An extension of path coupling and its application to the Glauber dynamics for graph colourings

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Sampling via Markov-chain simulation

Markov chain $\mathcal{M}$
finite state space Ω
stationary distribution π

Total variation distance of $\mathcal{M}$ from π at t
(with initial state x)

$$\begin{aligned} d(P_x^t, \pi) &= \frac{1}{2} \sum_{y \in \Omega} |P_x^t(y) - \pi(y)| \\ &= \max_{A \subseteq \Omega} |P_x^t(A) - \pi(A)|. \end{aligned}$$

Mixing time

$$\tau_x(\epsilon) = \min_t\{t \mid d(P_x^{t'}, \pi) \le \epsilon \text{ for all } t' \ge t\}.$$

$$\tau(\epsilon) = \max_x\{\tau_x(\epsilon)\}.$$

Proving Rapid Mixing
Coupling Method

Coupling for $\mathcal{M}$
Markov process (X_t, Y_t) on $\Omega \times \Omega$

- X_t is a faithful copy of $\mathcal{M}$

- Y_t is a faithful copy of $\mathcal{M}$

Coupling Lemma Suppose that Y_0 has the stationary distribution π. Then, if X_t has distribution P^t,

$$d(P^t, \pi) \leq \Pr(X_t \neq Y_t).$$

Proximity function

$\rho : \Omega \times \Omega \to \mathbb{N}$

$\rho(x, y) = 0$ implies $x = y$.

$$\Pr(X_t \neq Y_t) \leq E[\rho(X_t, Y_t)].$$

For some $\gamma < 1$,

$$E[\rho(X_1, Y_1) \mid (X_0, Y_0) = (x, y)] \leq \gamma\, \rho(x, y).$$

$$d(P^t, \pi) \leq \gamma^t \rho(X_0, Y_0).$$

$$\tau(\epsilon) \leq \max_{(x,y)} \frac{\log(\rho(x, y)/\epsilon)}{1 - \gamma}.$$

Example: Sampling colourings

state space $\Omega_k(G)$ is the set of proper k-colourings of an n-vertex graph G

"Glauber Dynamics" Markov Chain:

- Choose vertex v and colour i u.a.r.
- Recolour v with i iff this makes v properly coloured

Special Case: G is the edge-free graph

The coupling: Use the same random choice for both copies.

$\rho =$ Hamming distance.

$\gamma = 1 - 1/n$.

$\rho(x, y) \leq n$.

$$\tau(\epsilon) \leq n \log(n/\epsilon).$$

Path Coupling (Bubley and Dyer 1997)

- $S \subseteq \Omega \times \Omega$

- transitive closure of S
  (viewed as a relation) is $\Omega \times \Omega$

- Proximity function ρ for pairs in S

- Define coupling on pairs in S and "lift" to $\Omega \times \Omega$

A **path** from state x to state y

$$\overset{x}{\overset{\|}{w_1}} — w_2 — w_3 — w_4 — \qquad \cdots \qquad — \overset{y}{\overset{\|}{w_r}}$$

Adjacency: $(w_i, w_{i+1}) \in S$

- Define ρ on **all** pairs in $\Omega \times \Omega$, ensuring that the *triangle inequality* holds:

$$\rho(x,y) = \min \sum_{i=1}^{r-1} \rho(w_i, w_{i+1}),$$

  where the "min" is over all paths from x to y.

- Extending the coupling: For each pair (x,y) **fix** a shortest path from x to y and compose the couplings along the path.

By construction of the coupling on S, there is a $\gamma \leq 1$ such that for all $(\chi, \psi) \in S$ we have

$$E[\rho(X_1, Y_1) \mid (X_0, Y_0) = (\chi, \psi)] \leq \gamma\, \rho(\chi, \psi).$$

We want this for all $(x, y) \in \Omega \times \Omega$:

$$E[\rho(X_1, Y_1) \mid (X_0, Y_0) = (x, y)] \leq \gamma\, \rho(x, y).$$

This follows by linearity of expectation.

Delayed Path Coupling

Czumaj, Kanarek, Kutylowksi, Loryś

Instead of proving, for all $(\chi, \psi) \in S$,

$$E[\rho(X_1, Y_1) \mid (X_0, Y_0) = (\chi, \psi)] \leq \gamma\, \rho(\chi, \psi).$$

Choose a positive integer R and prove

$$E[\rho(X_R, Y_R) \mid (X_0, Y_0) = (\chi, \psi)] \leq \gamma\, \rho(\chi, \psi).$$

$$E[\rho(X_{R\,t}, Y_{R\,t}) \mid (X_0, Y_0) = (x, y)] \leq \gamma^t \rho(x, y).$$

$$\tau(\epsilon) \leq \max_{(x,y)} \frac{\log(\rho(x, y)/\epsilon)}{1 - \gamma} \cdot R.$$

Our Approach

If

A. for all $(\chi, \psi) \in S$,
$E[\rho(X_1, Y_1) \mid (X_0, Y_0) = (\chi, \psi)] \leq \beta\rho(\chi, \psi)$,
and

B. R is a positive integer, and

C. T is a stopping time such that for $0 \leq s \leq T$ we have $\rho(X_s, Y_s) = \rho(X_0, Y_0)$,

Then

$$
\begin{aligned}
E\ [\ \rho(X_R, Y_R) \mid (X_0, Y_0) = (\chi, \psi)\ ]
&\leq \Pr(T > R)\, \rho(\chi, \psi) \\
&+ \Pr(T \leq R)\, \beta^R \\
&\quad E[\rho(X_T, Y_T) \mid T \leq R \text{ and } (X_0, Y_0) = (\chi, \psi)].
\end{aligned}
$$

Suppose $\rho(\chi, \psi) = 1$. Let γ be

$$\begin{aligned} & \Pr(T > R) \\ + \; & \Pr(T \le R)\, \beta^R \\ & E[\rho(X_T, Y_T) \mid T \le R \text{ and } (X_0, Y_0) = (\chi, \psi)]. \end{aligned}$$

If $0 < \gamma < 1$, we get

$$\tau(\epsilon) \le \max_{(x,y)} \frac{\log(\rho(x,y)/\epsilon)}{1-\gamma} \cdot R.$$

The Glauber Dynamics

state space $\Omega_k(G)$ is the set of proper k-colourings of an n-vertex graph G

- Choose vertex v and colour i u.a.r.

- Recolour v with i iff this makes v properly coloured

- Jerrum; Salas, Sokal (1995)
  $O(n \log n)$ mixing time for $k > 2\Delta$

- Bubley, Dyer (1997)
  $O(n^3)$ mixing time for $k = 2\Delta$

- Vigoda (1999)

  - $O(n \log n)$ mixing time for more complicated chain when $k \geq 11\Delta/6$

  - applied comparison methods of Diaconis and Saloff-Coste (1993) to get mixing time $O(k \log(k) n^2 \log(n))$ for Glauber dynamics

Suppose

- The graph is Δ-regular (for $\Delta \geq 14$) and triangle-free

- $(2 - \eta)\Delta \leq k \leq 2\Delta$

Then the mixing time is $O(n \log n)$

state space $\Omega_k(G)$ is the set of proper k-colourings of a Δ-regular triangle-free n-vertex graph G.

Markov chain: Glauber dynamics.

$S \subseteq \Omega_k(G) \times \Omega_k(G)$.

ρ : Hamming distance.

Define coupling on pairs in S and “lift” to $\Omega_k(G) \times \Omega_k(G)$

Our Approach

If

A. for all $(\chi, \psi) \in S$,
$E[\rho(X_1, Y_1) \mid (X_0, Y_0) = (\chi, \psi)] \leq \beta\rho(\chi, \psi)$,
and

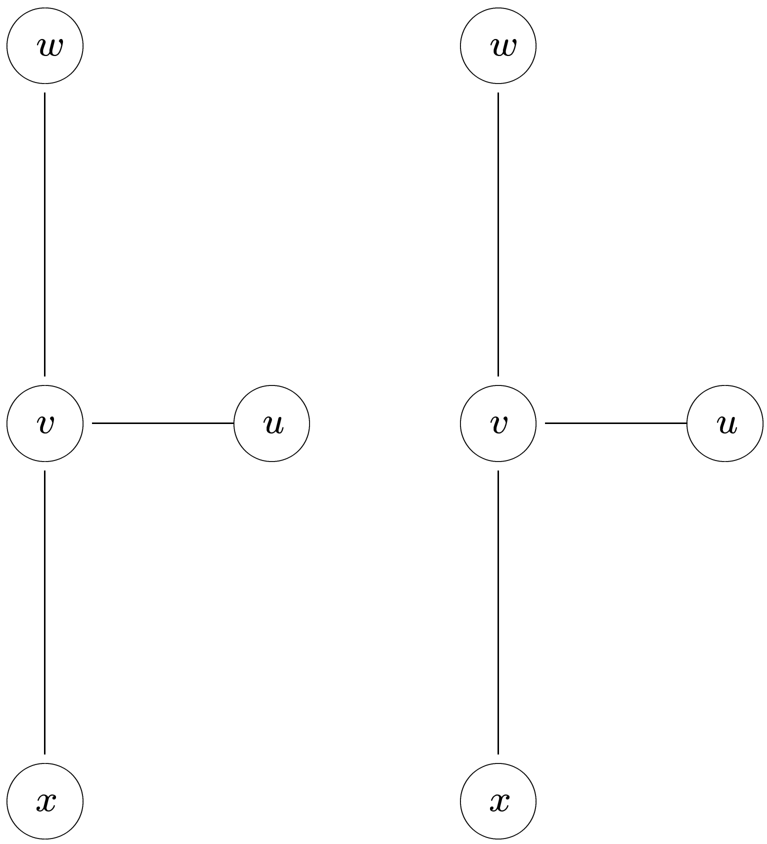
B. R is a positive integer, and

C. T is a stopping time such that for $0 \leq s \leq T$ we have $\rho(X_s, Y_s) = \rho(X_0, Y_0)$,

Then

$$
\begin{aligned}
E\ [\ \rho(X_R, Y_R) \mid (X_0, Y_0) = (\chi, \psi)\ ] \\
\leq\ & \Pr(T > R)\, \rho(\chi, \psi) \\
+\ & \Pr(T \leq R)\, \beta^R \\
& E[\rho(X_T, Y_T) \mid T \leq R \text{ and } (X_0, Y_0) = (\chi, \psi)].
\end{aligned}
$$

χ ψ



The Coupling for pairs in S

If vertex α and colour i are chosen then use this choice in both copies except

- $\alpha = w$, $i = \chi(v)$. Use $(w, \chi(v))$ in χ. Use $(w, \psi(v))$ in ψ. (Both fail, so ρ does not change.)

- $\alpha = w$, $i = \psi(v)$. Use $(w, \psi(v))$ in χ. Use $(w, \chi(v))$ in ψ. (ρ may increase by 1.)

The only other time that ρ could change is when $\alpha = v$. If the colour i is good for v then ρ decreases by 1.

Expected new value of ρ

$$\leq 1 - \frac{A}{kn} + \frac{\Delta}{kn} \leq 1 + \frac{\eta\Delta}{kn},$$

where A is the number of colours **not** used by v's neighbors.

Stopping time T: First time that either

- v is selected, or
- a neighbor w is selected with colour $\Psi(v)$.

Suppose $\rho(\chi, \psi) = 1$. Let γ be

$$\begin{aligned} & \Pr(T > R) \\ + \; & \Pr(T \leq R)\, \beta^R \\ & E[\rho(X_T, Y_T) \mid T \leq R \text{ and } (X_0, Y_0) = (\chi, \psi)]. \end{aligned}$$

If $0 < \gamma < 1$, we get

$$\tau(\epsilon) \leq \max_{(x,y)} \frac{\log(\rho(x,y)/\epsilon)}{1 - \gamma} \cdot R.$$

$$\begin{aligned} \gamma = 1 \;\; - \;\; & \Pr(T \leq R) \\ \Big(\;\; & 1 - \beta^R E[\rho(X_T, Y_T) \mid \\ & \quad T \leq R \text{ and } (X_0, Y_0) = (\chi, \psi)\;] \Big). \end{aligned}$$

Expected new value of ρ

$$\leq 1 - \frac{A}{kn} + \frac{\Delta}{kn}$$

$$E[C \mid T \leq n/8 \text{ and } (X_0, Y_0) = (\chi, \psi)] \geq 2\eta\Delta,$$

where C is the number of colours which occur *more than once* around v just before the stopping time.

$B_i =$ set of neighbors of v at which colour i is acceptable in both χ and ψ

There are at least $k/5$ colours i such that $|B_i| \geq \Delta/3$

Consider the event in which exactly *two* elements of B_i, u and w, are re-coloured with colour i before the stopping time.

- u and w are never re-chosen.

- no neighbor of u or w is chosen with colour i.

$\Pr[i \text{ repeated} \mid T \leq n/8, |B_i| \geq \Delta/3] \geq \frac{10\eta\Delta}{k}$.