Balanced Allocations: The Heavily Loaded Case

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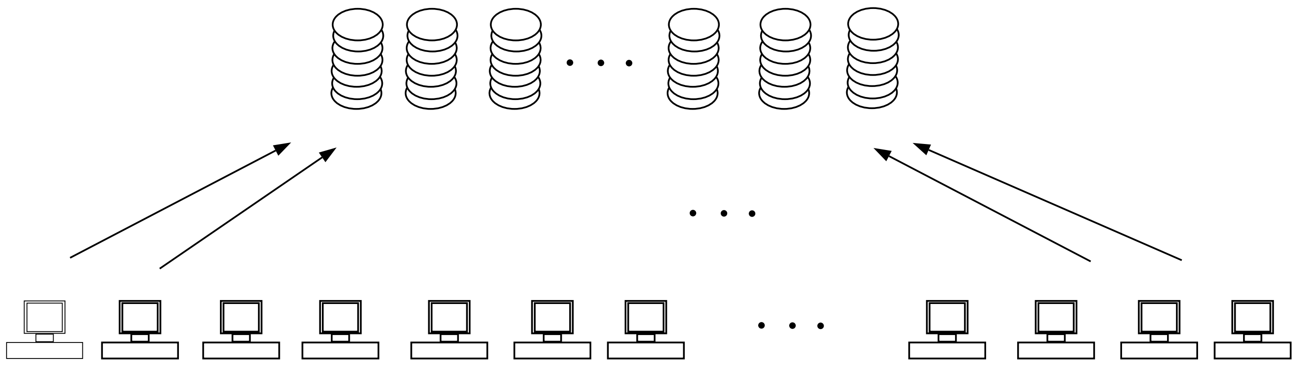
Ballanced Allocation



Motivation: Assignment of

jobs	$\hat{=}$	servers,
data	$\hat{=}$	disks,

[data $\hat{=}$ position in a hashing table]

Modeling

As a “Balls and Bins” game:

balls	$\hat{=}$	requests,
bins	$\hat{=}$	server,
max load	$\hat{=}$	max number of balls per urn

Notation: m balls, n bins

$X_i \quad :=$ number of balls in ith bin.

Minimize: $\max_{1\le i\le n} X_i$ or, more precisely, $\max_{1\le i\le n}\left(X_i - \frac{m}{n}\right)$

Note: in general $m \gg n$

The Classical “Balls and Bins” Game

Model: We place m balls in n bins, randomly and independently. Then

$$\Pr\left[\max_{1\le i\le n} X_i \ge k_\alpha\right] = \begin{cases} o(1) & \text{, if } \alpha > 1 \\ 1 - o(1) & \text{, if } \alpha < 1, \end{cases}$$

where

$$k_\alpha = \begin{cases} \frac{\log n}{\log \frac{n\log n}{m}} \left(1 + \alpha \frac{\log^{(2)} \frac{n\log n}{m}}{\log \frac{n\log n}{m}}\right), & \frac{n}{\operatorname{polylog}(n)} \le m \ll n\log n, \\ (d_c - 1 + \alpha)\log n, & m = c\cdot n\log n, \quad c \text{ constant}, \\ \frac{m}{n} + \alpha\sqrt{2\frac{m}{n}\log n}, & \text{if } n\log n \ll m \le n\cdot\operatorname{polylog}(n), \\ \frac{m}{n} + \sqrt{2\frac{m}{n}\log n\left(1 - \frac{1}{\alpha}\frac{\log^{(2)} n}{2\log n}\right)}, & m \gg n\cdot(\log n)^3. \end{cases}$$

Proof: First and second moment method, see e.g. Raab, St. ’98

Multiple Choice Strategy

Azar, Broder, Karlin, Upfal '94

Model: Place balls successively according to the following rule:

For each ball choose d bins uniformly at random and place the ball in that bin which contains the fewest number of balls. Then w.h.p.:

$$m = n: \quad \max_{1\le i\le n} X_i \le \frac{\ln\ln n}{\ln d} + \mathcal{O}(1)$$

$$m \gg n: \quad \max_{1\le i\le n} X_i = \Theta(\frac{m}{m})$$

Note: This is weaker than the bound for the classical single choice model.

Improvement by Vöcking '99:

$$m = n: \quad \max_{1\le i\le n} X_i \le \frac{\ln\ln n}{d^*} + \mathcal{O}(1), \quad \text{where } d^* \approx d\ln 2.$$

Multiple Choice Strategies

GREEDY[d]: (**Azar, Broder, Karlin, Upfal '94**)

For each ball choose d bins uniformly at random.

Place the ball in that bin which contains the fewest number of balls.

LEFT[d]: (**Vöcking '99**)

Divide the set of bins into d groups od equal size.

For each ball choose one bin uniformly at random from each group.

Place the ball in that bin which contains the fewest number of balls.

In case of ties, the ball is placed in the leftmost bin with smallest load.

Variants of Azar, Broder, Karlin, Upfal for $m \gg n$

Adler, Berenbrink, Schröder '97:

In each round:

- $k \leq \frac{n}{6e}$ many balls are placed,
- from each nonempty bin one ball is deleted.

Then we have for all $j \in \mathbb{N}$:

$$\Pr\left[\text{all balls of } j\text{th round are “out” after at most } \frac{\log\log n}{\log d} + c \text{ steps}\right] \;=\; 1 - o(1).$$

Variants of Azar, Broder, Karlin, Upfal for $m \gg n$ (2)

Cole, Frieze, Maggs, Mitzenmacher, Richa, Sitaraman, Upfal'98:

In each round:

– 1 ball is placed,

– a ball, chosen by an adversary *before* the start of the process, is deleted.

Then we have for all $j \in \mathbb{N}$:

$$\Pr\left[\text{After } j \text{ rounds each bin contains } \mathcal{O}(\log\log n) \text{ balls}\right] = 1 - o(1).$$

Proof techniques

layered induction

Azar, Broder, Karlin, Upfal ’94

witness trees

Cole et al ’98, Vöcking ’99

differential equations

Mitzenmacher ’96, Vvedenskaya et al ’96,’97

But: **All these techniques work only for $m = \Theta(n)$!!!**

Our Result

Berenbrink, Czumaj, St., Vöcking ’99

Let $m \geq n$. Then the process LEFT$[d]$ satisfies:

The maximum load is with w.h.p. at most

$$\frac{m}{n} + \frac{\ln \ln n}{d^*} + \Theta(1).$$

More precisely:

Number of bins with load $\geq \frac{m}{n} + i$ is $\leq exp(-\phi_d^{d \cdot i - \mathcal{O}(1)}) \cdot n$ and

number of bins with load $\leq \frac{m}{n} - i$ is $\leq 1.9^{-i+1} \cdot n$.

[Similar results hold for GREEDY$[d]$.]

Main parts of the proof

Part I: Show that:

$\forall m \geq n^k$: Distribution after m balls $\hat{=}$ distribution after n^k balls.

Main idea: Use coupling.

Part II: Show that:

$\forall t \leq n^{k-1}$: Distribution after $t \cdot n$ balls is as claimed in theorem.

Part II — The Distribution

Setup: Place balls in rounds, each round consists of n balls.

Notation: at time $t \mathrel{\hat{=}}$ after round t.

"light" bins (at time t) $\mathrel{\hat{=}}$ bins with load $< \frac{m}{n}$.

"heavy" bins (at time t) $\mathrel{\hat{=}}$ bins with load $> \frac{m}{n}$.

Distribution of Light Bins

For all $t \leq n^{k-1}$ the following invariants hold:

$L_1(t)$: number of bins with load $\leq t - i$ at time t is at most $\alpha_i \cdot n \quad \forall 1 \leq i \leq c_1 \ln n$,
$\alpha_1 = 0.8, \alpha_2 = 0.55, \alpha_i = 1.9^{-i+1} \;\; \forall i \geq 3.$

$L_2(t)$: number of bins with load $\leq t - i$ at time t is zero $\quad \forall i \geq c_2 \ln n$.

Note: ... exponential decrease

Corollary: Number of "holes" is at most $2n$.

Distribution of Heavy Bins

For all $t \leq n^{k-1}$ the following invariants hold:

$H_1(t)$: number of bins with load $\geq t + i$ at time t is at most $f(i)$ $\forall 1 \leq i \leq \lceil \log_d \ln n \rceil$.

$$f(i) = \max\left\{n \cdot \exp\left(-\frac{d^i+1}{2}\right), n^{1/4} \cdot 2^{-i}\right\}.$$

$H_2(t)$: $\sum_{i \geq \lceil \log_d \ln n \rceil} \big($ number of bins with load $\geq t + i$ at time $t \big) = \mathcal{O}(1)$

Note: ... double exponential decrease

Corollary: Maximum load is at most $t + \frac{\ln \ln n}{\ln d} + \mathcal{O}(1)$.

[This is for GREEDY[d]. The invariants for LEFT[d] are slightly more complicated.]