All Rational Polytopes Are Transportation Polytopes and All Polytopal Integer Sets Are Contingency Tables

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Abstract. We show that any rational polytope is polynomial-time representable as a "slim" $r \times c \times 3$ three-way line-sum transportation polytope. This universality theorem has important consequences for linear and integer programming and for confidential statistical data disclosure. It provides polynomial-time embedding of arbitrary linear programs and integer programs in such slim transportation programs and in bipartite biflow programs. It resolves several standing problems on 3-way transportation polytopes. It demonstrates that the range of values an entry can attain in any slim 3-way contingency table with specified 2-margins can contain arbitrary gaps, suggesting that disclosure of k-margins of d-tables for $2 \leq k < d$ is confidential.
Our construction also provides a powerful tool in studying concrete questions about transportation polytopes and contingency tables; remarkably, it enables to automatically recover the famous "real-feasible integer-infeasible" $6 \times 4 \times 3$ transportation polytope of M. Vlach, and to produce the first example of 2-margins for $6 \times 4 \times 3$ contingency tables where the range of values a specified entry can attain has a gap.

1 Introduction

Transportation polytopes, their integer points (contingency tables), and their projections, have been used and studied extensively in the operations research and mathematical programming literature, see e.g. [1,2,3,12,14,17,18,21,22] and references therein, and in the context of secure statistical data management by agencies such as the U.S. Census Bureau [20], see e.g. [5,6,9,10,13,16] and references therein.

We start right away with the statement of the main theorem of this article, followed by a discussion of some of its consequences for transportation polytopes,

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linear and integer programming, bipartite biflows, and confidential statistical data disclosure. Throughout, $\mathbb{R}_+$ denotes the set of nonnegative reals.

Theorem 1. *Any rational polytope $P = \{y \in \mathbb{R}^n_+ : Ay = b\}$ is polynomial-time representable as a "slim" (of smallest possible depth three) 3-way transportation polytope*

$$T \quad = \quad \{\, x \in \mathbb{R}^{r\times c\times 3}_+ \ : \ \sum_i x_{i,j,k} = w_{j,k}\,,\ \sum_j x_{i,j,k} = v_{i,k}\,,\ \sum_k x_{i,j,k} = u_{i,j}\,\}\ .$$

By saying that a polytope $P \subset \mathbb{R}^p$ is *representable* as a polytope $Q \subset \mathbb{R}^q$ we mean in the strong sense that there is an injection $\sigma : \{1, \dots, p\} \longrightarrow \{1, \dots, q\}$ such that the coordinate-erasing projection

$$\pi : \mathbb{R}^q \longrightarrow \mathbb{R}^p \ : \ x = (x_1, \dots, x_q) \mapsto \pi(x) = (x_{\sigma(1)}, \dots, x_{\sigma(p)})$$

provides a bijection between Q and P and between the sets of integer points $Q \cap \mathbb{Z}^q$ and $P \cap \mathbb{Z}^p$. In particular, if P is representable as Q then P and Q are isomorphic in any reasonable sense: they are linearly equivalent and hence all linear programming related problems over the two are polynomial-time equivalent; they are combinatorially equivalent hence have the same facial structure; and they are integer equivalent and hence all integer programming and integer counting related problems over the two are polynomial-time equivalent as well.

The polytope T in the theorem is a 3-way transportation polytope with specified *line-sums* $(u_{i,j}), (v_{i,j}), (w_{i,j})$ (2-*margins* in the statistical context to be elaborated upon below). The arrays in T are of size $(r, c, 3)$, that is, they have r rows, c columns, and "slim" depth 3, which is best possible: 3-way line-sum transportation polytopes of depth ≤ 2 are equivalent to ordinary 2-way transportation polytopes which are not universal.

An appealing feature of the representation manifested by Theorem 1 is that the defining system of T has only $\{0, 1\}$-valued coefficients and depends only on r and c. Thus, every rational polytope has a representation by one such system, where all information enters through the right-hand-side data $(u_{i,j}), (v_{i,j}), (w_{i,j})$.

We now discuss some of the consequences of Theorem 1.

1.1 Universality of Transportation Polytopes: Solution of the Vlach Problems

As mentioned above, there is a large body of literature on the structure of various transportation polytopes. In particular, in the comprehensive paper [21], M. Vlach surveys some ten families of necessary conditions published over the years by several authors (including Schell, Haley, Smith, Moràvek and Vlach) on the line-sums $(u_{i,j}), (v_{i,j}), (w_{i,j})$ for a transportation polytope to be nonempty, and raises several concrete problems regarding these polytopes. Specifically, [21, Problems 4,7,9,10] ask about the sufficiency of some of these conditions. Our results show that transportation polytopes (in fact already of slim $(r, c, 3)$ arrays) are universal and include all polytopes. This indicates that the answer to each of Problems 4,7,9,10 is negative. Details will appear elsewhere. Problem 12 asks whether all dimensions can occur as that of a suitable transportation polytope: the affirmative answer, given very recently in [10], follows also at once from our universality result.

Our construction also provides a powerful tool in studying concrete questions about transportation polytopes and their integer points, by allowing to write down simple systems of equations that encode desired situations and lifting them up. Here is an example to this effect.

Example 1. Vlach's rational-nonempty integer-empty transportation: using our construction, we automatically recover the smallest known example, first discovered by Vlach [21], of a rational-nonempty integer-empty transportation polytope, as follows. We start with the polytope $P = \{y \geq 0 : 2y = 1\}$ in one variable, containing a (single) rational point but no integer point. Our construction represents it as a transportation polytope T of $(6,4,3)$-arrays with line-sums given by the three matrices below; by Theorem 1, T is integer equivalent to P and hence also contains a (single) rational point but no integer point.

$$(u_{i,j}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad (v_{i,k}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad (w_{j,k}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Returning to the Vlach problems, [21, Problem 13] asks for a characterization of line-sums that guarantees an integer point in T. In [13], Irving and Jerrum show that deciding $T \cap \mathbb{Z}^{r \times c \times h} \neq \emptyset$ is NP-complete, and hence such a characterization does not exist unless P=NP. Our universality result strengthens this to *slim* arrays of smallest possible constant depth 3; we have the following immediate corollary of Theorem 1.

Corollary 1. *Deciding if a slim $r \times c \times 3$ transportation polytope has an integer point is NP-complete.*

A comprehensive complexity classification of this decision problem under various assumptions on the array size and on the input, as well as of the related counting problem and other variants, are in [4].

The last Vlach problem [21, Problem 14] asks whether there is a *strongly-polynomial-time* algorithm for deciding the (real) feasibility $T \neq \emptyset$ of a transportation polytope. Since the system defining T is $\{0,1\}$-valued, the results of Tardos [19] provide an affirmative answer. However, the existence of a strongly-polynomial-time algorithm for linear programming in general is open and of central importance; our construction embeds any linear program in an $r \times c \times 3$ transportation program in polynomial-time, but unfortunately this process is *not* strongly-polynomial. Nonetheless, our construction may shed some light on the problem and may turn out useful in sharpening the boundary (if any) between strongly and weakly polynomial-time solvable linear programs.

1.2 Universality of Bipartite Biflows

Another remarkable immediate consequence of Theorem 1 is the universality of the two-commodity flow (biflow) problem for bipartite digraphs where all arcs are oriented the same way across the bipartition. The problem is the following: given *supply* vectors $s^1, s^2 \in \mathbb{R}^r_+$, *demand* vectors $d^1, d^2 \in \mathbb{R}^c_+$, and *capacity* matrix $u \in \mathbb{R}^{r\times c}_+$, find a pair of nonnegative "flows" $x^1, x^2 \in \mathbb{R}^{r\times c}_+$ satisfying supply and demand requirements $\sum_j x^k_{i,j} = s^k_i$, $\sum_i x^k_{i,j} = d^k_j$, $k = 1, 2$, and capacity constraints $x^1_{i,j} + x^2_{i,j} \leq u_{i,j}$. In network terminology, start with the directed bipartite graph with vertex set $R \uplus C$, $|R| = r$, $|C| = c$, and arc set $R \times C$ with capacities $u_{i,j}$, and augment it with two sources a^1, a^2 and two sinks b^1, b^2, and with arcs (a^k, i), $i \in R$, (j, b^k), $j \in C$, $k = 1, 2$ with capacities $u(a^k, i) := s^k_i$, $u(j, b^k) := d^k_j$. We have the following universality statement.

Corollary 2. *Any rational polytope $P = \{y \in \mathbb{R}^n_+ : Ay = b\}$ is polynomial-time representable as a bipartite biflow polytope*

$$F = \{ (x^1, x^2) \in \mathbb{R}^{r\times c}_+ \oplus \mathbb{R}^{r\times c}_+ \ : \ x^1_{i,j} + x^2_{i,j} \leq u_{i,j} \ , \quad \sum_j x^k_{i,j} = s^k_i \ , \ \sum_i x^k_{i,j} = d^k_j \ , \quad k = 1, 2 \} \ .$$

Proof. By an easy adjustment of the construction in §2.3 used in the proof of Theorem 3 (see §2.3 for details): take the capacities to be $u_{I,J}$ as defined in §2.3; take the supplies to be $s^1_I := V_{I,1} = U$ and $s^2_I := V_{I,3} = U$ for all I; and take the demands to be $d^1_J := W_{J,1}$ and $d^2_J := W_{J,3}$ for all J. Note that by taking s^2_I and d^2_J to be $V_{I,3}$ and $W_{J,3}$ instead of $V_{I,2}$ and $W_{J,2}$ we get that all supplies have the same value U, giving a stronger statement. □

As noted in the proof of Corollary 2, it remains valid if we take all supplies to have the same value $s^k_i = U$, $i = 1, \ldots, r$, $k = 1, 2$; further (see discussion in the beginning of §2.3 on encoding forbidden entries by bounded entries), all capacities $u_{i,j}$ can be taken to be $\{0, U\}$-valued, giving a stronger statement.

Moreover, as the next example shows, using our construction we can systematically produce simple bipartite biflow acyclic networks of the above form, with integer supplies, demands and capacities, where any feasible biflow must have an arbitrarily large prescribed denominator, in contrast with Hu's celebrated half-integrality theorem for the undirected case [11].

Example 2. Bipartite biflows with arbitrarily large denominator: Fix any positive integer q. Start with the polytope $P = \{y \geq 0 : qy = 1\}$ in one variable containing the single point $y = \frac{1}{q}$. Our construction represents it as a bipartite biflow polytope F with integer supplies, demands and capacities, where y is embedded as the flow $x^1_{1,1}$ of the first commodity from vertex $1 \in R$ to $1 \in C$. By Corollary 2, F contains a single biflow with $x^1_{1,1} = y = \frac{1}{q}$. For $q = 3$, the data for the biflow problem is below, resulting in a unique, $\{0, \frac{1}{3}, \frac{2}{3}\}$-valued, biflow.

$$(u_{i,j}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad (s_i^1) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad (s_i^2) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$(d_j^1) = (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0), \quad (d_j^2) = (0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 2 \ 1).$$

By Corollary 2, any (say, feasibility) linear programming problem can be encoded as such a bipartite biflow problem (unbounded programs can also be treated by adding to the original system a single equality $\sum_{j=0}^{n} y_j = U$ with y_0 a new "slack" variable and U derived from the Cramer's rule bound). Thus, any (hopefully combinatorial) algorithm for such bipartite biflow will give an algorithm for general linear programming. There has been some interest lately [15] in combinatorial approximation algorithms for (fractional) multiflows, e.g. [7,8]; these yield, via Corollary 2, approximation algorithms for general linear programming, which might prove a useful and fast solution strategy in practice. Details of this will appear elsewhere.

1.3 Confidential Statistical Data Disclosure: Models, Entry-Range, and Spectrum

A central goal of statistical data management by agencies such as the U.S. Census Bureau is to allow public access to as much as possible information in their data base. However, a major concern is to protect the confidentiality of individuals whose data lie in the base. A common practice [5], taken in particular by the U.S. Census Bureau [20], is to allow the release of some margins of tables in the base but not the individual entries themselves. The security of an entry is closely related to the range of values it can attain in any table with the fixed released collection of margins: if the range is "simple" then the entry may be exposed, whereas if it is "complex" the entry may be assumed secure.

In this subsection only, we use the following notation, which is common in statistical applications. A *d-table of size* $n = (n_1, \ldots, n_d)$ is an array of non-negative integers $x = (x_{i_1,\ldots,i_d})$, $1 \leq i_j \leq n_j$. For any $0 \leq k \leq d$ and any k-subset $J \subseteq \{1, \ldots, d\}$, the *$k$-margin* of x corresponding to J is the k-table $x^J := (x^J_{i_j : j \in J}) := (\sum_{i_j : j \notin J} x_{i_1,\ldots,i_d})$ obtained by summing the entries over all indices *not in* J. For instance, the 2-margins of a 3-table $x = (x_{i_1,i_2,i_3})$ are its *line-sums* x^{12}, x^{13}, x^{23} such as $x^{13} = (x^{13}_{i_1,i_3}) = (\sum_{i_2} x_{i_1,i_2,i_3})$, and its 1-margins are its *plane-sums* x^1, x^2, x^3 such as $x^2 = (x^2_{i_2}) = (\sum_{i_1,i_3} x_{i_1,i_2,i_3})$.

A *statistical model* is a triple $\mathcal{M} = (d, \mathcal{J}, n)$, where $\mathcal{J}$ is a set of subsets of $\{1, \ldots, d\}$ none containing the other and $n = (n_1, \ldots, n_d)$ is a tuple of positive integers. The model dictates the collection of margins for d-tables of size n to be specified. The models of primary interest in this article are $(3, \{12, 13, 23\}, (r, c, h))$, that is, 3-tables of size (r, c, h) with all three of their 2-margins specified.

Fix any model $\mathcal{M} = (d, \mathcal{J}, n)$. Any specified collection $u = (u^J : J \in \mathcal{J})$ of margins under this model gives rise to a corresponding set of *contingency tables* with these margins (where $\mathbb{N}$ is the set of nonnegative integers),

$$C(\mathcal{M}; u) \quad := \quad \{ x \in \mathbb{N}^{n_1 \times \cdots \times n_d} \ : \ x^J = u^J \ , \ J \in \mathcal{J} \} \ .$$

Clearly, this set is precisely the set of integer points in the corresponding transportation polyhedron.

We make several definitions related to the range a table entry can assume. Permuting coordinates if necessary, we may always consider the first entry $x_{\mathbf{1}}$, where $\mathbf{1} := (1, \ldots, 1)$. The *entry-range* of a collection of margins u under model $\mathcal{M}$ is the set $R(\mathcal{M}; u) := \{x_{\mathbf{1}} : x \in C(\mathcal{M}; u)\} \subset \mathbb{N}$ of values $x_{\mathbf{1}}$ can attain in any table with these margins. The *spectrum* of a model $\mathcal{M}$ is the set of all entry-ranges of all collections of margins under $\mathcal{M}$,

$$\mathrm{Spec}(\mathcal{M}) := \{R(\mathcal{M}; u) \, : \, u = (u^J : J \in \mathcal{J}) \text{ some margin collection under } \mathcal{M}\} \, .$$

The *spectrum of a class* $\mathcal{C}$ of models is the union $\mathrm{Spec}(\mathcal{C}) := \bigcup_{\mathcal{M} \in \mathcal{C}} \mathrm{Spec}(\mathcal{M})$ of spectra of its models.

The following proposition characterizes the spectra of all-1-margin models and show they always consist precisely of the collection of all intervals, hence "simple". The proof is not hard and is omitted.

Proposition 1. *Let $\mathcal{C}_d^1$ be the class of models $(d, \{1, 2, \ldots, d\}, (n_1, \ldots, n_d))$ of all-1-margins of d-tables. Then for all $d \geq 2$, the spectrum of $\mathcal{C}_d^1$ is the set of intervals in $\mathbb{N}$, Spec$(\mathcal{C}_d^1) = \{[l, u] := \{r \in \mathbb{N} : l \leq r \leq u\} : l, u \in \mathbb{N}\}$.*

An important consequence of our Theorem 1 is the following strikingly surprising result, in contrast with Proposition 1 and with recent attempts by statisticians to better understand entry behavior of slim 3-tables (see [5] and references therein). It says that the class of models of all-2-margin slim 3-tables is *universal* - its spectrum consists of all finite subsets of $\mathbb{N}$ - and hence "complex" and presumably secure.

Corollary 3. *Let $\mathcal{S}_3^2$ be the class of models $(3, \{12, 13, 23\}, (r, c, 3))$ of all-2-margins of slim 3-tables. Then $\mathcal{S}_3^2$ is* universal*, namely, its spectrum Spec$(\mathcal{S}_3^2)$ is the set of all finite sets of nonnegative integers.*

Proof. Let $S = \{s_1, \ldots, s_n\} \subset \mathbb{N}$ be any finite set of nonnegative integers. Consider the polytope

$$P \quad := \quad \{y \in \mathbb{R}_+^{n+1} \ : \ y_0 - \sum_{j=1}^{n} s_j \cdot y_j = 0 \, , \ \sum_{j=1}^{n} y_j = 1 \} \ .$$

By Theorem 1, there are r, c and 2-margins $u = (u^{12}, u^{13}, u^{23})$ for the model $\mathcal{M} := (3, \{12, 13, 23\}, (r, c, 3))$, such that the corresponding line-sum slim transportation polytope T represents P. By suitable permutation of coordinates, we can assume that the injection σ giving that representation satisfies $\sigma(0) =$

$(1,1,1)$, embedding y_0 as $x_{\mathbf{1}} = x_{1,1,1}$. The set $C(\mathcal{M};u)$ of contingency tables is the set of integer points in T, and by Theorem 1, T is integer equivalent to P. The corresponding entry-range is therefore

$$\begin{aligned} R(\mathcal{M};u) &= \{x_{\mathbf{1}} \,:\, x \in C(\mathcal{M};u)\} \\ &= \{x_{\mathbf{1}} \,:\, x \in T \cap \mathbb{Z}^{r\times c\times 3}\} \;=\; \{y_0 \,:\, y \in P \cap \mathbb{Z}^{n+1}\} \;=\; S \ . \end{aligned}$$

So, indeed, any finite $S \subset \mathbb{N}$ satisfies $S = R(\mathcal{M};u)$ for some margin collection u under some model $\mathcal{M} \in \mathcal{S}_3^2$. ☐

We conclude with a (possibly smallest) example of 2-margins of $(6,4,3)$-tables with "complex" entry-range.

Example 3. Gap in entry-range and spectrum of 2-margined 3-tables: we start with the polytope $P = \{y \geq 0 : y_0 - 2y_1 = 0,\ y_1 + y_2 = 1\}$ in three variables, where the set of integer values y_0 attains is $\{0,2\}$. Our construction (with a suitable shortcut that gives a smaller array size) represents it as the transportation polytope corresponding to the model $\mathcal{M} = (3, \{12, 13, 23\}, (6,4,3))$, with 2-margins $u = (u^{12}, u^{13}, u^{23})$ given by the three matrices below, where the variable y_0 is embedded as the entry $x_{1,1,1}$. By Theorem 1, the corresponding entry-range is $R(\mathcal{M};u) = \{0,2\} \in \mathrm{Spec}(\mathcal{M})$ and has a gap.

$$(u^{12}_{i_1,i_2}) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} , \ (u^{13}_{i_1,i_3}) = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix} , \ (u^{23}_{i_2,i_3}) = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix} .$$

2 The Three-Stage Construction

Our construction consists of three stages which are independent of each other as reflected in Lemma 1 and Theorems 2 and 3 below. Stage one, in §2.1, is a simple preprocessing in which a given polytope is represented as another whose defining system involves only small, $\{-1,0,1,2\}$-valued, coefficients, at the expense of increasing the number of variables. This enables to make the entire construction run in polynomial-time. However, for systems with small coefficients, such as in the examples above, this may result in unnecessary blow-up and can be skipped. Stage two, in §2.2, represents any rational polytope as a 3-way transportation polytope with specified *plane-sums* and *forbidden-entries*. In the last stage, in §2.3, any plane-sum transportation polytope with upper-bounds on the entries is represented as a slim 3-way line-sum transportation polytope. In §2.4 these three stages are integrated to give Theorem 1, and a complexity estimate is provided.

2.1 Preprocessing: Coefficient Reduction

Let $P = \{y \geq 0 : Ay = b\}$ where $A = (a_{i,j})$ is an integer matrix and b is an integer vector. We represent it as a polytope $Q = \{x \geq 0 : Cx = d\}$, in polynomial-time, with a $\{-1,0,1,2\}$-valued matrix $C=(c_{i,j})$ of coefficients, as follows. Consider any variable y_j and let $k_j := \max\{\lfloor \log_2 |a_{i,j}| \rfloor : i=1,\dots m\}$ be the maximum number of bits in the binary representation of the absolute value of any $a_{i,j}$. We introduce variables $x_{j,0},\dots,x_{j,k_j}$, and relate them by the equations $2x_{j,i}-x_{j,i+1}=0$. The representing injection σ is defined by $\sigma(j):=(j,0)$, embedding y_j as $x_{j,0}$. Consider any term $a_{i,j}\,y_j$ of the original system. Using the binary expansion $|a_{i,j}|=\sum_{s=0}^{k_j} t_s 2^s$ with all $t_s\in\{0,1\}$, we rewrite this term as $\pm\sum_{s=0}^{k_j} t_s x_{j,s}$. To illustrate, consider a system consisting of the single equation $3y_1-5y_2+2y_3=7$. Then the new system is

$$\begin{array}{cccccccr}
2x_{1,0} & -x_{1,1} & & & & & & =0\\
 & & 2x_{2,0} & -x_{2,1} & & & & =0\\
 & & & 2x_{2,1} & -x_{2,2} & & & =0\\
 & & & & & 2x_{3,0} & -x_{3,1} & =0\\
x_{1,0} & +x_{1,1} & -x_{2,0} & & -x_{2,2} & & +x_{3,1} & =7
\end{array} .$$

It is easy to see that this procedure provides the sought representation, and we get the following.

Lemma 1. *Any rational polytope $P=\{y\ge 0 : Ay=b\}$ is polynomial-time representable as a polytope $Q=\{x\ge 0 : Cx=d\}$ with $\{-1,0,1,2\}$-valued defining matrix C.*

2.2 Representing Polytopes as Plane-Sum Entry-Forbidden Transportation Polytopes

Let $P=\{y\ge 0 : Ay=b\}$ where $A=(a_{i,j})$ is an $m\times n$ integer matrix and b is an integer vector: we assume that P is bounded and hence a polytope, with an integer upper bound U (which can be derived from the Cramer's rule bound) on the value of any coordinate y_j of any $y\in P$.

For each variable y_j, let r_j be the largest between the sum of the positive coefficients of y_j over all equations and the sum of absolute values of the negative coefficients of y_j over all equations,

$$r_j \quad := \quad \max\left(\sum_k \{a_{k,j} : a_{k,j}>0\}\,,\ \sum_k \{|a_{k,j}| : a_{k,j}<0\}\right) .$$

Let $r:=\sum_{j=1}^n r_j$, $R:=\{1,\dots,r\}$, $h:=m+1$ and $H:=\{1,\dots,h\}$. We now describe how to construct vectors $u,v\in\mathbb{Z}^r, w\in\mathbb{Z}^h$, and a set $E\subset R\times R\times H$ of triples - the "enabled", non-"forbidden" entries - such that the polytope P is represented as the corresponding transportation polytope of $r\times r\times h$ arrays with plane-sums u,v,w and only entries indexed by E enabled,

$$T \;=\; \{\, x\in\mathbb{R}_+^{r\times r\times h} \;:\; x_{i,j,k}=0 \text{ for all } (i,j,k)\notin E\,, \text{ and } \sum_{i,j} x_{i,j,k}=w_k\,,\ \sum_{i,k} x_{i,j,k}=v_j\,,\ \sum_{j,k} x_{i,j,k}=u_i \,\} .$$

We also indicate the injection $\sigma : \{1,\ldots,n\} \longrightarrow R\times R\times H$ giving the desired embedding of the coordinates y_j as the coordinates $x_{i,j,k}$ and the representation of P as T (see paragraph following Theorem 1).

Basically, each equation $k=1,\ldots,m$ will be encoded in a "horizontal plane" $R\times R\times\{k\}$ (the last plane $R\times R\times\{h\}$ is included for consistency and its entries can be regarded as "slacks"); and each variable y_j, $j=1,\ldots,n$ will be encoded in a "vertical box" $R_j\times R_j\times H$, where $R=\biguplus_{j=1}^n R_j$ is the natural partition of R with $|R_j|=r_j$, namely with $R_j := \{1+\sum_{l<j} r_l,\ldots,\sum_{l\le j} r_l\}$.

Now, all "vertical" plane-sums are set to the same value U, that is, $u_j := v_j := U$ for $j=1,\ldots,r$. All entries not in the union $\biguplus_{j=1}^n R_j\times R_j\times H$ of the variable boxes will be forbidden. We now describe the enabled entries in the boxes; for simplicity we discuss the box $R_1\times R_1\times H$, the others being similar. We distinguish between the two cases $r_1=1$ and $r_1\ge 2$. In the first case, $R_1=\{1\}$; the box, which is just the single line $\{1\}\times\{1\}\times H$, will have exactly two enabled entries $(1,1,k^+),(1,1,k^-)$ for suitable k^+, k^- to be defined later. We set $\sigma(1) := (1,1,k^+)$, namely embed $y_1 = x_{1,1,k^+}$. We define the *complement* of the variable y_1 to be $\bar y_1 := U-y_1$ (and likewise for the other variables). The vertical sums u,v then force $\bar y_1 = U-y_1 = U-x_{1,1,k^+} = x_{1,1,k^-}$, so the complement of y_1 is also embedded. Next, consider the case $r_1\ge 2$. For each $s=1,\ldots,r_1$, the line $\{s\}\times\{s\}\times H$ (respectively, $\{s\}\times\{1+(s \bmod r_1)\}\times H$) will contain one enabled entry $(s,s,k^+(s))$ (respectively, $(s,1+(s \bmod r_1),k^-(s))$. All other entries of $R_1\times R_1\times H$ will be forbidden. Again, we set $\sigma(1) := (1,1,k^+(1))$, namely embed $y_1 = x_{1,1,k^+(1)}$; it is then not hard to see that, again, the vertical sums u,v force $x_{s,s,k^+(s)} = x_{1,1,k^+(1)} = y_1$ and $x_{s,1+(s \bmod r_1),k^-(s)} = U - x_{1,1,k^+(1)} = \bar y_1$ for each $s=1,\ldots,r_1$. Therefore, both y_1 and $\bar y_1$ are each embedded in r_1 distinct entries.

To clarify the above description it may be helpful to visualize the $R\times R$ matrix $(x_{i,j,+})$ whose entries are the vertical line-sums $x_{i,j,+} := \sum_{k=1}^h x_{i,j,k}$. For instance, if we have three variables with $r_1=3, r_2=1, r_3=2$ then $R_1=\{1,2,3\}, R_2=\{4\}, R_3=\{5,6\}$, and the line-sums matrix $x=(x_{i,j,+})$ is

$$\begin{pmatrix} x_{1,1,+} & x_{1,2,+} & 0 & 0 & 0 & 0 \\ 0 & x_{2,2,+} & x_{2,3+} & 0 & 0 & 0 \\ x_{3,1,+} & 0 & x_{3,3,+} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{4,4,+} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{5,5,+} & x_{5,6,+} \\ 0 & 0 & 0 & 0 & x_{6,5,+} & x_{6,6,+} \end{pmatrix} = \begin{pmatrix} y_1 & \bar y_1 & 0 & 0 & 0 & 0 \\ 0 & y_1 & \bar y_1 & 0 & 0 & 0 \\ \bar y_1 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & y_3 & \bar y_3 \\ 0 & 0 & 0 & 0 & \bar y_3 & y_3 \end{pmatrix}.$$

We now encode the equations by defining the horizontal plane-sums w and the indices $k^+(s),k^-(s)$ above as follows. For $k=1,\ldots,m$, consider the kth equation $\sum_j a_{k,j}y_j = b_k$. Define the index sets $J^+ := \{j : a_{k,j}>0\}$ and $J^- := \{j : a_{k,j}<0\}$, and set $w_k := b_k + U\cdot\sum_{j\in J^-}|a_{k,j}|$. The last coordinate of w is set for consistency with u,v to be $w_h = w_{m+1} := r\cdot U - \sum_{k=1}^m w_k$. Now, with $\bar y_j := U-y_j$ the complement of variable y_j as above, the kth equation can be rewritten as

$$\sum_{j\in J^+} a_{k,j}y_j + \sum_{j\in J^-} |a_{k,j}|\bar{y}_j = \sum_{j=1}^{n} a_{k,j}y_j + U\cdot\sum_{j\in J^-}|a_{k,j}| = b_k + U\cdot\sum_{j\in J^-}|a_{k,j}| = w_k.$$

To encode this equation, we simply "pull down" to the corresponding kth horizontal plane as many copies of each variable y_j or $\bar{y}_j$ by suitably setting $k^+(s) := k$ or $k^-(s) := k$. By the choice of r_j there are sufficiently many, possibly with a few redundant copies which are absorbed in the last hyperplane by setting $k^+(s) := m+1$ or $k^-(s) := m+1$. For instance, if $m = 8$, the first variable y_1 has $r_1 = 3$ as above, its coefficient $a_{4,1} = 3$ in the fourth equation is positive, its coefficient $a_{7,1} = -2$ in the seventh equation is negative, and $a_{k,1} = 0$ for $k \neq 4,7$, then we set $k^+(1) = k^+(2) = k^+(3) := 4$ (so $\sigma(1) := (1,1,4)$ embedding y_1 as $x_{1,1,4}$), $k^-(1) = k^-(2) := 7$, and $k^-(3) := h = 9$.

This way, all equations are suitably encoded, and we obtain the following theorem.

Theorem 2. *Any rational polytope $P = \{y \in \mathbb{R}^n_+ : Ay = b\}$ is polynomial-time representable as a plane-sum entry-forbidden 3-way transportation polytope*

$$T \;=\; \{\, x\in\mathbb{R}_+^{r\times r\times h} \;:\; x_{i,j,k} = 0 \;\textit{ for all }\; (i,j,k)\notin E\,, \quad \textit{and} \\ \sum_{i,j} x_{i,j,k} = w_k\,,\; \sum_{i,k} x_{i,j,k} = v_j\,,\; \sum_{j,k} x_{i,j,k} = u_i\,\}\;.$$

Proof. Follows from the construction outlined above and Lemma 1. □

2.3 Representing Plane-Sum Entry-Bounded as Slim Line-Sum Entry-Free

Here we start with a transportation polytope of plane-sums and *upper-bounds* $e_{i,j,k}$ on the entries,

$$P = \{\, y\in\mathbb{R}_+^{l\times m\times n} : \sum_{i,j} y_{i,j,k} = c_k\,, \sum_{i,k} y_{i,j,k} = b_j\,, \sum_{j,k} y_{i,j,k} = a_i\,,\; y_{i,j,k}\le e_{i,j,k}\,\}\,.$$

Clearly, this is a more general form than that of T appearing in Theorem 2 above - the forbidden entries can be encoded by setting a "forbidding" upper-bound $e_{i,j,k} := 0$ on all forbidden entries $(i,j,k) \notin E$ and an "enabling" upper-bound $e_{i,j,k} := U$ on all enabled entries $(i,j,k) \in E$. Thus, by Theorem 2, any rational polytope is representable also as such a plane-sum entry-bounded transportation polytope P. We now describe how to represent, in turn, such a P as a slim line-sum (entry-free) transportation polytope of the form of Theorem 1,

$$T \;=\; \{\, x\in\mathbb{R}_+^{r\times c\times 3} \;:\; \sum_I x_{I,J,K} = w_{J,K}\,,\; \sum_J x_{I,J,K} = v_{I,K}\,,\; \sum_K x_{I,J,K} = u_{I,J}\,\}\,.$$

We give explicit formulas for $u_{I,J}, v_{I,K}, w_{J,K}$ in terms of a_i, b_j, c_k and $e_{i,j,k}$ as follows. Put $r := l\cdot m$ and $c := n+l+m$. The first index I of each entry $x_{I,J,K}$ will be a pair $I = (i,j)$ in the r-set

$$\{(1,1),\ldots,(1,m),(2,1),\ldots,(2,m),\ldots,(l,1),\ldots,(l,m)\}\;.$$

The second index J of each entry $x_{I,J,K}$ will be a pair $J = (s,t)$ in the c-set

$$\{(1,1),\ldots,(1,n),(2,1),\ldots,(2,l),(3,1),\ldots,(3,m)\}\ .$$

The last index K will simply range in the 3-set $\{1,2,3\}$. We represent P as T via the injection σ given explicitly by $\sigma(i,j,k) := ((i,j),(1,k),1)$, embedding each variable $y_{i,j,k}$ as the entry $x_{(i,j),(1,k),1}$. Let U denote the minimal between the two values $\max\{a_1,\ldots,a_l\}$ and $\max\{b_1,\ldots,b_m\}$. The line-sums are the matrices

$$(u_{I,J}) = \begin{array}{cc}
 & \begin{array}{ccccccccccccccc} 11 & 12 & \cdots & 1n & 21 & 22 & \cdots & 2l & 31 & 32 & \cdots & 3m \end{array} \\
\begin{array}{c} 11 \\ 12 \\ \vdots \\ 1m \\ 21 \\ 22 \\ \vdots \\ 2m \\ \vdots \\ l1 \\ l2 \\ \vdots \\ lm \end{array} &
\begin{pmatrix}
e_{1,1,1} & e_{1,1,2} & \cdots & e_{1,1,n} & U & 0 & \cdots & 0 & U & 0 & \cdots & 0 \\
e_{1,2,1} & e_{1,2,2} & \cdots & e_{1,2,n} & U & 0 & \cdots & 0 & 0 & U & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{1,m,1} & e_{1,m,2} & \cdots & e_{1,m,n} & U & 0 & \cdots & 0 & 0 & 0 & \cdots & U \\
e_{2,1,1} & e_{2,1,2} & \cdots & e_{2,1,n} & 0 & U & \cdots & 0 & U & 0 & \cdots & 0 \\
e_{2,2,1} & e_{2,2,2} & \cdots & e_{2,2,n} & 0 & U & \cdots & 0 & 0 & U & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{2,m,1} & e_{2,m,2} & \cdots & e_{2,m,n} & 0 & U & \cdots & 0 & 0 & 0 & \cdots & U \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{l,1,1} & e_{l,1,2} & \cdots & e_{l,1,n} & 0 & 0 & \cdots & U & U & 0 & \cdots & 0 \\
e_{l,2,1} & e_{l,2,2} & \cdots & e_{l,2,n} & 0 & 0 & \cdots & U & 0 & U & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e_{l,m,1} & e_{l,m,2} & \cdots & e_{l,m,n} & 0 & 0 & \cdots & U & 0 & 0 & \cdots & U
\end{pmatrix}
\end{array}$$

$$(v_{I,K}) = \begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 11 \\ 12 \\ \vdots \\ 1m \\ 21 \\ 22 \\ \vdots \\ 2m \\ \vdots \\ l1 \\ l2 \\ \vdots \\ lm \end{array} &
\begin{pmatrix}
U & e_{1,1,+} & U \\
U & e_{1,2,+} & U \\
\vdots & \vdots & \vdots \\
U & e_{1,m,+} & U \\
U & e_{2,1,+} & U \\
U & e_{2,2,+} & U \\
\vdots & \vdots & \vdots \\
U & e_{2,m,+} & U \\
\vdots & \vdots & \vdots \\
U & e_{l,1,+} & U \\
U & e_{l,2,+} & U \\
\vdots & \vdots & \vdots \\
U & e_{l,m,+} & U
\end{pmatrix}
\end{array}
\qquad
(w_{J,K}) = \begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 11 \\ 12 \\ \vdots \\ 1n \\ 21 \\ 22 \\ \vdots \\ 2l \\ 31 \\ 32 \\ \vdots \\ 3m \end{array} &
\begin{pmatrix}
c_1 & e_{+,+,1} - c_1 & 0 \\
c_2 & e_{+,+,2} - c_2 & 0 \\
\vdots & \vdots & \vdots \\
c_n & e_{+,+,n} - c_n & 0 \\
m \cdot U - a_1 & 0 & a_1 \\
m \cdot U - a_2 & 0 & a_2 \\
\vdots & \vdots & \vdots \\
m \cdot U - a_l & 0 & a_l \\
0 & b_1 & l \cdot U - b_1 \\
0 & b_2 & l \cdot U - b_2 \\
\vdots & \vdots & \vdots \\
0 & b_m & l \cdot U - b_m
\end{pmatrix}
\end{array}$$

Theorem 3. *Any rational plane-sum entry-bounded* 3*-way transportation polytope*

$$P = \{\, y \in \mathbb{R}_+^{l\times m\times n} : \sum_{i,j} y_{i,j,k} = c_k \,, \sum_{i,k} y_{i,j,k} = b_j \,, \sum_{j,k} y_{i,j,k} = a_i \,,\ y_{i,j,k} \le e_{i,j,k} \} \,.$$

is strongly-polynomial-time representable as a line-sum slim transportation polytope

$$T \;=\; \{\, x \in \mathbb{R}_+^{r\times c\times 3} \;:\; \sum_I x_{I,J,K} = w_{J,K} \,,\; \sum_J x_{I,J,K} = v_{I,K} \,,\; \sum_K x_{I,J,K} = u_{I,J} \,\}.$$

Proof. We outline the proof; complete details are in [4]. First consider any $y = (y_{i,j,k}) \in P$; we claim the embedding via σ of $y_{i,j,k}$ in $x_{(i,j),(1,k),1}$ can be extended uniquely to $x = (x_{I,J,K}) \in T$. First, the entries $x_{I,(3,t),1}$, $x_{I,(2,t),2}$ and $x_{I,(1,t),3}$ for all $I = (i,j)$ and t are zero since so are the line-sums $w_{(3,t),1}$, $w_{(2,t),2}$ and $w_{(1,t),3}$. Next, consider the entries $x_{I,(2,t),1}$: since all entries $x_{I,(3,t),1}$ are zero, examining the line-sums $u_{I,(2,t)}$ and $v_{I,1} = U$, we find $x_{(i,j),(2,i),1} = U - \sum_{t=1}^{n} x_{(i,j),(1,t),1} = U - y_{i,j,+} \ge 0$ whereas for $t \ne i$ we get $x_{(i,j),(2,t),1} = 0$. This also gives the entries $x_{I,(2,t),3}$: we have $x_{(i,j),(2,i),3} = U - x_{(i,j),(2,i),1} = y_{i,j,+} \ge 0$ whereas for $t \ne i$ we have $x_{(i,j),(2,t),3} = 0$. Next, consider the entries $x_{I,(1,t),2}$: since all entries $x_{I,(1,t),3}$ are zero, examining the line-sums $u_{(i,j),(1,k)} = e_{i,j,k}$ we find $x_{(i,j),(1,k),2} = e_{i,j,k} - y_{i,j,k} \ge 0$ for all i,j,k. Next consider the entries $x_{I,(3,t),2}$: since all entries $x_{I,(2,t),2}$ are zero, examining the line-sums $u_{(i,j),(3,t)}$ and $v_{(i,j),2} = e_{i,j,+}$, we find $x_{(i,j),(3,j),2} = e_{i,j,+} - \sum_{k=1}^{l} x_{(i,j),(1,k),2} = y_{i,j,+} \ge 0$ whereas for $t \ne j$ we get $x_{(i,j),(3,t),2} = 0$. This also gives the entries $x_{I,(3,t),3}$: we have $x_{(i,j),(3,j),3} = U - x_{(i,j),(3,j),2} = U - y_{i,j,+} \ge 0$ whereas for $t \ne j$ we get $x_{(i,j),(3,t),3} = 0$.

Conversely, given any $x = (x_{I,J,K}) \in T$, let $y = (y_{i,j,k})$ with $y_{i,j,k} := x_{(i,j),(1,k),1}$. Since x is nonnegative, so is y. Further, $e_{i,j,k} - y_{i,j,k} = x_{(i,j),(1,k),2} \ge 0$ for all i,j,k and hence y obeys the entry upper-bounds. Finally, using the relations established above $x_{(i,j),(3,t),2} = 0$ for $t \ne j$, $x_{(i,j),(2,t),3} = 0$ for $t \ne i$, and $x_{(i,j),(3,j),2} = x_{(i,j),(2,i),3} = y_{i,j,+}$, we obtain

$$\sum_{i,j} y_{i,j,k} \;=\; \sum_{i,j} x_{(i,j),(1,k),1} \;=\; w_{(1,k),1} \;=\; c_k \,, \qquad 1 \le k \le n \;;$$

$$\sum_{i,k} y_{i,j,k} \;=\; \sum_{i} x_{(i,j),(3,j),2} \;=\; w_{(3,j),2} \;=\; b_j \,, \qquad 1 \le j \le m \;;$$

$$\sum_{j,k} y_{i,j,k} \;=\; \sum_{j} x_{(i,j),(2,i),3} \;=\; w_{(2,i),3} \;=\; a_i \,, \qquad 1 \le i \le l \;.$$

This shows that y satisfies the plane-sums as well and hence is in P. Since integrality is also preserved in both directions, this completes the proof. □

2.4 The Main Theorem and a Complexity Estimate

Call a class $\mathcal{P}$ of rational polytopes *polynomial-time representable* in a class $\mathcal{Q}$ if there is a polynomial-time algorithm that represents any given $P \in \mathcal{P}$ as some $Q \in \mathcal{Q}$. The resulting binary relation on classes of rational polytopes is clearly transitive. Thus, the composition of Theorem 2.2 (which incorporates Lemma 1) and Theorem 2.3 gives at once Theorem 1 stated in the introduction. Working out the details of our three-stage construction, we can give the following estimate on the number of rows r and columns c in the resulting representing transportation polytope, in terms of the input. The computational complexity of the construction is also determined by this bound, but we do not dwell on the details here.

Theorem 1 (with complexity estimate). *Any polytope* $P = \{y \in \mathbb{R}^n_+ : Ay = b\}$ *with integer* $m \times n$ *matrix* $A = (a_{i,j})$ *and integer vector* b *is polynomial-time representable as a slim transportation polytope*

$$T \quad = \quad \{ \, x \in \mathbb{R}_+^{r \times c \times 3} \ : \ \sum_i x_{i,j,k} = w_{j,k} \, , \ \sum_j x_{i,j,k} = v_{i,k} \, , \ \sum_k x_{i,j,k} = u_{i,j} \, \} \ ,$$

with $r = O(m^2(n+L)^2)$ *rows and* $c = O(m(n+L))$ *columns, where* $L := \sum_{j=1}^n \max_{i=1}^m \lfloor \log_2 |a_{i,j}| \rfloor$.

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