An application of wavelet series to Brownian motion

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Let $\{\lambda_n\}$ be any orhonormal basis in $L^2(0,1)$, let

$$\Lambda_n(t) = \int_0^t \lambda_n(x)dx$$

and let (X_n) be a sequence of independent, identically distributed standard normal random variables. By Ito-Nisio theorem the series

$$\sum_{n=1}^{\infty} \Lambda_n(t) X_n$$

is almost surely uniformly convergent and the limit is the Brownian motion process. Among all such representations of the Brownian motion process the Haar basis representation exhibits special properties. It is the only basis for which the series is finite on the dyadic rationals and the filtration at time t is generated by a sequence of independent, identically distributed standard normal random variables that unfolds with time t.

Using the Haar representation of Brownian motion, which is also known as Levy-Ciesielski representation, we have obtained a simple and insightful proof of both Levy's modulus of continuity and the law of iterated logarithms with additional new results. Specifically, for $\varepsilon > 0$ and $\delta < 1/2$, we have determined the confidence interval for the maximal deviation of increments of a Brownian motion process (W_t), that is, we have shown

$$P(\sup_{0 \leq |t-s \leq \delta|} |W_t - W_s| \leq g(\delta) f(\varepsilon, \delta)) \geq h(\varepsilon, \delta)$$

where $g(t) = \sqrt{2t \ln \frac{1}{t}}$ is the modulus of continuity function for the standard Brownian motion and f and h can be written explicitly. Using the same method we have proved the Ȟinčin's law of iterated logarithms for the standard Brownian motion, with the same type of gains, that is

$$P(\sup_{0 \leq |t-s \leq \delta|} |W_t - W_s| \leq g(\delta) k(\varepsilon, \delta)) \geq l(\varepsilon, \delta)$$

where $g(t) = \sqrt{2t \ln \ln \frac{1}{t}}$and k and l can be written explicitly in a closed form. In both cases the Levy-Ciesielski representation reveals how and why the logarithmic terms appear in the corresponding g functions. Based on those results and an appropriate wavelet expansion for the drift term, we have found an early warning method for volatility changes in a stock price if the price is modeled by the eponential Brownian motion. We have also proved that our method, within the same confidence interval, detects sooner then the existing methods the volatility changes.

Furthermore we have introduced a new stock price model based on the exponential finite Levy-Ciesielski representation and developed the option pricing theory in this new setting. This approach reveals a path to a more general "wavelet based method" for modeling stock prices.