) X'_2(x_5) Y'_{011}(x_2) X'_1(x_4)}{X'_1(x_4) Y'_{000}(x_5) X'_2(x_2) Y'_{011}(x_4)} \right)$$

(using (TG1) for $x_4 \in L'_{13,00} \setminus X'_2$, $x_5 \in L'_{13,00} \setminus Y'_{010}$ and $x_2 \in L'_{13,00} \setminus Y'_{000}$)

$$= \xi\rho' \left(\varepsilon \frac{Y'_{010}(x_4) Y'_{011}(x_2)}{Y'_{010}(x_2) Y'_{011}(x_4)} \right)$$

(using (TG1) twice for $x_2, x_4 \in L'_{12,01} \setminus X'_3$, $x_3 \in L'_{12,01} \setminus Y'_{010}$ and $x_6 \in L'_{12,01} \setminus Y'_{011}$)

$$= \xi\rho'(\varepsilon) = 1. \qquad \square$$

For every connected component $\mathbf{M}'_i$ of $\mathbf{M}'$ with not all its points in $\mathcal{C}$ we now choose some point C'_i in it, not belonging to $\mathcal{C}$, and set $\rho C_i(a) = 1$. Then, using Lemma 3.3, we define uniquely $\rho C(a)$ for any $C \in \mathfrak{C}$ with $a \in C$ in such a way that for any two distinct points $C'_1, C'_2 \in \mathfrak{C}_{\mathbf{M}'} \setminus \mathcal{C}$, lying on the same connected line, we have

$$\frac{\rho C_2(a)}{\rho C_1(a)} = t(C_1, C_2). \tag{3-7}$$

Thus for every $C \in \mathfrak{C}$ with $(C \setminus \{a\}) \in \mathfrak{C}_{M'}$ we have defined $\rho C(x)$ for all $x \in C$.

Now let $C \in \mathfrak{C}$ with $(C \setminus \{a\}) \notin \mathfrak{C}_{M'}$. In this case $a \notin C$ and C is a line in $\mathbf{M}'$ with $L = \theta C = C \cup \{a\}$. We choose $x_0 \in C$ and set $\rho C(x_0) = 1$. Let $C_0 = \langle L \setminus \{x_0\} \rangle$. We set

$$\rho C(x_1) = \varepsilon \frac{\rho C_0(x_1) \rho C_1(a)}{\rho C_1(x_0) \rho C_0(a)}\ , \tag{3-8}$$

for any $x_1 \in C \cap C_0$, where $C_1 = \langle L \setminus \{x_1\} \rangle$, and

$$\rho C(x_1) = \frac{\rho C_2(x_1)}{\rho C_2(x_0)}\ , \tag{3-9}$$

for any $x_1 \in C \setminus C_0$, where $C_2 = \langle L \setminus \{x_2\} \rangle$ for some $x_2 \in C \cap C_0$ ($\neq \emptyset$, because a is not a loop).

Lemma 3.4. *The map ρ from the set of generators of $\mathbb{T}^{\mathfrak{C}}_{\mathbf{M}}$ to $\mathbb{T}^{(2)}_{\mathbf{M}}$, defined above, gives rise to a homomorphism $\rho : \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \to \mathbb{T}^{(2)}_{\mathbf{M}}$, making the diagram (3-5) commutative.*

Proof. We must show that under the conditions of (TG1)

$$\frac{\rho C_1(x_2) \cdot \rho C_2(x_3) \cdot \rho C_3(x_1)}{\rho C_1(x_3) \cdot \rho C_2(x_1) \cdot \rho C_3(x_2)} = \varepsilon. \tag{3-10}$$

Let us first consider the case when $L' = L \setminus \{a\}$ is a line in $\mathbf{M}'$. Let $C_i = \theta C_i'$ for $C_i' \in \mathfrak{C}_{\mathbf{M}'}$ $(i = 1,2,3)$. If $x_1, x_2, x_3 \neq a$, then (3-10) follows from the corresponding identity for $\rho' C_i'(x_j)$ in $\mathbb{T}^{(2)}_{\mathbf{M}'}$. If one of x_i, say x_1, equals a, then $a \in C_2 \cap C_3$ and L' is a connected line in $\mathbf{M}'$ (because it has three pairwise distinct points). By (3-6), (3-7) we have

$$\frac{\rho C_3(a)}{\rho C_2(a)} = t(C_2, C_3) = \begin{bmatrix} C_2 & C_3 \\ C_1 & C_1 \end{bmatrix} \cdot \xi\rho' \left(\frac{C_3'(x)}{C_2'(x)} \right) = \frac{\rho C_3(x)}{\rho C_2(x)} \ ,$$

where $x \in L' \setminus C_1$ ($\neq \emptyset$, because $\dim_{\mathbf{M}'}(C_1) = 0$). Hence (3-10) follows from the same equality with $x_1 = x$ instead of $x_1 = a$.

If one of the points C_1, C_2 and C_3, say C_1, is not in the image of θ, then $C_1 = L'$ and $x_1 = a$. If $x_0 \in C_2 \cap C_3$ (where $x_0 \in C_1$ was used to define $\rho C_1(x_i)$ by (3-8)–(3-9)), then we have

$$\rho C_1(x_i) = \varepsilon \frac{\rho C_0(x_i) \rho C_i(a)}{\rho C_i(x_0) \rho C_0(a)} \tag{3-11}$$

for $i = 2,3$, thus

$$\frac{\rho C_1(x_2) \cdot \rho C_2(x_3) \cdot \rho C_3(x_1)}{\rho C_1(x_3) \cdot \rho C_2(x_1) \cdot \rho C_3(x_2)} = \frac{\rho C_0(x_2) \cdot \rho C_2(x_3) \cdot \rho C_3(x_0)}{\rho C_0(x_3) \cdot \rho C_2(x_0) \cdot \rho C_3(x_2)} = \varepsilon,$$

because $a \in C_i$ for $i = 0,2,3$, and this case has already been considered. If $x_0 \in C_2 \setminus C_3$, then we still have (3-11) for $i = 2$ and

$$\rho C_1(x_3) = \frac{\rho C_2(x_3)}{\rho C_2(x_0)} \ ,$$

thus

$$\frac{\rho C_1(x_2) \cdot \rho C_2(x_3) \cdot \rho C_3(x_1)}{\rho C_1(x_3) \cdot \rho C_2(x_1) \cdot \rho C_3(x_2)} = \varepsilon \frac{\rho C_0(x_2) \cdot \rho C_3(a)}{\rho C_0(a) \cdot \rho C_3(x_2)} = \varepsilon,$$

because $C_3 = C_0$. The same arguments work for $x_0 \in C_3 \setminus C_2$.

In the remaining case $L \setminus \{a\}$ is a plane in $\mathbf{M}'$, thus $a \notin L$ and $P = \theta L = L \cup \{a\}$. Let $L_i = \langle P \setminus \{x_i\} \rangle$ and $C_{i+3} = \langle L_j \cap L_k \rangle$ $(\{i,j,k\} = \{1,2,3\})$. If C_4, C_5 and C_6 are pairwise distinct, then from (3-10) for the line $L_i \ni a$ (already proved) we have

$$\frac{\rho C_i(x_j)}{\rho C_i(x_k)} = \varepsilon \frac{\rho C_{j+3}(x_j) \cdot \rho C_{k+3}(a)}{\rho C_{j+3}(a) \cdot \rho C_{k+3}(x_k)}$$

and (3-10) follows. If $C_4 = C_5$ $(= C_6)$, then of course $C_4' = C_4 \setminus \{a\} \in \mathfrak{C}_{\mathbf{M}'}$ and, according to [7, Theorem 4.27], there is a line L_4' on the plane $P' = L$ of $\mathbf{M}'$ with $C_4' \not\subset L_4'$, thus $C_4 \not\subset L_4$. As L_4' is a line on P', there exists $x_4 \in P' \setminus L_4'$, hence $L_4 = \langle P \setminus \{x_4\} \rangle \neq L$ and $a \in L_4$. Let $C_{i+6} = \langle L_4 \cap L_i \rangle$ for $i = 1,2,3$. If $C_i \neq C_{i+6}$, then L_i is a connected line with $a \in L_i$. ¿From the case of (3-10) already proved we have

$$\frac{\rho C_i(x_j)}{\rho C_{i+6}(x_j)} = \varepsilon \frac{\rho C_i(x_4) \cdot \rho C_4(a)}{\rho C_{i+6}(a) \cdot \rho C_4(x_4)}$$

and

$$\frac{\rho C_i(x_k)}{\rho C_{i+6}(x_k)} = \varepsilon \frac{\rho C_i(x_4)\cdot \rho C_4(a)}{\rho C_{i+6}(a)\cdot \rho C_4(x_4)} ,$$

hence

$$\frac{\rho C_i(x_j)}{\rho C_i(x_k)} = \frac{\rho C_{i+6}(x_j)}{\rho C_{i+6}(x_k)}$$

for $\{i,j,k\} = \{1,2,3\}$. Therefore (3-10) follows from the same equality for the line L_4, which also belongs to the type already considered. □

Lemma 3.5. *$\rho \circ \lambda = \mathrm{id}$ for the homomorphism $\rho : \mathbb{T}^{\mathfrak{C}}_{\mathbf{M}} \to \mathbb{T}^{(2)}_{\mathbf{M}}$ constructed above.*

Proof. We must prove that

$$\rho\lambda \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}, \tag{3-12}$$

where $C_i \in \mathfrak{C}$ for $i = 1,2,3,4$, $L = C_1 \cup C_2 \cup C_3 \cup C_4 = C_k \cup C_l$ for $k = 1,2$, $l = 3,4$ and $\dim(L) = 1$. It is obvious if $C_1 = C_2$ or $C_3 = C_4$. It is also clear if $C_i = \theta C_i'$ for $i = 1,2,3,4$. In the sequel we assume that C_1, C_2, C_3 and C_4 are pairwise distinct. Let $L = \theta L'$ and $C_i \neq \theta(C_i \setminus \{a\}$ for some $i \in \{1,2,3,4\}$, thus $C_i = L \setminus \{a\}$. ¿From (CR1)–(CR3) it is clear that we need consider only the case $i = 4$. But in this case

$$\begin{aligned} \rho\lambda \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} =& \rho \left(\frac{C_1(x_3)C_2(a)}{C_1(a)C_2(x_3)} \right) \\ &= \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \xi\rho' \left(\frac{C_2'(x_3)}{C_1'(x_3)} \right) \rho \left(\frac{C_1(x_3)}{C_2(x_3)} \right) \\ &= \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}, \end{aligned}$$

where $C_i' = C_i \setminus \{a\}$ and $x_3 \in L \setminus C_3$.

Now let $L \setminus \{a\}$ be a plane in $\mathbf{M}'$; thus $a \notin L$ and $P = \theta L = L \cup \{a\}$. The plane P is connected, because a is not a loop, hence there are connected lines $L_1, \ldots, L_4$ on P with $\langle L_i \cap L \rangle = C_i$. If $\langle L_1 \cap L_2 \rangle \not\subset L_3, L_4$, then we set $C_5 = \langle L_1 \cap L_2 \rangle$, $C_6 = \langle L_1 \cap L_3 \rangle$, $C_7 = \langle L_1 \cap L_4 \rangle$, $C_8 = \langle L_3 \cap L_2 \rangle$, $C_9 = \langle L_4 \cap L_2 \rangle$. ¿From (CR6) we get

$$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_5 \\ C_6 & C_7 \end{bmatrix} \begin{bmatrix} C_5 & C_2 \\ C_8 & C_9 \end{bmatrix}.$$

On both factors of the right-hand side $\rho \circ \lambda$ acts as the identity, hence it acts as the identity on the left-hand side. If $\langle L_3 \cap L_4 \rangle \not\subset L_1, L_2$, then we use (CR3) and the same arguments. The remaining case is $\langle L_1 \cap L_2 \rangle = \langle L_3 \cap L_4 \rangle$. Then from the fact that P is connected it follows that there exists a line L_5 on P with $L_5 \neq L$ and $\langle L_1 \cap L_2 \rangle \not\subset L_5$. If $C_5 = \langle L_5 \cap L \rangle \neq C_1, C_2$, then we use (CR2) and apply to $\begin{bmatrix} C_1 & C_2 \\ C_4 & C_5 \end{bmatrix}$ and $\begin{bmatrix} C_1 & C_2 \\ C_5 & C_3 \end{bmatrix}$ the previous arguments. If $C_5 \in \{C_1, C_2\}$, then we use (CR3) again and do the same for $\begin{bmatrix} C_3 & C_4 \\ C_2 & C_5 \end{bmatrix}$ and $\begin{bmatrix} C_3 & C_4 \\ C_5 & C_1 \end{bmatrix}$. In every case we express

$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ as a product of factors, the invariance of which under $\rho \circ \lambda$ is already established. □

To complete the proof of Theorem 4 we need only use induction on the number of elements of E. □

Corollary 2. *Let* $\mathbf{M}$ *be a fixed matroid without minors of the Fano-type or its dual. Then the set* $\mathcal{R}(\mathbf{M})$ *of reorientation classes of oriented matroids with the underlying ordinary matroid* $\mathbf{M}$ *is in one-to-one correspondence with the set of functions* $\psi(C_1, C_2 | C_3, C_4)$, *defined for* $C_1, C_2, C_3, C_4 \in \mathfrak{C}$ *with* $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$ *and* $\{C_1, C_2\} \cap \{C_3, C_4\} = \emptyset$, *ranging in the set* $\{+1, -1\}$, *and satisfying the following conditions:*

(1) $\psi(C_1, C_2 | C_3, C_3) = 1$,
(2) $\psi(C_1, C_2 | C_3, C_4) = \psi(C_3, C_4 | C_1, C_2)$,
(3) *for pairwise distinct circuits* C_1, C_2, C_3 *and* C_4 *with* $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$, *there is exactly one* -1 *among* $\psi(C_1, C_2 | C_3, C_4)$, $\psi(C_1, C_4 | C_2, C_3)$ *and* $\psi(C_1, C_3 | C_4, C_2)$.
(4) $\psi(C_1, C_2 | C_3, C_4)\psi(C_1, C_2 | C_4, C_5)\psi(C_1, C_2 | C_5, C_3) = 1$,
(5) $\psi(C_1, C_2 | C_6, C_9)\psi(C_2, C_3 | C_4, C_7)\psi(C_3, C_1 | C_5, C_8) = 1$ *for any circuits* $C_1, \ldots, C_9$ *as in (CR6).*

Proof. It is obvious that such maps ψ are in one-to-one correspondence with group homomorphisms of $\mathbb{T}_{\mathbf{M}}^{(2)} \cong \mathbb{T}_{\mathbf{M}}^{(0)}$ to $\{+1, -1\}$, satisfying the conditions of Theorem 2. □

Note that no matroid having a minor of Fano-type or its dual is orientable. Hence we have lost no oriented matroids by excluding this case from the very beginning.

Example 2. We illustrate the Corollary for the case of an oriented matroid $\tilde{\mathbf{M}}$ whose dual $\tilde{\mathbf{M}}^*$ is represented by a configuration $v : E \to \mathbf{R}^n$ of vectors in $\mathbf{R}^n$.

For $A \subseteq E$, set $\overline{A} = E \setminus A$; and let $\mathbf{R}(A)$ denote the subspace of $\mathbf{R}^n$ spanned by $\{v(x) | x \in A\}$. Under the hypotheses in which $\psi(C_1, C_2 | C_3, C_4)$ is defined, $\mathbf{R}(\overline{C_i})$ is a hyperplane for $i = 1, 2, 3, 4$, and $\mathbf{R}(\overline{L})$ has codimension 2. Set $V = \mathbf{R}^n / \mathbf{R}(\overline{L})$, $V_i = \mathbf{R}^n / \mathbf{R}(\overline{C_i})$; so $V_1, \ldots, V_4$ are lines through the origin in the two-dimensional V. Let $\psi(C_1, C_2 | C_3, C_4)$ be the sign of the cross ratio of these lines, taken in that order, as in Example 1. Then the reorientation class of $\tilde{\mathbf{M}}$ corresponds to the map ψ. We omit the verification that ψ satisfies the conditions of Corollary 2 except to point out that the last one holds by the classical theorem of Menelaus [11, Theorem 7.5.5]. □

This example explains why the term "generalized cross ratio" is used for the generators $\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ of $\mathbb{T}_{\mathbf{M}}^{(2)}$.

Hence we could use the description provided by Corollary 2 to define projective orientations, instead of the definition given in section 2. The advantage of doing so is that far fewer data are required to specify a projective orientation of a matroid (compare Example 1); however, we have found these data less easy to work with than the method we have chosen.

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