Nonpolyhedral relaxations of graph-bisection problems

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March 22, 1993

*Visitor of the DIMACS Special Year in Combinatorial Optimization.

†The second author acknowledges partial financial support from the 'Christian Doppler Labor Diskrete Optimierung'

Abstract

We study the problem of finding the minimum bisection of a graph into two parts of prescribed sizes. We formulate two lower bounds on the problem by relaxing node- and edge- incidence vectors of cuts. We prove that both relaxations provide the same bound. The main fact we prove is that the duality between the relaxed edge- and node- vectors preserves very natural cardinality constraints on cuts.

We present an analogous result also for the max-cut problem, and show a relation between the edge relaxation and some other optimality criteria studied before. Finally, we briefly mention possible applications for a practical computational approach.

1 Introduction

We consider the problem of decomposing a weighted graph G on n nodes into two parts of prescribed sizes n_1 and n_2, $n_1 + n_2 = n$, so that the total sum of the edge weights between the parts is minimum. For an edge weight function c, and $k := n_1 - n_2$, we denote the minimum by $b(G, c, k)$.

We consider two lower bounds $\nu(G, c, k)$ and $\eta(G, c, k)$ on the bisection number $b(G, c, k)$, and call them the *node-* and *edge-relaxation*, respectively. We prove that these two bounds are equal by means of duality. An interesting fact that we want to emphasize is that the duality preserves certain cardinality constraints which are trivially satisfied by the integer vectors. To be specific, let $(S, V \setminus S)$ be a bipartition of the node set V, and let $x = (x_i) \in \Re^n$ and $y = (y_{ij}) \in \Re^{\binom{n}{2}}$ be the *node-* and *edge-incidence* vectors of this bipartition, respectively, defined by

$$x_i = \begin{cases} 1 & i \in S \\ -1 & i \notin S \end{cases} \tag{1}$$

and

$$y_{ij} = \begin{cases} 1 & i \in S, j \notin S \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

Then the partition classes have sizes n_1 and n_2 if and only if either

$$|\sum_{i=1}^{n} x_i| = |n_1 - n_2| \tag{3}$$

or

$$\sum_{1 \le i < j \le n} y_{ij} = n_1 n_2 \ . \tag{4}$$

We will show in Section 2 that the correspondence between an x satisfying (3) and a y satisfying (4) remains preserved even if (1) is relaxed to $\sum_{i=1}^{n} x_i^2 = n$ and (2) is relaxed to

$$\sum_{1 \le i < j \le n} b_i b_j y_{ij} \le \frac{1}{4} \left(\sum_{i=1}^{n} b_i \right)^2 \tag{5}$$

for all $b_1, \ldots, b_n$. This condition will correspond to a positive semidefiniteness constraint of a matrix derived from y.

In Section 3 we extend the results of Section 2 to the case when the sizes n_1 and n_2 are merely constrained by lower and upper bounds on the difference $|n_1 - n_2|$. In Section 4 we formulate the duality statement for the max-cut problem. We also discuss a relation to the optimality certificate introduced in [5]. In the last section we propose a computational framework where the two approaches are combined.

Notation. We fix some notation that will be used throughout paper. $G = (V, E)$ denotes a graph on n nodes, with node set V and edge set E. We assume

$V = \{1, \ldots, n\}$. An edge-weight function is denoted by c. We consider c as a vector in $\Re^{\binom{n}{2}}$ where c_{ij} is the weight of an edge $e = ij$ for $ij \in E$, and $c_{ij} = 0$ for $ij \notin E$, $i < j$. Occasionally, we use also the symmetric weight matrix $C = (C_{ij})$ where $C_{ij} = C_{ji} = c_{ij}$ and $C_{ii} = 0$ for all i. The pair (G, c) denotes the weighted graph G with edge weights c.

The *weighted degree* d_i of a node i is defined by setting $d_i := \sum_{j=1}^{n} C_{ij}$. The vector of weighted degrees is denoted as $d = (d_i)$, and $D = diag(d)$ denotes the corresponding diagonal matrix. The matrix $L = L(G, c) := D - C$ is called the *Laplacian matrix* of the weighted graph (G, c). Let us recall a useful identity satisfied by the Laplacian matrix for every vector $x \in \Re^n$

$$x^t L x = \sum_{1 \le i < j \le n} c_{ij}(x_i - x_j)^2 \ . \tag{6}$$

Given a symmetric matrix M, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimum and maximum eigenvalue of M, respectively. We will frequently use the Rayleigh expression for the extreme eigenvalues

$$\lambda_{\min}(M) = \min_{\|x\|=1} x^t M x \quad and \quad \lambda_{\max}(M) = \max_{\|x\|=1} x^t M x \ . \tag{7}$$

The eigenspace of an eigenvalue $\lambda(M)$ will be denoted as $Eig(\lambda(M))$. We write $M \succeq 0$ to denote that M is positive semi-definite.

Finally, we will use $\mathcal{U}$ and $\mathcal{X}_k$ to denote the following special subsets of vectors of $\Re^n$,

$$\mathcal{U} := \{u \in \Re^n \mid \sum_{i=1}^{n} u_i = 0\} \tag{8}$$

$$\mathcal{X}_k := \{x \in \Re^n \mid \|x\| = 1 \ and \ \sum_{i=1}^{n} x_i = \frac{k}{\sqrt{n}}\} \ . \tag{9}$$

Capital letters will be used sometimes to denote the corresponding diagonal matrix, as $U = \mathrm{diag}(u)$ or $Z = \mathrm{diag}(z)$.

2 Graph bisection into fixed sizes

Let (G, c) be a weighted graph on n nodes an n_1, n_2 be a pair of positive integers satisfying $n_1 + n_2 = n$. Let k be such that $n_1 = \frac{n-k}{2}$ and $n_2 = \frac{n+k}{2}$ We define the *bisection width* $b(G, c, k)$ as the minimum total weight of an edge-cut induced by a bipartition of G into two parts of sizes n_1 and n_2, respectively. That is,

$$b(G, c, k) := \min_{|S|=n_1, S \subset V} \sum_{i \in S, j \notin S} c_{ij}$$

Clearly, the integer k has to satisfy $0 \le k \le n$, and $n - k \equiv 0 \pmod 2$. We will call any such k *admissible*.

Rendl and Wolkowicz [17] introduced in an equivalent form (see the remark at the end of this Section) the following lower bound $\nu(G,c,k)$ on the bisection width. Let

$$\nu(G,c,k) := \max_{u\in\mathcal{U}} \min_{x\in\mathcal{X}_k} \frac{n}{4} x^t(L+\mathrm{diag}(u))x \tag{10}$$

where $\mathcal{U}$, $\mathcal{X}_k$ and the Laplacian matrix $L = L(G,c)$ were defined above.

Lemma 2.1 *[17] Let (G,c) be a weighted graph. Then $\nu(G,c,k) \le b(G,c,k)$ for every admissible k.*

Proof. Let $S \subset V, |S| = \frac{1}{2}(n-k)$, be such that $b(G,c,k) = \sum_{i\in S, j\notin S} c_{ij}$. Let $\overline{x}$ be the node-incidence vector of the bipartition $(S, V\setminus S)$ given by (1). Using (6), we have

$$\sum_{i\in S, j\notin S} c_{ij} = \frac{1}{4}\sum_{i\in S, j\notin S} c_{ij}(\overline{x}_i - \overline{x}_j)^2 = \frac{1}{4}\overline{x}^t L\overline{x}\ .$$

Given $u\in\mathcal{U}$, we have $\sum_{i=1}^n u_i\overline{x}_i^2 = \sum_{i=1}^n u_i = 0$ since $\overline{x}$ is a ±1-vector. Hence

$$b(G,c,k) = \frac{1}{4}\overline{x}^t(L+\mathrm{diag}(u))\overline{x} \ge \nu(G,c,k)\ .$$

□

Our main result is that $\nu(G,c,k)$ can alternatively be obtained as the optim um of another optimization problem. The importance of the result consists in the fact that our new formulations allow adding further constraints which have already proved to be useful in the approximation of graph partition problems. We postpone the more detailed discussion to the last section.

Let $y = (y_{ij}) \in \Re^{\binom{n}{2}}$, $1 \le i < j \le n$, be a vector of $\binom{n}{2}$ variables, and let $Y = (y_{ij})$ denote the corresponding $n\times n$ symmetric matrix with zero diagonal and entries ij and ji equal y_{ij}, $i<j$. Let J be the all 1's matrix. We introduce $\eta(G,c,k)$ as the optimum value of the following semi-infinite linear program.

$$\begin{aligned} \eta(G,c,k) := \quad & \min c^t y && (11)\\ & \tfrac{1}{2}J - Y \succeq 0 && (12)\\ & e^t y = m && (13) \end{aligned}$$

where $m := n_1 n_2 = \frac{1}{4}(n^2-k^2)$.

The following lemma shows that $\eta(G,c,k)$ is indeed a lower bound on the bisection width.

Lemma 2.2 *Let (G,c) be a weighted graph. Then $\eta(G,c,k) \le b(G,c,k)$ for every admissible k.*

Proof. It is useful to present a direct proof although the statement is a consequence of Lemma 2.1 and Theorem 2.1.

Let $S \subset V, |S| = \frac{1}{2}(n-k)$, be such that $b(G,c,k) = \sum_{i \in S, j \notin S} c_{ij}$. Let y be the edge-incidence vector of the bipartition $(S, V \setminus S)$ given by (2), and Y be the corresponding matrix. Further let x be the corresponding node-incidence vector defined in (1). Then

$$\frac{1}{2}J - Y = \frac{1}{2}xx^t \succeq 0,$$

and y clearly satisfies (13). Thus, y is a feasible solution of the above program, and hence $\eta(G,c,k) \leq b(G,c,k)$.

□

In subsection 5.4 the condition (12), which is equivalent to (5) is further strengthened.

We also point out that A. Schrijver [19] proposed formulation (12) with Y instead of y to derive tractable relaxations of the max-cut problem. This formulation of Schrijver was in fact the motivation for the approach described in the present paper.

It is also useful to mention that the constraint (12) implies that $0 \leq y \leq 1$ by considering 2×2 submatrices of Y.

We now formulate our main theorem.

Theorem 2.1 *Let (G,c) be a weighted graph. Then $\nu(G,c,k) = \eta(G,c,k)$ for any admissible k.*

Before giving the proof of Theorem 2.1, we need to recall the general duality theorem (Lemma 2.3), and derive some auxiliary technical results.

For a closed convex cone $C \subset \Re^n$, let $C^* = \{x \in \Re^n \mid x^t y \geq 0 \ \forall y \in C\}$. It is well known that $C^{**} = C$ for every closed convex cone.

Let A be a matrix, b, c vectors, and Ω and S closed convex cones. Consider the following pair of problems.

$$\min c^t y \qquad \textit{subject to} \quad Ay - b \in S, \quad y \in \Omega \tag{14}$$

$$\max b^t x \qquad \textit{subject to} \quad c - A^t x \in \Omega^*, \quad x \in S^* \tag{15}$$

Problem (14) is said to satisfy *generalized Slater condition* if there exists some $y \in$ rel-int Ω such that $Ay - b \in$ rel-int S (where rel-int denotes the *relative interior*).

The following duality lemma is a special case of a more general result of [20, Theorem 4.1].

Lemma 2.3 *[20]*

(i) (weak duality) If y is a feasible solution of (14) and x is a feasible solution of (15), then $b^t x \leq c^t y$.

(ii) (strong duality) Assume that (14) satisfies generalized Slater condition. If both (14) and (15) have a feasible solution, then $\min c^t y = \max b^t x$.

(iii) (complementary slackness) If y *is an optimal solution of (14) and* x *is an optimal solution of (15), then both* $y^t(c - A^t x) = 0$ *and* $x^t(Ay - b) = 0$.

□

We have to formulate an auxiliary result on a subcone K of positive semifinite matrices, which will be used in the proof of our main theorem. Let K denote the set of positive semidefinite $n \times n$ matrices M satisfying $Me = 0$. (Thus, $\lambda_{\min}(M) = 0$ with e as the corresponding eigenvector.) Some properties of K and its dual K^* are gethered in the next lemma.

Lemma 2.4 *(i)* K *is a closed convex cone,*

(ii) $(\frac{1}{2} - p)J + pI \in$ *rel-int* K *where* $p = \frac{n}{2(n-1)}$

(iii) $K^* = \{M + tJ \mid M \succeq 0, t \in \Re\}$.

Proof. (i) K is the intersection of two closed convex cones $C_1 = Psd$ (the cone of positive semidefinite matrices) and $C_2 = \{M \mid e^t M e = 0\}$. Hence K is a closed convex cone as well.

(ii) Let $\bar{A} := (\frac{1}{2} - p)J + pI$ where $p = \frac{n}{2(n-1)}$. The eigenvalues of $(\frac{1}{2} - p)J$ are $(0, \ldots, 0, n(\frac{1}{2} - p))$ ($n - 1$ times the eigenvalue 0). Therefore the eigenvalues of $\bar{A}$ are $(p, p, \ldots, p, \frac{n}{2} - np + p)$, and they are positive except the last one which is 0 due to the definition of p. Thus $\bar{A} = (\bar{a}_{ij})$ is positive semidefinite, and $\bar{A} \in K$. Let $A = (a_{ij})$ be an arbitrary matrix such that $Ae = 0$ and $|\bar{a}_{ij} - a_{ij}| \leq \varepsilon$ where $\varepsilon := \frac{\lambda_2(\bar{A})}{n} = \frac{1}{2(n-1)}$. Given a vector w, satisfying $w^t e = 0$ and $\|w\| = 1$, we have

$$w^t A w = w^t(\bar{A} - (\bar{A} - A))w = w^t \bar{A} w - w^t(\bar{A} - A)w \geq$$
$$\lambda_2(\bar{A}) - \varepsilon \left(\sum_{i=1}^{n} |w_i| \right)^2 \geq \lambda_2(\bar{A}) - \varepsilon n \geq 0.$$

Hence, every matrix A in the ε-neighbourhood of $\bar{A}$ is positive semidefinite as well. This proves that $\bar{A} \in$ rel-int K.

(iii) In order to prove part (iii), we need three claims. Let L denote the set of matrices M such that $M + tJ$ is positive semidefinite for some $t \in \Re$.

Claim 1. $A \notin L$ if and only if $\exists x \in \Re^n$ such that $e^t x = 0$ and $x^t A x < 0$.

Assume that such x exists. Then

$$x^t(A + tJ)x = x^t A x + t \left(\sum_{i=1}^{n} x_i \right)^2 = x^t A x < 0$$

and hence $A + tJ$ is not positive semidefinite for any t. Thus $A \notin L$ is established.

Conversely, assume that $x^t Ax \geq 0$ for all x such that $e^t x = 0$. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k < 0 \leq \lambda_{k+1} \leq \ldots \leq \lambda_n$ be the eigenvalues of A, and $y_j = (y_{ij})$, $j = 1, \ldots, n$, be the collection of orthonormal eigenvectors. Set

$$\bar{t} := \max \left(\frac{\lambda_1}{(\sum_{i=1}^n y_{i1})^2}, \ldots, \frac{\lambda_k}{(\sum_{i=1}^n y_{ik})^2} \right).$$

It is straightforward to check that $A + \bar{t}J$ is positive semidefinite. Hence $A \in L$.

Claim 2. L is a closed convex cone.

It is easy to see that L is a convex cone. We prove that L is closed. Let $A \notin L$ be given. By Claim 1, there exists x such that $e^t x = 0$ and $x^t Ax < 0$. It is easy to see that there exists some sufficiently small $\varepsilon > 0$ such that $x^t A' x < 0$ (with the same x) also for every matrix A' in the ε-neighbourhood of A. This proves that L is closed.

Claim 3. If C_1 and C_2 are two closed convex cones, then

$$C_1^* + C_2^* \subset (C_1 \cap C_2)^* \subset \overline{C_1^* + C_2^*}$$

where $\overline{C}$ denotes the closure of C.

The proof of Claim 3 follows by a direct application of the definition of a dual cone.

Now we are able to conclude the proof of part (iii). Let us denote $C_1 = Psd$ (the cone of positive semidefinite matrices), and $C_2 = \{A \mid e^t Ae = 0\}$. Thus $K = C_1 \cap C_2$. We have $C_1^* = Psd$ and $C_2^* = \{tJ \mid t \in \Re\}$. Since $L = C_1^* + C_2^*$ is closed by Claim 2, we have $C_1^* + C_2^* = (C_1 \cap C_2)^*$ by Claim 3. Thus $L = K^*$ is established. □

Now we introduce an auxiliary optimization problem in two real variables s and t. Let M be a symmetric matrix of size $n \times n$, $n \geq 2$, and let J denote the matrix of all ones. Let $0 < k < \frac{n}{2}$ (k not necessarily integer).

$$\max \; ns - k^2 t \tag{16}$$

$$M + tJ - sI \succeq 0 \tag{17}$$

$$s, t \in \Re \tag{18}$$

Lemma 2.5 *The maximum in (16) is attained for some $\bar{s}, \bar{t} \in \Re$. Moreover, $\bar{s} = \lambda_{\min}(M + \bar{t}J)$.*

Proof. Clearly, $s \leq \lambda_{\min}(M + tJ)$ for any feasible s and t due to (17). Hence we may restrict our choice to $s = \lambda_{\min}(M + tJ)$, since increasing s increases the value of the objective function as well. Let us denote $\rho = \rho(M)$ the spectral radius of M,

and $\lambda_0 = \lambda_{\min}(M)$ the minimum eigenvalue of M. Let us set $f(t) := n\lambda_{\min}(M + tJ) - k^2 t$. We claim that $f(t) \geq f(0) = n\lambda_0$ only for $t \in [-\frac{n(\rho-\lambda_0)}{n^2-k^2}, \frac{n(\rho-\lambda_0)}{k^2}]$.

(i) *Lower bound.* Take $e = (1, \ldots, 1)^t$. Then $\lambda_{\min}(M + tJ) \leq \frac{e^t(M+tJ)e}{e^t e} \leq \rho(M) + tn$. Assume that $f(t) \geq f(0)$, i.e. $n(\rho + tn) - k^2 t \geq n\lambda_0$, which yields $t \geq -\frac{n(\rho-\lambda_0)}{n^2-k^2}$.

(ii) *Upper bound.* Take $x = (1, -1, 0, \ldots, 0)^t$. (Here the assumption $n \geq 2$ is used.) Then $\lambda_{\min}(M + tJ) \leq \frac{x^t(M+tJ)x}{x^t x} \leq \rho(M)$. Assume that $f(t) \geq f(0)$, i.e. $n\rho - k^2 t \geq n\lambda_0$, which yields $t \leq \frac{n(\rho-\lambda_0)}{k^2}$.

Since $f(t) \geq f(0)$ may hold only for t belonging to a closed interval containing zero, and the objective function (16) is continuous, the maximum is attained for some $\overline{t}$ and $\overline{s} = \lambda_{\min}(M + \overline{t}J)$. □

Lemma 2.6 *Let $\mathcal{E} \subset \Re^n$ be a linear subspace and d be a constant, $0 < d < n$, such that $(\sum_{i=1}^n y_i)^2 \leq d\|y\|^2$ for every $y \in \mathcal{E}$. Then, for every $D > d$, there exists a constant c such that, for every $x \in \Re^n$,*

$$(\sum_{i=1}^n x_i)^2 \leq D\|x\|^2 + c\|z\|^2 \tag{19}$$

where $z = x - proj(x)$ and $proj(x)$ denotes the projection of x on the subspace $\mathcal{E}$.

Proof. Let us first recall that $(\sum_{i=1}^n x_i)^2 \leq n\|x\|^2$ for every $x \in \Re^n$. Let $x \in \Re^n$ be given. Let us decompose it as $x = y + z$ where $y = proj(x)$. We have

$$\begin{gathered}(\textstyle\sum_{i=1}^n x_i)^2 = (\sum_{i=1}^n y_i + \sum_{i=1}^n z_i)^2 = \\ (\textstyle\sum_{i=1}^n y_i)^2 + 2(\sum_{i=1}^n y_i)(\sum_{i=1}^n z_i) + (\sum_{i=1}^n z_i)^2 \leq \\ d\|y\|^2 + 2\sqrt{dn}\|y\|\|z\| + n\|z\|^2 \ .\end{gathered} \tag{20}$$

Set $\delta := \frac{D-d}{2\sqrt{nd}}$ and $c := n + \frac{2\sqrt{nd}}{\delta}$. We distinguish two cases.

(i) *(z "small")* Assume $\|z\| \leq \delta\|x\|$. We have (using $\|y\| \leq \|x\|$)

$$\begin{aligned} d\|y\|^2 + 2\sqrt{dn}\|y\|\|z\| + n\|z\|^2 \leq \\ d\|x\|^2 + 2\sqrt{dn}\|x\|\delta\|x\| + n\|z\|^2 \leq \\ (d + 2\delta\sqrt{dn})\|x\|^2 + n\|z\|^2 \leq \\ D\|x\|^2 + c\|z\|^2. \end{aligned}$$

(ii) *(z "large")* Assume $\|z\| \geq \delta\|x\|$. We have (using $\|y\| \leq \|x\| \leq \frac{1}{\delta}\|z\|$)

$$\begin{aligned} d\|y\|^2 + 2\sqrt{dn}\|y\|\|z\| + n\|z\|^2 \leq \\ d\|y\|^2 + \frac{2\sqrt{dn}}{\delta}\|z\|^2 + n\|z\|^2 \leq \\ D\|x\|^2 + c\|z\|^2. \end{aligned}$$

□

Remark 2.1 *The statement of Lemma 2.6 does not hold for $D = d$. A counterexample was found by Chun-Wa Ko.*

The next two lemmas are crucial for the proof of Theorem 2.1. We recall that $\mathcal{X}_k := \{x \in \Re^n \mid \|x\| = 1 \text{ and } \sum_{i=1}^n x_i = \frac{k}{\sqrt{n}}\}$

Lemma 2.7 *Let $\overline{s}$ and $\overline{t}$ be the optimum for the program (16)-(18) Then $\mathcal{X}_k \cap Eig(\lambda_{\min}(M + \overline{t}J - \overline{s}I))$ is nonempty.*

Proof. Let us denote $M(s,t) := M + tJ - sI$, and set

$$f(s,t) := ns - k^2 t$$

and

$$g(s,t) := \lambda_{\min}(M(s,t)).$$

Hence, the problem (16)–(18) can be re-written as

$$\max f(s,t) \tag{21}$$
$$g(s,t) \geq 0 \tag{22}$$

Let $\overline{s}$ and $\overline{t}$ denote the optimum solution, which exists by Lemma 2.5. We distinguish three cases.

Case (i). Assume that the eigenvalue $\lambda_{\min} M(\overline{s}, \overline{t})$ is simple.

Using e.g. Theorem 2 of [7], we get that the function $g(s,t)$ is differentiable at the point $(\overline{s}, \overline{t})$, and its partial derivatives are given by

$$\begin{aligned} \frac{\partial g(\overline{s},\overline{t})}{\partial s} &= -\sum_{i=1}^n \overline{x}_i^2 = -1 \\ \frac{\partial g(\overline{s},\overline{t})}{\partial t} &= (\sum_{i=1}^n \overline{x}_i)^2 \end{aligned}$$

where $\overline{x} = (\overline{x}_i)$ is the eigenvector of $\lambda_{\min} M(\overline{s}, \overline{t})$ of norm one. The partial derivatives of the objective function f are

$$\frac{\partial f(\overline{s},\overline{t})}{\partial s} = n$$

$$\frac{\partial f(\overline{s},\overline{t})}{\partial t} = -k^2$$

Using the Kuhn-Tucker optimality condition, by which

$$\nabla f(\overline{s},\overline{t}) + \alpha \nabla g(\overline{s},\overline{t}) = 0$$

for some $\alpha \in \Re$, we get $\alpha = n$, and hence $(\sum_{i=1}^n \overline{x}_i)^2 = \frac{k^2}{n}$.

Case (ii). Assume that the eigenvalue $\lambda_{\min} M(\overline{s}, \overline{t})$ is multiple, and that there exists an eigenvector x such that $\|x\|^2 = 1$ and $\sum_{i=1}^n x_i > \frac{k}{\sqrt{n}}$. Clearly, $y := -x$ is an eigenvector of unit length satisfying $\sum_{i=1}^n y_i < -\frac{k}{\sqrt{n}}$. Since $\{x \in \Re^n \mid \|x\| = 1\} \cap Eig(\lambda_{\min}(M + \overline{t}J - \overline{s}I))$ is compact and connected, we conclude that there exists an eigenvector $\overline{x}$ of norm one satisfying $\sum_{i=1}^n \overline{x}_i = \frac{k}{\sqrt{n}}$.

Case (iii). Let $\mathcal{E}$ denote the eigenspace of $\lambda_{\min} M(\overline{s}, \overline{t})$. Assume that there exists some $\ell < k$ such that $|\sum_{i=1}^n y_i| \leq \frac{\ell}{\sqrt{n}}$ for all eigenvectors $y \in \mathcal{E}$ of unit length. Let us apply Lemma 2.6 to the eigenspace $\mathcal{E}$, and with parameters $d := \frac{\ell^2}{n}$ and $D := \frac{\ell^2 + k^2}{2n}$, and let c denote the constant given by the Lemma 2.6. Let us fix some positive ε such that $\varepsilon c < \lambda_2(M(\overline{s}, \overline{t}))$ where λ_2 denotes the smallest eigenvalue larger than $\lambda_{\min}(M(\overline{s}, \overline{t}))$, which we have just shown to be 0. Let us set

$$\overline{\overline{t}} := \overline{t} - \varepsilon$$

and

$$\overline{\overline{s}} := \overline{s} - \frac{1}{n}\varepsilon\frac{k^2 + \ell^2}{2}.$$

We claim that the matrix $\overline{\overline{M}} := M(\overline{\overline{s}}, \overline{\overline{t}})$ is positive semidefinite, and the value of the objective function (21) increases for $(\overline{\overline{s}}, \overline{\overline{t}})$ as compared to $(\overline{s}, \overline{t})$. Clearly,

$$f(\overline{\overline{s}}, \overline{\overline{t}}) = n\overline{\overline{s}} - k^2\overline{\overline{t}} = n\overline{s} - k^2\overline{t} + \varepsilon\left(k^2 - \frac{k^2 + \ell^2}{2}\right) > n\overline{s} - k^2\overline{t} = f(\overline{s}, \overline{t})$$

since $\ell^2 < k^2$.

Now, we check that $\overline{\overline{M}}$ is positive semidefinite. Let an arbitrary $x \in \Re^n$ be given. Let us decompose x as $x = y + z$ where $y \in \mathcal{E}$, and $z^t y = 0$. We have

$$x^t\overline{\overline{M}}x = x^t\overline{M}x + \varepsilon x^t\left(-J + \frac{\ell^2 + k^2}{2n}I\right)x \tag{23}$$

We will estimate each term on the right-hand-side of (23) separately. We have

$$x^t\overline{M}x = (y + z)^t\overline{M}(y + z) \geq \lambda_2(\overline{M})\|z\|^2 \tag{24}$$

and, using Lemma 2.6 (with the parameters defined above),

$$x^t(-J + \frac{\ell^2 + k^2}{2n}I)x = -(\sum_{i=1}^n x_i)^2 + \frac{\ell^2 + k^2}{2n}\|x\|^2 \geq \tag{25}$$
$$-D\|x\|^2 - c\|z\|^2 + \frac{\ell^2 + k^2}{2n}\|x\|^2 = -c\|z\|^2.$$

Hence $x^t\overline{\overline{M}}x$ can be estimated, using (24,25), as

$$x^t\overline{\overline{M}}x \geq \lambda_2(\overline{M})\|z\|^2 - \varepsilon c\|z\|^2 > 0$$

by the choice of ε. This contradiction shows that case (iii) cannot occur, proving the lemma. □

Lemma 2.8 *Let $\overline{s}, \overline{t}$ be the optimum of (16)–(18). Then*

$$\overline{s} - \frac{k^2}{n}\overline{t} = \min_{x \in \mathcal{X}_k} x^t M x.$$

Proof. (i) Let $\overline{x} \in \mathcal{X}_k \cap Eig(\lambda_{\min}(M + \overline{t}J - \overline{s}I))$. We have

$$0 = \lambda_{\min}(M + \overline{t}J - \overline{s}I) = \lambda_{\min}(M + \overline{t}J) - \overline{s} =$$

$$\overline{x}^t(M + \overline{t}J)\overline{x} - \overline{s} = \overline{x}^t M \overline{x} + \overline{t}(\sum_{i=1}^{n} \overline{x}_i)^2 - \overline{s} =$$

$$\overline{x}^t M \overline{x} + \frac{\overline{t}k^2}{n} - \overline{s} \geq \frac{\overline{t}k^2}{n} - \overline{s} + \min_{x \in \mathcal{X}_k} x^t M x.$$

Hence

$$\min_{x \in \mathcal{X}_k} x^t M x \leq \overline{s} - \frac{\overline{t}k^2}{n}.$$

(ii) Let $\tilde{x} \in \mathcal{X}_k$ be such that $\tilde{x}^t M \tilde{x} = \min_{x \in \mathcal{X}_k} x^t M x$. We have

$$\tilde{x}^t M \tilde{x} + \frac{\overline{t}k^2}{n} = \tilde{x}^t(M + \overline{t}J)\tilde{x} \geq \lambda_{\min}(M + \overline{t}J) = \overline{s}.$$

Hence $\tilde{x}^t M \tilde{x} \geq \overline{s} - \frac{\overline{t}k^2}{n}$, which proves the opposite inequality.

□

Proof of Theorem 2.1. We first formulate the program (11)–(13) which defines the bound $\eta(G, c, k)$ as a program of the form (14). Let Psd denote the cone of symmetric positive semi-definite matrices, considered as vectors in $\Re^{n^2}$. We set $\Omega := \Re^{\binom{n}{2}}$ and $S := \{(w, 0) \in \Re^{n^2+1} \mid w \in Psd\}$. Then clearly $\Omega^* = \{0\} \subset \Re^{\binom{n}{2}}$ and $S^* = \{(x, t) \mid x \in Psd, t \in \Re\}$, because $Psd = Psd^*$. Further let M be the $n^2 \times \binom{n}{2}$ matrix defined such that $w = My$ is the symmetric matrix corresponding to y with main diagonal zero, and written as a vector in $\Re^{n^2}$. In other words

$$M_{(i,j),\{k,\ell\}} := \begin{cases} 1 & \text{if } \{i,j\} = \{k,\ell\} \\ 0 & \text{otherwise.} \end{cases}$$

Finally $j \in \Re^{n^2}$ denotes the matrix J written as a vector. Then the program (11)–(13) is equivalent to the following problem.

$$\eta(G, c, k) = \min c^t y \tag{26}$$

$$-My + \frac{1}{2}j \in Psd \tag{27}$$

$$e^t y - m = 0 \tag{28}$$

$$y \in \Omega \tag{29}$$

We have to distinguish two cases according whether $k > 0$ or $k = 0$. It will be shown that these two cases correspond to whether or not problem (27)–(29) satisfies generalized Slater condition.

Case (i). $k > 0$. We claim that (27)–(29) satisfy generalized Slater condition.

In order to prove the claim, let us define the matrix $\bar{Y}$ as a matrix with the zero diagonal, and all off-diagonal entries equal $p := \frac{n^2-k^2}{2n(n-1)}$. Let us consider $\frac{1}{2}J - \bar{Y} = (\frac{1}{2} - p)J + pI$. The eigenvalues of this matrix are (cf. part (ii) of Lemma 2.4) $(p, p, \ldots, p, \frac{n}{2} - np + p)$, and they are all positive due to the definition of p. Thus the matrix $\frac{1}{2}J - \bar{Y}$ is positive definite, and hence it lies in the interior of the cone of positive semidefinite matrices.

The dual of (26)–(29) now reads as follows.

$$\max \; mt - \frac{1}{2} \sum_{1 \le i,j \le n} x_{ij} \tag{30}$$

$$c_{ij} + x_{ij} + x_{ji} - t = 0 \quad i < j \tag{31}$$

$$x \in Psd \tag{32}$$

$$t \in \Re \tag{33}$$

Here x is the vector of the variables dual to the constraint (12), and t is the variable dual to the constraint (13).

Case (ii). $k = 0$. In this case, we have $e^t(\frac{1}{2}J - Y)e = 0$ for every Y satisfying (13). Hence no $\frac{1}{2}J - Y$ is positive definite, and generalized Slater condition is not satisfied. Let K denote the cone of positive semidefinite matrices M satisfying $Me = 0$, which is studied in Lemma 2.4. The program (26)–(29) can be replaced by

$$\eta(G, c, 0) = \min c^t y \tag{34}$$

$$-My + \frac{1}{2}j \in K \tag{35}$$

$$y \in \Omega \tag{36}$$

Now we will apply Lemma 2.4. By part (i), K is a closed convex cone. By part (ii), the problem (34)–(36) satisfies the generalized Slater condition. In order to show it, consider matrix $\bar{Y}$ with the zero diagonal, and all off-diagonal entries equal $p := \frac{n}{2(n-1)}$. Hence $\bar{Y} = (\frac{1}{2} - p)J + pI \in$ rel-int K.

Using Lemma 2.3, the dual problem of (34)–(36) reads

$$\max \; -\frac{1}{2} \sum_{1 \le i,j \le n} w_{ij} \tag{37}$$

$$c_{ij} + w_{ij} + w_{ji} = 0 \quad i < j \tag{38}$$

$$W \in K^* \tag{39}$$

where $W = (w_{ij})$ is a symmetric matrix of dual variables. Using the characterization of K^* given in part (iii) of Lemma 2.4, we can write

$$W = X - \frac{1}{2}tJ \tag{40}$$

for some t. After this substitution, (37)–(39) turn into the form (30)–(33) with $m = \frac{1}{4}n^2$.

Thus, we have shown that independently of whether $k = 0$ or $k > 0$, the dual of the problem (27)–(29) can be reduced to the form (30)–(33).

The dual program (30)–(33) can be further simplified in the following way. $x \in Psd$ implies $x_{ij} = x_{ji}$ for all $i \neq j$, thus $x_{ij} = \frac{1}{2}(t - c_{ij})$ for every $i, j, i \neq j$. Let us introduce new variables $z_i, i = 1, \ldots, n$, by

$$z_i := d_i - 2x_{ii} + t \tag{41}$$

where $d_i := \sum_{j=1}^n c_{ij}$, i.e. d_i is the 'weighted degree' of node i. Now, we can write the dual program as

$$\begin{aligned} \eta(G, c, k) = \quad & \max \tfrac{1}{4}\textstyle\sum_{i=1}^n z_i - \tfrac{1}{4}k^2 t && (42)\\ & tJ + L - diag(z) \succeq 0 && (43)\\ & t \in \Re,\ z \in \Re^n && (44) \end{aligned}$$

where $L = L(G, c)$ is the Laplacian matrix of (G, c).

We prove that the programs (42)–(44) and (10) defining $\nu(G, c, k)$ provide the same optimum.

Given $z = (z_i)$, set $s := \frac{1}{n}\sum_{i=1}^n z_i$, and $u_i := -z_i + s, i = 1, \ldots, n$. Clearly, $u \in \mathcal{U}$ since $\sum u_i = 0$. Program (42)–(44) is equivalent to

$$\max_{u \in \mathcal{U}} \ \frac{1}{4}\ f(u)$$

where

$$\begin{aligned} f(u) := \quad & \max \ ns - k^2 t && (45)\\ & M(u) + tJ - sI \succeq 0 \\ & t, s \in \Re. \end{aligned}$$

Here $M(u) := L + \mathrm{diag}(u)$. Since

$$\nu(G, c, k) := \max_{u \in \mathcal{U}} \ \min_{x \in \mathcal{X}_k} \frac{n}{4} x^t (L + \mathrm{diag}(u)) x$$

we can apply Lemma 2.8 to any matrix $M := M(u)$. The equality

$$\nu(G, c, k) = \eta(G, c, k)$$

follows by taking $M(\overline{u})$ where $\overline{u}$ is the optimum of (10).

□

Corollary 2.1 *The relaxation $\eta(G, c, k)$ of the bisection width is monotone with respect to k, i.e. $\eta(G, c, k_1) \geq \eta(G, c, k_2)$ for $k_1 < k_2$.*

Proof. Let $\overline{z}$ and $\overline{t}$ be optimum for the program (42) – (44) defining $\eta(G, c, k_2)$. We have

$$\eta(G, c, k_2) = \frac{1}{4}\sum_{i=1}^{n} \overline{z}_i - \frac{1}{4}k_2^2\overline{t} \leq \frac{1}{4}\sum_{i=1}^{n} \overline{z}_i - \frac{1}{4}k_1^2\overline{t} \leq \eta(G, c, k_1) \ .$$

□

Remark 2.2 In the special case of $n_1 = n_2 = n/2$, the relaxation (46) has been first given by R. Boppana [4] (in an equivalent form). Since $k := n_1 - n_2 = 0$, the node relaxation (10) takes form

$$\nu(G, c, 0) := \max_{u \in \mathcal{U}} \min_{x \in \mathcal{X}_0} \frac{n}{4} x^t (L + \mathrm{diag}(u))x \tag{46}$$

Let us remark that the case with $k = 0$ is somewhat easier than the general case with $k > 0$. One reason is that $\mathcal{X}_0$ becomes a linear subspace, and hence $\min_{x \in \mathcal{X}_0} \frac{n}{4} x^t (L + \mathrm{diag}(u))x$ can be reduced to an eigenvalue problem by projecting on the subspace $\mathcal{X}_0$. Another reason is that the proof of Theorem 2.1 can be made much simpler for this case. We sketch some details. The dual definition (42)–(44) of $\eta(G, c, 0)$ reads

$$\begin{aligned} \eta(G, c, 0) := \quad & \max \tfrac{1}{4}\textstyle\sum_{i=1}^{n} z_i \\ & tJ + L - diag(z) \succeq 0 \\ & t \in \Re,\ z \in \Re^n \end{aligned}$$

The crucial part in the proof of Theorem 2.1 consists of the following claim.

If $\overline{t}$ and $\overline{z}$ are optimal for the above program, then

$$Eig(\lambda_{\min}(\overline{t}J + L - diag(\overline{z}))) \cap \mathcal{X}_0 \neq \emptyset \ .$$

Since $k = 0$, case (iii) of Lemma 2.7 cannot occur. Hence, one does not need Lemma 2.6 either. On the other hand, we needed Lemma 2.4 in order to establish the correctness of the dual problem in case $k = 0$.

□

Remark 2.3 *In [17] the general partitioning problem into subsets of specified sizes is investigated. In the special case of partitioning into just two sets the approach chosen in [17] is equivalent to the relaxation (10). To see the equivalence we note that in [17] a partition (S_1, S_2) with $|S_1| = n_1$ is represented by an $n \times 2$ 0-1-matrix $X = (x_{ij})$ where $x_{ij} = 1$ if and only if $i \in S_j$. The relaxation in [17] for bisection is then obtained by optimizing over all matrices $X = (y\ z)$, satisfying the constraints*

$$y^t y = n_1, \quad z^t z = n - n_1, y^t z = 0,$$

$$y + z = e, \quad e^t y = n_1, e^t z = n - n_1.$$

(The first set of constraints describes X as having orthogonal columns of specified lengths, while the second set describes the bisection into specified sizes $n_1, n - n_1$. Defining $x := y - z$ shows that x normalized to unit length lies in $\mathcal{X}_k$. Conversely taking $x \in \mathcal{X}_k$ we can set $w := \sqrt{n}x$, and $y := (e - w)/2, \ z = (e + w)/2$. Then the matrix $X = (y\ z)$ can easily be shown to satisfy the constraints above. This shows the equivalence of the two approaches.

Remark 2.4 *One of the early relaxations of graph partitioning was proposed in [9]. In the case of bisections, this approach amounts to optimizing over all matrices X satisfying just the orthogonality constraints described above. Therefore this relaxation is never better than the relaxation ¿from [17] or (10).*

3 Graph bisection under inequality constraints

The results of the previous section can be formulated also for a slightly more general version of graph bisection where the sizes n_1 and n_2 are not exactly prescribed but rather constrained by a condition $\ell \leq n_1 - n_2 \leq k$ where the bounds ℓ and k are given in advance. Obviously, the exact bisection corresponds to the case when $\ell = k$.

Given a weighted graph (G, c) and the bounds $\ell, k, 0 \leq \ell \leq k \leq n/2$, let $b(G, c, \ell, k)$ denote the minimum total weight of an edge-cut induced by a bipartition $(S, V \setminus S)$ satisfying $\ell \leq |S| - |V \setminus S| \leq k$.

Analogously, we can introduce lower bounds $\nu(G, c, \ell, k)$ and $\eta(G, c, \ell, k)$ on $b(G, c, \ell, k)$, which are obtained by relaxation of node and edge incidence vectors, respectively. Namely, let us introduce

$$\nu(G, c, \ell, k) := \max_{u \in \mathcal{U}} \min_{x \in \mathcal{X}_{\ell k}} \frac{n}{4} x^t (L + \mathrm{diag}(u)) x \tag{47}$$

where

$$\mathcal{U} := \{u \in \Re^n \mid \sum_{i=1}^{n} u_i = 0\} \tag{48}$$

$$\mathcal{X}_{\ell k} := \{x \in \Re^n \mid \|x\| = 1 \text{ and } \frac{\ell}{\sqrt{n}} \leq \sum_{i=1}^{n} x_i \leq \frac{k}{\sqrt{n}}\} \tag{49}$$

and $L = L(G,c)$ is the Laplacian matrix of the weighted graph (G,c), and

$$\eta(G,c,\ell,k) := \quad \min c^t y \tag{50}$$

$$\tfrac{1}{2}J - Y \succeq 0 \tag{51}$$

$$\tfrac{1}{4}(n^2 - k^2) \leq e^t y \leq \tfrac{1}{4}(n^2 - \ell^2) \tag{52}$$

where Y denotes the symmetric matrix with zero diagonal and the off-diagonal entries y_{ij} given by the vector y.

Theorem 3.1 *Let (G,c) be a weighted graph. Then*

$$\nu(G,c,\ell,k) = \eta(G,c,\ell,k)$$

for any bounds $\ell \leq k$.

Proof. Using the duality given by Lemma 2.3, the dual problem to the definition of $\eta(G,c,\ell,k)$ can be transformed to the following problem.

$$\max \ \tfrac{1}{4}\left(-k^2 t_1 + \ell^2 t_2 + \textstyle\sum_{i=1}^{n} z_i\right) \tag{53}$$

$$(t_1 - t_2)J + L - diag(z) \succeq 0 \tag{54}$$

$$t_1, t_2 \geq 0, z \in \Re^n. \tag{55}$$

We omit further details of the proof, since it is quite analogous to that of Theorem 2.1.

□

4 Max-cut problem

Given a weighted graph (G,c), the *max-cut problem* asks to find a bipartition $(S, V \setminus S)$ of V such that the total weight of the edges between S and $V \setminus S$ is maximum. We denote

$$mc(G,c) := \max_{S \subset V} \sum_{i \in S \ j \notin S} c_{ij} \ .$$

Thus, the max-cut problem can be viewed as a special case of the constrained bisection problem formulated in Section 3 by setting the bounds $\ell = 0$ and $k = n/2$, and the objective function is $c' := -c$. It is advantagous however to focus on the max-cut problem independently. The max-cut problem is an *unconstrained*

optimization problem, and hence the main ideas of our paper become also most transparant for it.

We will use the notation $\varphi(G,c)$ and $\psi(G,c)$ to denote the node- and edge-relaxation of the max-cut problem. We use a notation different from the previous sections for two reasons; (i) to avoid confusion with the minimization problem, and (ii) the notation $\varphi(G,c)$ has already been used in [5].

The *node-relaxation* $\varphi(G,c)$ was introduced in [5] as

$$\varphi(G,c) := \min_{u\in\mathcal{U}} \frac{n}{4}\lambda_{\max}(L+\mathrm{diag}(u)) \tag{56}$$

Let us note that (56) is nothing else but

$$\varphi(G,c) := \min_{u\in\mathcal{U}} \frac{n}{4} \max_{\|x\|=1} x^t(L+\mathrm{diag}(u))x \tag{57}$$

when using the Rayleigh quotient to express $\lambda_{\max}$, and thus (57) is quite analogous to (10) and (47).

The *edge-relaxation* $\psi(G,c)$ of the max-cut is defined by

$$\psi(G,c) := \max\{\sum_{1\le i<j\le n} c_{ij}y_{ij} \mid \frac{1}{2}J - Y \succeq 0\} \tag{58}$$

where $y=(y_{ij}) \in \Re^{\binom{n}{2}}$ is a vector of variables, and Y is the corresponding $n\times n$ matrix defined as before so that

$$Y_{ij} := \begin{cases} y_{ij} & i<j \\ y_{ji} & i>j \\ 0 & i=j \end{cases}$$

Lemma 4.1 *Both $\varphi(G,c)$ and $\psi(G,c)$ are upper bounds on $mc(G,c)$.*

□

Actually, φ and ψ define the same bound.

Theorem 4.1 *Let (G,c) be a weighted graph. Then $\varphi(G,c)=\psi(G,c)$.*

Proof. Let us first re-write the definition of the bound $\psi(G,c)$ in a form which allows dualization by Lemma 2.3. We have (in our previous notation)

$$\begin{aligned} \psi(G,c) = \quad & \max c^t y && (59)\\ & \tfrac{1}{2}j - My \in Psd && (60)\\ & y \in \Re^{\binom{n}{2}} && (61) \end{aligned}$$

The dual of (59)–(61) reads

$$\psi(G,c) = \quad \min \tfrac{1}{2} j^t x \tag{62}$$
$$M^t x - c = 0 \tag{63}$$
$$x \in Psd \tag{64}$$

Equivalently,

$$\psi(G,c) = \quad \min \tfrac{1}{2} \textstyle\sum_{1 \le i,j \le n} x_{ij} \tag{65}$$
$$x_{ij} + x_{ji} - c_{ij} = 0, \quad i < j \tag{66}$$
$$x \in Psd. \tag{67}$$

Since all entries x_{ij} but x_{ii} are determined by (66) as $x_{ij} = \frac{1}{2} c_{ij}$, the actual variables are only x_{ii}. Let us express them in the form

$$2x_{ii} = -d_i - u_i + s \tag{68}$$

where d_i, $i = 1, \ldots, n$, is the weighted degree, and $s := \frac{1}{n} \sum_{i=1}^{n} (x_{ii} + d_i)$ and $u_i = -d_i + s - 2x_{ii}$. Clearly, $u \in \mathcal{U}$. The constraint (67) is equivalent to

$$C - diag(d) - diag(u) + sI \succeq 0$$

or, equivalently, to

$$s \ge \lambda_{\max}(L + diag(u)) \tag{69}$$

since $L = diag(d) - C$. The objective function (65) becomes, after substituting (68)

$$\frac{1}{2} \sum_{i \ne j} x_{ij} + \frac{1}{2} \sum_{i=1}^{n} x_{ii} = \frac{1}{4} \sum_{i \ne j} c_{ij} + \frac{1}{4} \sum_{i=1}^{n} (-d_i - u_i + s) = \frac{n}{4} s \tag{70}$$

since $\sum u_i = 0$ and $\sum_{i \ne j} c_{ij} = \sum_{i=1}^{n} d_i$. Now it is easy to see that (62) and (63) are equivalent to (56). □

In the rest of this section, we show the way in which the optimum solution Y of (58) is related to the optimality certificate formulated in [5].

Let (G,c) be a weighted graph and $u \in \mathcal{U}$. Let $(z_1, \ldots, z_m) = (z_{im}) \in Eig(\lambda_{\max}(L + \mathrm{diag}(u))$ be a collection of m eigenvectors. We say that the collection $z_1, \ldots, z_m$ is an *optimality certificate for* u, if

$$\sum_{j=1}^{m} z_{ij}^2 = 1 \quad \text{for every } i = 1, \ldots, n. \tag{71}$$

Let us say that $u^* \in \mathcal{U}$ is an *optimum correcting vector* for (G,c) if $\varphi(G,c) := \frac{n}{4} \lambda_{\max}(L + diag(u^*))$.

Theorem 4.2 *[5] Let $u \in \mathcal{U}$. Then u is the optimum correcting vector if and only if there exists an optimality certificate for u.*

□

The next theorem describes the mutual relation between an optimal Y in the edge-relaxation (58) and an optimality certificate.

Theorem 4.3 *Let (G, c) be a weighted graph, and $u \in \mathcal{U}$ be the optimum correcting vector.*

(i) Let $z_1, \ldots, z_m$ be an optimality certificate for u. Let $Z = [z_1, \ldots, z_m]$ be the $n \times m$ matrix with columns $z_1, \ldots, z_m$. Then $Y := \frac{1}{2}(J - ZZ^t)$ is the optimum for (58).

(ii) Let Y be the optimum for (58). Then there exists a collection of pairwise orthogonal vectors $z_1, \ldots, z_m$ satisfying

$$J - 2Y = z_1 z_1^t + \ldots + z_m z_m^t \tag{72}$$

which form an optimality certificate for u.

Proof. (i) The matrix Y has zero diagonal, since ZZ^t has all 1's on the main diagonal due to (71). The matrix $\frac{1}{2}J - Y$ is positive semi-definite, since $\frac{1}{2}J - Y = ZZ^t$. Thus, Y is feasible for (58). It remains to show the optimality of Y.

Let $U = \mathrm{diag}(u)$. We have

$$tr((L + U)ZZ^t) = n\lambda_{\max} \tag{73}$$

since the columns of Z are eigenvectors of $\lambda_{\max}(L + U)$, and ZZ^t has all 1's on the main diagonal. Further, we have

$$tr(CJ) - \mathrm{tr}((\mathrm{D} + \mathrm{U})\mathrm{ZZ}^{\mathrm{t}}) = \sum_{\mathrm{i}\neq\mathrm{j}} \mathrm{c_{ij}} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d_i} = 0 \ . \tag{74}$$

Using (73) and (74) we get

$$\begin{gathered} tr(2CY) = \mathrm{tr}(\mathrm{C}(\mathrm{J} - \mathrm{ZZ}^{\mathrm{t}})) = tr(CJ) - \mathrm{tr}(\mathrm{CZZ}^{\mathrm{t}}) = \\ tr(CJ) - \mathrm{tr}((\mathrm{C} - \mathrm{D} - \mathrm{U})\mathrm{ZZ}^{\mathrm{t}}) - tr((D + U)ZZ^t) = \\ tr((L + U)ZZ^t) = n\lambda_{\max} \ . \end{gathered}$$

(ii) Using the complementary slackness (cf. part (iii) of Lemma 2.3) for (60) and (67), we have

$$x^t(\frac{1}{2}j - My) = 0 \tag{75}$$

or equivalently

$$tr(X^t(\frac{1}{2}J - Y)) = 0 \tag{76}$$

when re-written back in the notation using matrices. Since $J - 2Y$ is positive semi-definite by (60), there exist $z_1, \ldots, z_m$ such that (72) holds.

Claim. The vectors $z_1, \ldots, z_m$ are eigenvectors of $\lambda_{\min}(X)$, and $\lambda_{\min}(X) = 0$.

Let x_{ij} denote the entries of X, and $z_{i\ell}$ the entries of z_ℓ, $\ell = 1, \ldots, m$, respectively. When substituting (72) into (76), we get

$$0 = tr(X^t(\frac{1}{2}J - Y)) = \sum_{i,j} x_{ij} \left(\sum_{\ell=1}^{m} z_{i\ell} z_{j\ell} \right) = \sum_{\ell=1}^{m} z_\ell^t X z_\ell. \tag{77}$$

Since $X \succeq 0$, we have $z_\ell^t X z_\ell \geq 0$ for every ℓ, and hence $z_\ell^t X z_\ell = 0$ for every ℓ, since the sum (77) of these nonnegative terms is zero. This proves that $z_1, \ldots, z_m$ are eigenvectors of X corresponding to eigenvalue zero, which is the minimum eigenvalue of X. This proves the claim.

Since the constraints (67) and (69) are equivalent (they differ only by a diagonal shift), we conclude that $z_1, \ldots, z_m$ are eigenvectors of $\lambda_{\max}(L + \mathrm{diag}(u))$. The property (71) follows immediately from (72).

□

Corollary 4.1 *There always exists an optimality certificate $z_1, \ldots, z_m$, $m \leq t$, where the vectors z_i are pairwise orthogonal and t is the dimension of the eigenspace $Eig(\lambda_{\max}(L + U))$.*

□

Let us remark that m can be substantially smaller than t. In particular, $m = 1$ if $Eig(\lambda_{\max}(L + U)) \cap \{-1, 1\}^n \neq \emptyset$, i.e. the eigenspace contains a ± 1-vector x. Then x determines a cut of size $\varphi(G, c)$, and hence $mc(G, c) = \varphi(G, c)$. However, it is NP-hard to determine the minimum size of an optimality certificate ([6]).

The optimality certificate from [5] recalled in Theorem 4.2 is related to an optimality criterion of Overton, formulated in [12] for a more general problem. We will re-phrase this criterion.

Let (G, c) be a weighted graph, and $u \in \mathcal{U}$. Let $q_1, \ldots, q_t$ be an orthonormal basis of the eigenspace $Eig(\lambda_{\max}(L + U))$ where t denotes the dimension of the eigenspace. Let $r_1, \ldots, r_n$ denote the rows of the matrix $Q = [q_1, \ldots, q_t]$.

Theorem 4.4 (Overton's criterion.) *The vector u is an optimum correcting vector for (G, c) if and only if there exists a $t \times t$ positive semidefinite matrix W, such that $r_i^t W r_i = 1$ for every $i = 1, \ldots, n$.*

□

The connection between Overton's criterion and the optimality certificate of Theorem 4.2 is the following.

Let $\mathcal{E} = Eig(\lambda_{\max}(L+U))$. Let $z_1, \ldots, z_m \in \mathcal{E}$ be an optimality certificate, and $q_1, \ldots, q_t \in \mathcal{E}$ be an orthonormal basis of $\mathcal{E}$. Let Δ denote the $m \times t$ matrix $\Delta = (\delta_{ij})$ of coefficients such that

$$z_\ell = \sum_{j=1}^{t} \delta_{\ell j} q_j \tag{78}$$

for $\ell = 1, \ldots, m$. Let $W = (w_{ij})$ be the $t \times t$ matrix defined by

$$w_{ij} = \sum_{\ell=1}^{m} \delta_{\ell i} \delta_{\ell j}. \tag{79}$$

Theorem 4.5 *The matrix W defined by (79) is suitable for Overton's criterion formulated in Theorem 4.4. Conversely, let W be a positive semidefinite matrix given by Theorem 4.4, and $\Delta = (\delta_{ij})$ be a $t \times t$ matrix such that*

$$W = \Delta^t \Delta \ . \tag{80}$$

(The existence of Δ follows from the fact that W is positive semi-definite.) Then $z_1, \ldots, z_t$ defined by (78) constitute an optimality certificate.

□

Remark 4.1 *Let Q and W be as in Theorem 4.4. Then $Y = \frac{1}{2}(J - QWQ^t)$ is the optimum Y for the definition of $\psi(G, c)$ in (58).*

5 Computational Aspects

Two different practical approaches to the graph partition problems have been pursued so far; (i) 'polyhedral' approach, based on solving a linear relaxation, and (ii) 'eigenvalue' approach based on optimizing a convex function involving the maximum (or minimum) eigenvalue of a matrix. The theory developed in this paper suggests how to naturally merge these two approaches into a more powerful computational scheme. Let us first recall some details about the individual techniques.

5.1 Polyhedral Approach

Let (G, c) be a weighted graph and $x = (x_{ij}) \in \Re^{\binom{n}{2}}$ a vector of variables. Let $\pi(G, c)$ (π for 'polyhedral') denote the following bound.

$$\pi(G, c) := \qquad \max c^t x \tag{81}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \quad \text{for every } i < j < k \tag{82}$$

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \quad \text{for every triple } i, j, k \tag{83}$$

Clearly, $\pi(G,c) \leq mc(G,c)$ because every incidence vector $x = (x_{ij})$ of an edge-cut satisfies (82) and (83). Computational experiments with this bound are reported e.g. in [3].

Since $\pi(G,c) = mc(G,c)$ for planar graphs G (see [2]), the bound can be expected to behave well on nearly planar instances. On the other hand, it was proved in [15] that the ratio $\frac{\pi(G,c)}{mc(G,c)}$ tends to 2 for a certain class of random graphs (with $c_e = 1$ for $e \in E(G)$).

5.2 Eigenvalue Approach

The earliest experiments based on the eigenvalue approach are reported in [9]. However, the bound considered there was not the best possible for the approach. The computational experiments with the bound $\nu(G,c,0)$ (graph bisection into equal sizes) are reported in [10], and the bound $\varphi(G,c)$ on the max-cut problem is computed in [14]. Lower bounds are obtained by rounding a suitable eigenvector to a $\pm 1-$vector, and a consecutive local improvement of the cuts. The approach provides solutions with about 5% relative error between the upper bound and a cut found.

The eigenvalue bound $\varphi(G,c)$ has several interesting properties which resemble the behaviour of the actual value $mc(G,c)$ ([5, 6]). In particular, the ratio $\frac{\varphi(G,c)}{mc(G,c)}$ tends to 1 for random graphs G, i.e. the bound is asymptotically optimal. However, the worst case ratio is not yet known.

The bounds $\nu(G,c,k)$ and $\varphi(G,c)$ can be computed, for arbitrary required precision, in polynomial time by using the ellipsoid method. However, for practical experiments we have used the Bundle Trust algorithm ([18]) in combination with a Lanczos routine for computing the maximum eigenvalue. Recently, several other methods have been proposed for minimization of the maximum eigenvalue of a parametrized matrix ([1, 13]), but their practical efficiency has not yet been investigated thoroughly.

5.3 Semi-infinite programs

Our main theorem opens new ways to derive even tighter relaxations of the graph bisection problem, by combining the polyhedral approach relying on a (partial) description of the cut polytope with the semi-infinite edge relaxation introduced in this paper. Specifically, we propose to merge the above approaches in the following semi-infinite program, which presents a lower bound on the bisection $b(G,c,k)$.

$$\min c^t x \tag{84}$$

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \quad \leq \quad \frac{1}{4}(\sum_{i=1}^{n} b_i)^2 \quad \text{for every } b_1, \ldots, b_n \tag{85}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \quad \text{for every } i,j,k \tag{86}$$

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \quad \text{for every } i,j,k \tag{87}$$

$$\sum_{1 \leq i < j \leq n} x_{ij} = \frac{n^2 - k^2}{4} \tag{88}$$

The program (84),(85), and (88) computes $\eta(G,c,k)$ since constraints (85) are equivalent to $\frac{1}{2}J - X \succeq 0$. Let us also remark that (85) can be efficiently tested by computing the minimum eigenvalue of $\frac{1}{2}J - X$, and if the eigenvalue is negative, then its eigenvector presents a b for which the constraint is violated.

5.4 Hypermetric and gap inequalities

For a particular choice of a vector b, a class of stronger inequalities than (85) can be considered. Let $b_1, \ldots, b_n$ be integers with $\sum_{i=1}^n b_i$ odd. Let us consider the following inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{1}{4}((\sum_{i=1}^n b_i)^2 - 1) \tag{89}$$

It is not difficult to see that the inequalities (89) are valid for the edge-cut incidence vectors. (*Proof.* Let $x = (x_{ij})$ be the edge-cut incidence vector of a partition $(S, V \setminus S)$. Then, say, $\sum_{i \in S} b_i \leq \frac{1}{2}(\sum_{i=1}^n b_i - 1)$, and $\sum_{i \notin S} b_i \geq \frac{1}{2}(\sum_{i=1}^n b_i + 1)$. Hence $\sum b_i b_j x_{ij} = \sum_{i \in S} \sum_{j \notin S} b_i b_j = (\sum_{i \in S} b_i)(\sum_{j \notin S} b_j) \leq \frac{1}{4}((\sum_{i=1}^n b_i)^2 - 1)$.)

A more general class of related inequalites, called *gap inequalities*, was proposed in [11]. Let $b_1, \ldots, b_n$ be integers, and γ be the maximum integer such that, for any partition of V into S and $V \setminus S$, the difference between the sums $(\sum_{i \in S} b_i)$ and $(\sum_{j \notin S} b_j)$ is at least γ. (For example $\gamma \geq 1$ if $\sum_{i=1}^n b_i$ is odd.) Then the inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{1}{4}((\sum_{i=1}^n b_i)^2 - \gamma) \tag{90}$$

is valid for all edge-cut incidence vectors by an argument similar to that above for the inequalities (89).

Clearly, every inequality (89) is dominated by a gap- inequality. However, the class (89) might be easier to handle, since it is NP-complete to determine the gap γ for given $b_1, \ldots, b_n$.

The gap-inequalities with $\gamma = 1$ are called the *hypermetric inequalities*, and have been quite intensively studied in the literature. An important theoretical result about the hypermetric inequalities was proved in [8], telling that, for every n, the hypermetric inequalities define a polytope.

The use of hypermetric inequalities in max-cut computation, and a heuristic search for violated ones, was proposed by G. Rinaldi and C. De Simone [16] at the workshop on graph partition problems at Rome 1991.

The class of inequalities (89) contains all hypermetric inequalities. We propose to search for possible violated inequalitites (89) in the 'neighbourhood' of the eigenspace of $\lambda_{\min}(\frac{1}{2}J - X)$, but details will be elaborated elsewhere.

Acknowledgement. We thank Monique Laurent, Chun-Wa Ko and Lex Schrijver for a discussion and helpful comments. In particular, we are indebted to Henry Wolkowicz for pointing out our partly uncorrect use of duality in an earlier version of the manuscript.

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