Information-theoretic Issues Concerning Broadcasting: Capacity Region Outer Bounds

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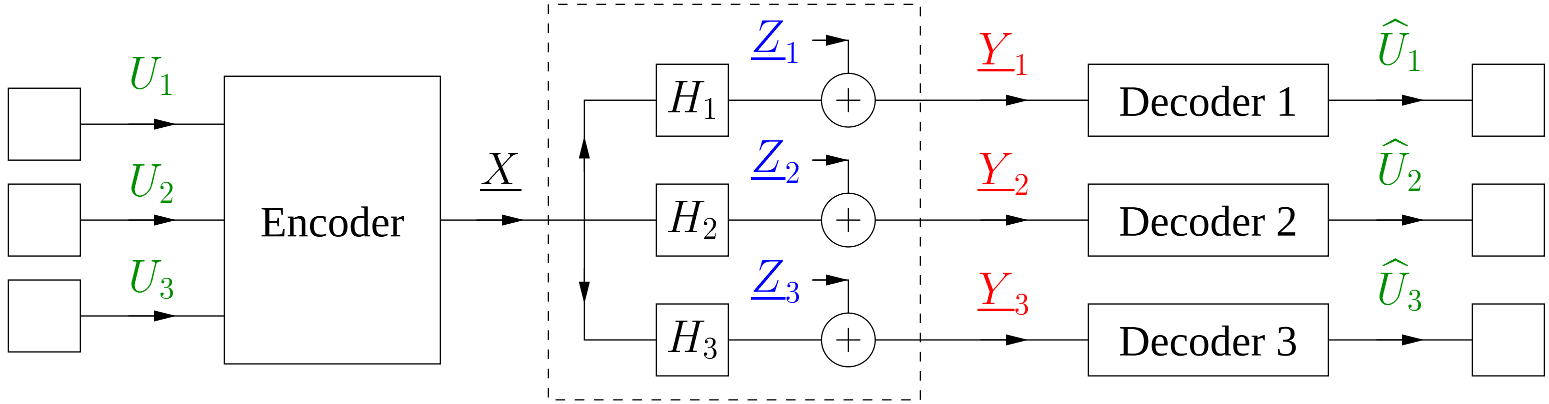
Outline:

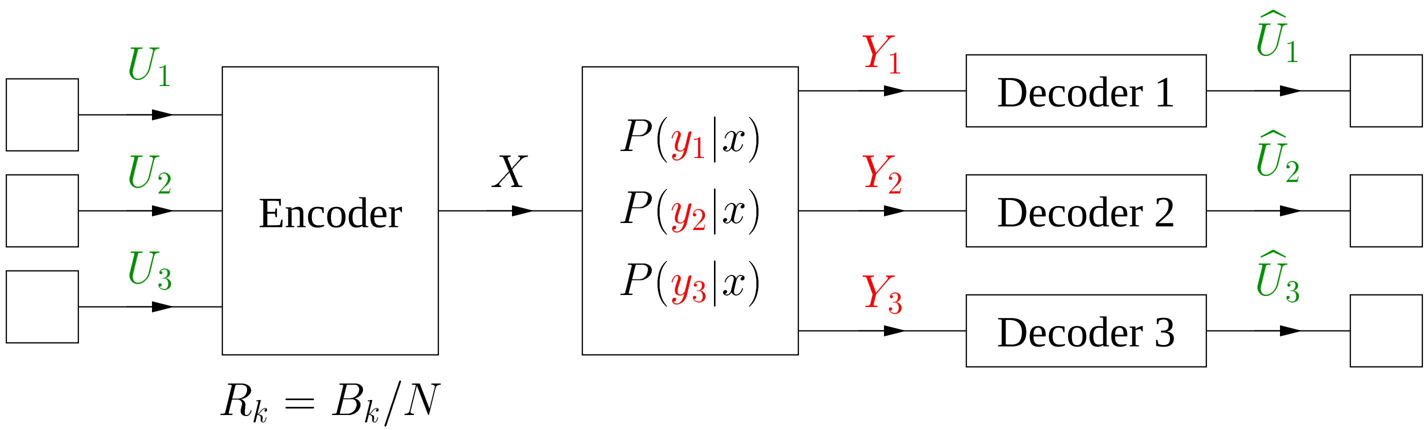
1. A capacity region outer bound for the general (discrete memoryless, nice continuous) broadcast channel
2. A capacity region outer bound for the Gaussian vector broadcast channel

The Gaussian Vector Broadcast Channel



- Problem: Find the set $\mathcal{C}$ of (R_1, R_2, R_3) for which there is an encoder and decoders such that $\Pr(U_k \neq \widehat{U}_k) < \epsilon$ for $k = 1, 2, 3$ and any $\epsilon > 0$.
- Partial results to date: best sum rate is known

1. A Capacity Region Outer Bound for the General Broadcast Channel



- Cover (1972): $\mathcal{C}$ depends only on the marginals of $P(y_1, y_2, y_3|x)$
- Cover's outer bound: $\mathcal{C}$ is in the convex hull of (R_1, R_2, R_3) satisfying

$$0 \leq R_1 \leq I(X;Y_1)$$
$$0 \leq R_2 \leq I(X;Y_2)$$
$$0 \leq R_3 \leq I(X;Y_3)$$

for some P_X

Sato's Outer Bound

- Add the co-operative bound

$$R_1 + R_2 + R_3 \leq I(X; Y_1\, Y_2\, Y_3)$$

  and consider the convex hull of regions obtained with different P_X

- Take the intersection over channels $\pi(y_1, y_2, y_3|x)$ having the same marginals as $P(y_1, y_2, y_3|x)$

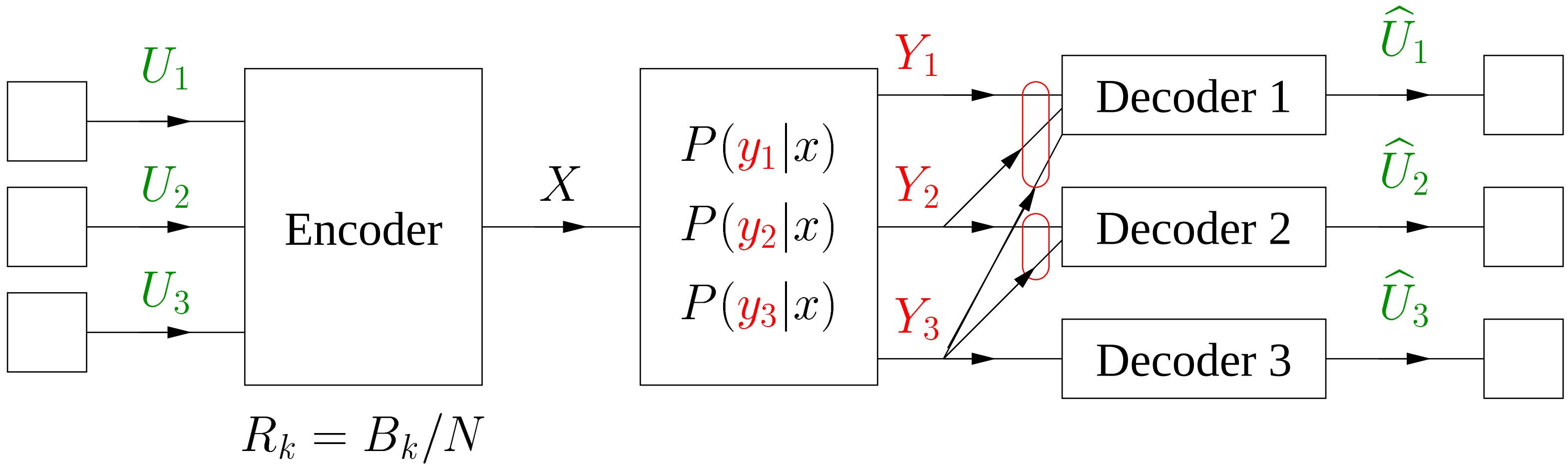
- Could call this the Co-operative, Same Marginals (CSM) bound

A Stronger Outer Bound

- make the channel physically degraded as follows:
  - ◇ receiver 1 gets $\pi(y_1, y_2, y_3|x)$ having marginals of $P(y_1, y_2, y_3|x)$
  - ◇ receiver 2 gets $\pi(y_2, y_3|x)$
  - ◇ receiver 3 gets $\pi(y_3|x) = P(y_3|x)$
- Capacity region is the convex hull of the (R_1, R_2, R_3) satisfying

$$\begin{aligned} 0 &\le R_1 \le I(X; Y_1 Y_2 Y_3 | V_2 V_3) \\ 0 &\le R_2 \le I(V_2; Y_2 Y_3 | V_3) \\ 0 &\le R_3 \le I(V_3; Y_3) \end{aligned}$$

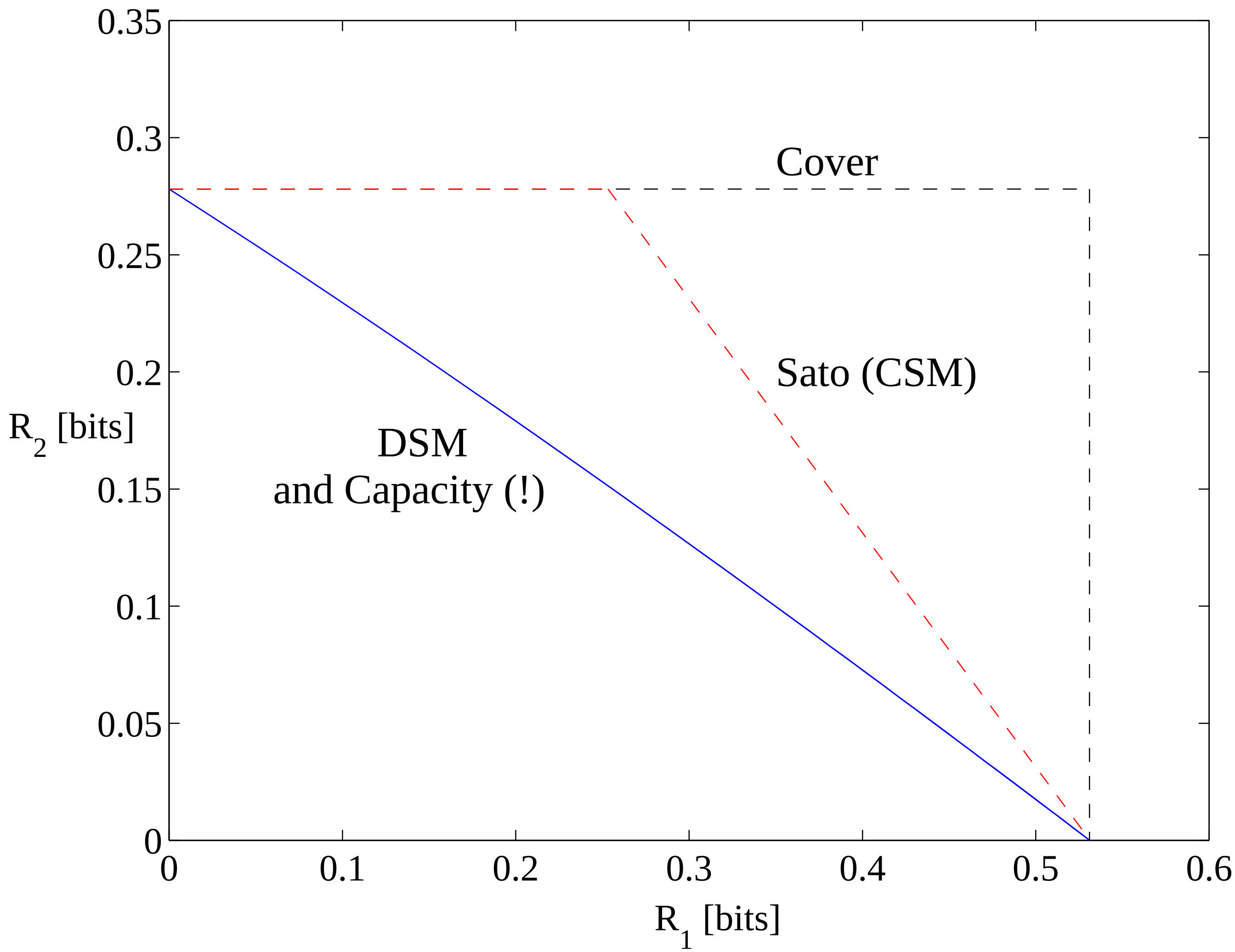
  where $[V_2, V_3] - X - [Y_1, Y_2, Y_3]$ forms a Markov chain
- Take the intersection over the $\pi(y_1, y_2, y_3|x)$
  Take the intersection over the $3!$ degraded orderings
- Call this the Degraded, Same Marginals (DSM) bound



U_1
U_2
U_3
Encoder
$R_k = B_k/N$
X
$P(y_1|x)$
$P(y_2|x)$
$P(y_3|x)$
Y_1
Y_2
Y_3
Decoder 1
Decoder 2
Decoder 3
$\widehat{U}_1$
$\widehat{U}_2$
$\widehat{U}_3$

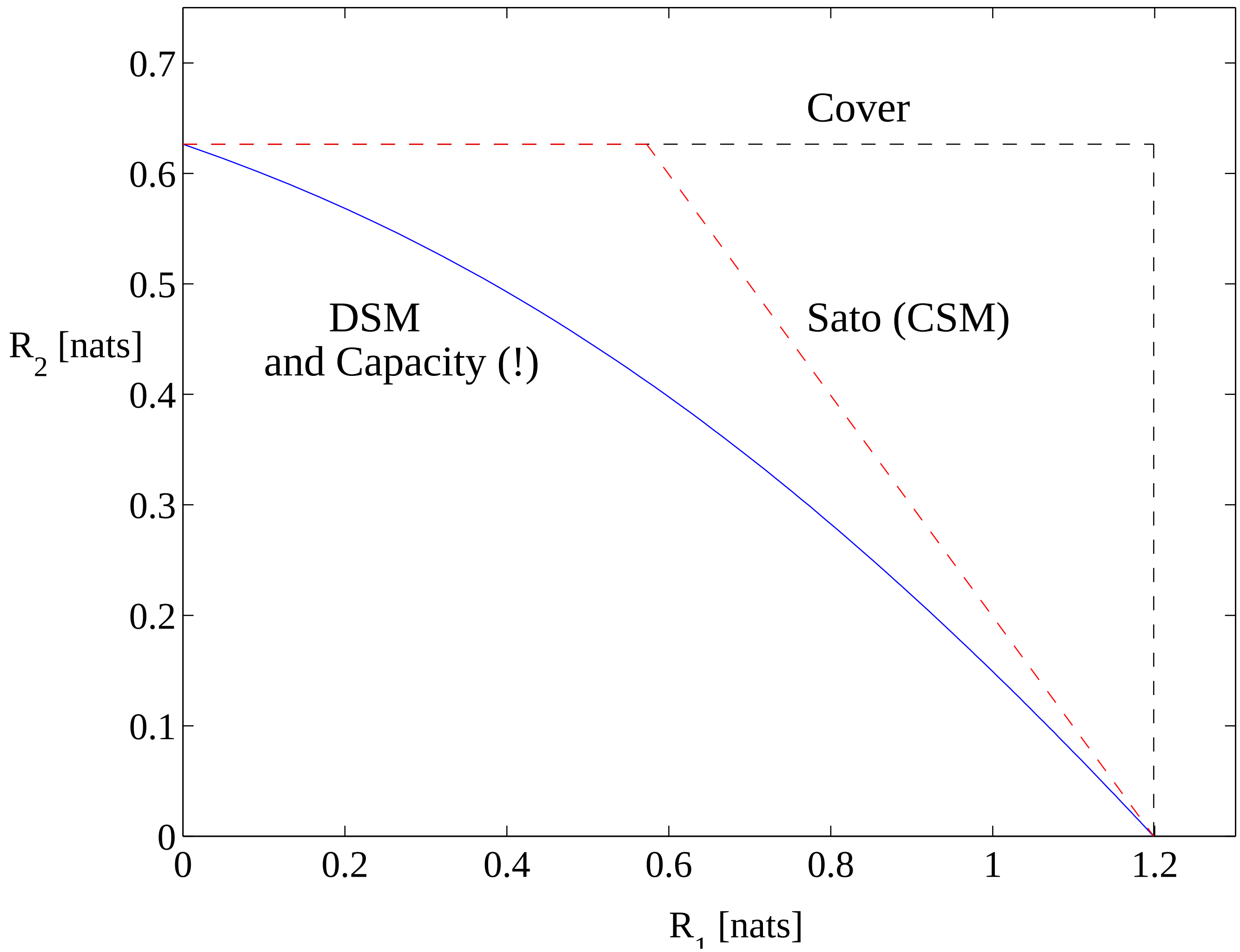
Example: A Binary Symmetric Broadcast Channel

- 2 receivers, $Y_k = X \oplus Z_k$ where $\Pr(Z_1 = 1) = 0.1, \Pr(Z_2 = 1) = 0.2$

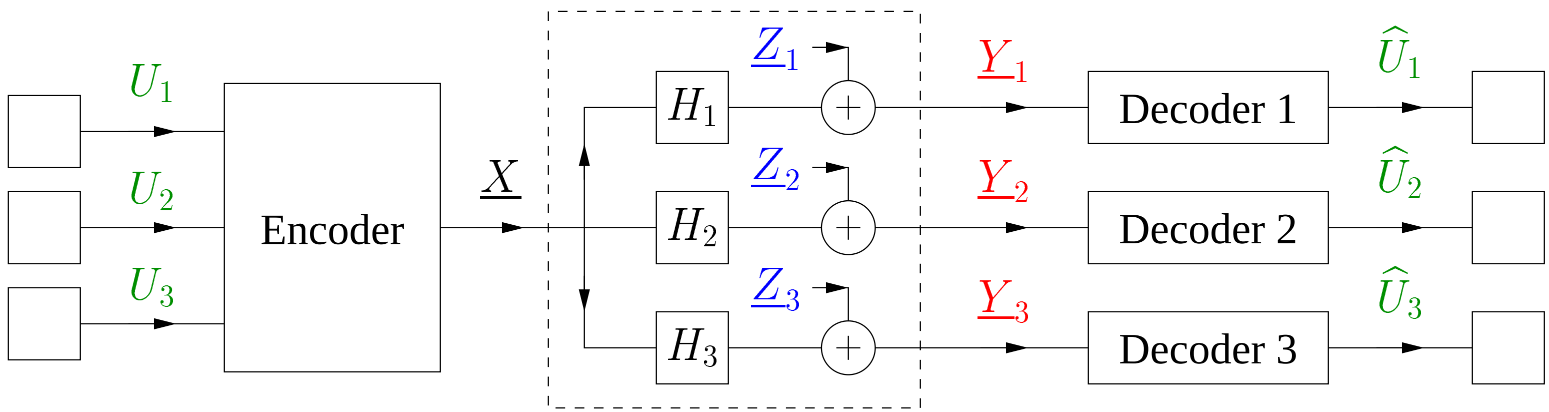


Example: A Gaussian Scalar Broadcast Channel

- 2 receivers, $Y_k = h_k X + Z_k$ where $h_1 = 1, h_2 = 0.5$ and Z_k is zero-mean, unit variance Gaussian noise, $P = \mathrm{E}[\mathrm{X}^2] \leq 10$



2. Application to the Gaussian Vector Broadcast Channel



- DSM Model:

$$\underline{Y}_{\pi 1} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \underline{X} + \underline{Z}_{\pi 1}, \quad \underline{Y}_{\pi 2} = \begin{bmatrix} H_2 \\ H_3 \end{bmatrix} \underline{X} + \underline{Z}_{\pi 2}, \quad \underline{Y}_{\pi 3} = \underline{Y}_3$$

where $\underline{Z}_{\pi 1}$ and $\underline{Z}_{\pi 2}$ have “the same marginals”

- Example: suppose $\underline{Y}_1, \underline{Y}_2, \underline{Y}_3$ have respective lengths $2, 3, 1$. Then

$$Z_{\pi 1} = \mathrm{E}[\underline{Z}_{\pi 1}\underline{Z}_{\pi 1}^{\dagger}] = \left[\begin{array}{cc|ccc|c} 1 & 0 & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 1 & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{13}^* & a_{23}^* & 1 & 0 & 0 & a_{36} \\ a_{14}^* & a_{24}^* & 0 & 1 & 0 & a_{46} \\ a_{15}^* & a_{25}^* & 0 & 0 & 1 & a_{56} \\ \hline a_{16}^* & a_{26}^* & a_{36}^* & a_{46}^* & a_{56}^* & 1 \end{array}\right]$$

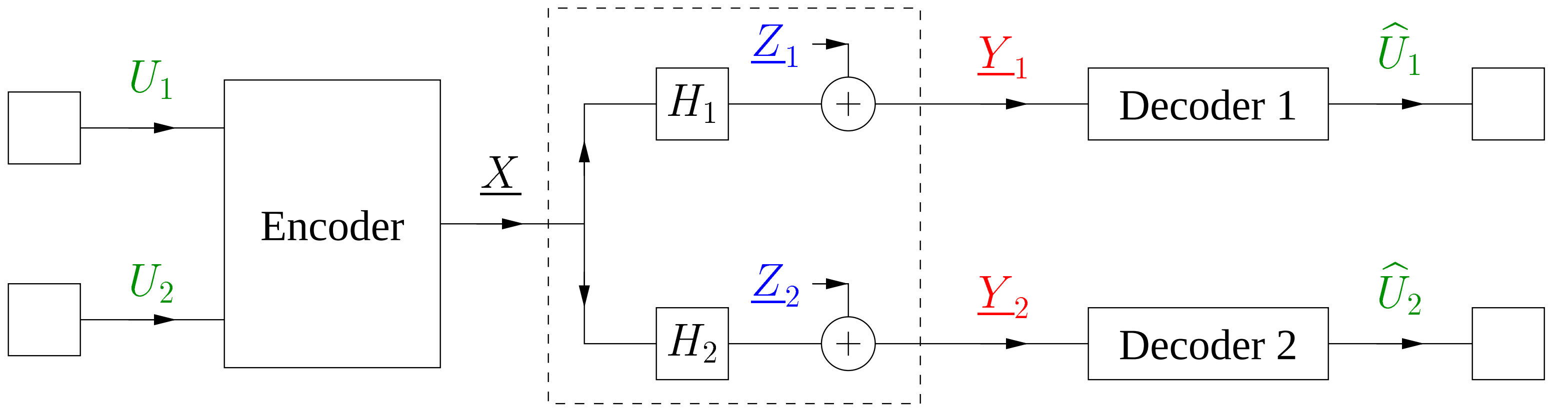
$$Z_{\pi 2} = \mathrm{E}[\underline{Z}_{\pi 2}\underline{Z}_{\pi 2}^{\dagger}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{36} \\ 0 & 1 & 0 & a_{46} \\ 0 & 0 & 1 & a_{56} \\ \hline a_{36}^* & a_{46}^* & a_{56}^* & 1 \end{array}\right]$$

- We know that the capacity region has the form

$$
\begin{aligned}
0 \le R_1 &\le I(\underline{X}; \underline{Y}_{\pi 1} | \underline{V}_2 \underline{V}_3) \\
0 \le R_2 &\le I(\underline{V}_2; \underline{Y}_{\pi 2} | \underline{V}_3) \\
0 \le R_3 &\le I(\underline{V}_3; \underline{Y}_{\pi 3})
\end{aligned}
$$

- Question: are Gaussian $\underline{X}, \underline{V}_2, \underline{V}_3$ optimal ?
  Approaches:
  - ◇ try using Bergmans' approach with the entropy-power inequality: problem is how to generalize ?
  - ◇ try working directly with the above bounds:
    fails because there are $\underline{V}_k$ in the informations
  - ◇ other bounding techniques ?

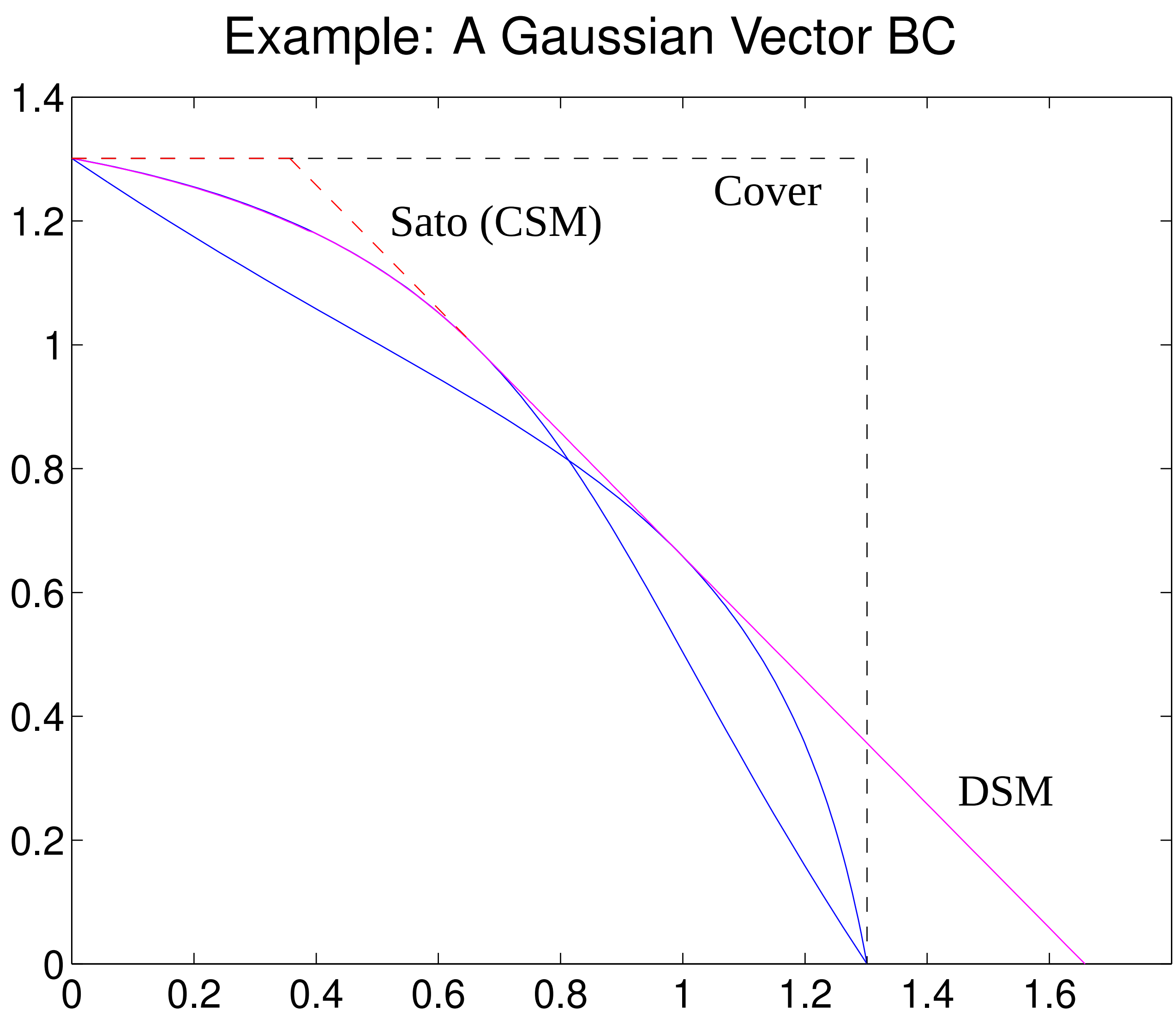
Example: A Gaussian Vector BC



- Channels: $H_1 = [1 \;\; 0.5]^T$ and $H_2 = [0.5 \;\; 1]^T$
- DSM Outer Bound with receiver 1 getting more information:

$$R_1 \leq \frac{1}{2}\log\left| I + Z_{\pi 1}^{-1} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} Q_1 \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right|$$

$$R_2 \leq \frac{1}{2}\log\left| I + \left(I + H_2 Q_1 H_2^{\dagger} \right)^{-1} H_2 Q_2 H_2^{\dagger} \right|$$


Example: A Gaussian Vector BC
Cover
Sato (CSM)
DSM
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Summary

- New results:
  - an outer bound (DSM) for general broadcast channels
  - dirty paper coding is optimal for Gaussian vector broadcast channels with Gaussian input distributions
    (see later talk)