Approximating permanents of complex matrices

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$\boxed{\textbf{Permanents}}$

$$\det(A) = \sum_{\pi} \mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)}$$

$$\mathrm{per}(A) = \sum_{\pi} \underbrace{\prod_{i=1}^{n} a_{i\,\pi(i)}}_{\text{monomial}}$$

Permanents defined by Cauchy and Binet [1812].

Important Facts

- The number of perfect matchings in a bipartite graph is equal to the permanent of the bipartite adjacency matrix.

- The permanent is #P complete [Valiant 1979], and the permanent of random 0-1 matrices is not in P, unless PH collapses [Feige and Lund 1992].

- The permanent can be computed in time $O(2^n n)$ [Ryser 1963] by the inclusion-exclusion method.

- The van der Waerden conjecture has been solved [D.I. Falikman 1981] and [G.P. Egoritsjev 1981].

$\boxed{\textbf{Definitions}}$

fpras (fully polynomial time randomized approximation scheme):
The relative error is at most ϵ.
The running time is polynomial in n and $1/\epsilon$.

(ϵ, δ)*-approximation*: The probability of a relative error greater than ϵ is less than δ.

efficient unbiased estimator for $f(x)$: A randomized polynomial time algorithm with input x, output y and $E(y) = f(x)$.

Approximation

- Broder [1986] proposes a Markov chain approach.

- Jerrum and Sinclair [1989] provide an *fpras* for random 0-1 matrices.

- Godsil and Gutman [1981] provide a (determinant based) efficient *unbiased estimator* for permanents of non-negative matrices.

- Kamarkar, Karp, Lipton, Lovász and Luby [KKLLL 1993] use a similar unbiased estimator to get an $\tilde{O}(2^{n/2}\epsilon^{-2}\log 1/\delta)$ time (ϵ,δ)-approximation for non-negative matrices.

$\boxed{\text{Random Walks}}$

After n steps of a random walk, the expected square of your distance from home is n (sum of the lengths of the steps).

Examples:
(a) N S (b) E W N S (c) any direction

Requirements:
(a) Expected value of each step is 0.
(b) Steps are **pairwise** independent.

Proof (Random Walk Distance)

Let s_i be the ith step (of absolute value 1) in the complex plane.

$$
\begin{aligned}
E\left(\left|\sum_{i=1}^{n} s_i\right|^2\right) &= E\left(\left(\sum_{i=1}^{n} \bar{s}_i\right)\left(\sum_{i=1}^{n} s_i\right)\right) \\
&= E\left(\sum_{i=1}^{n} \bar{s}_i s_i + \sum_{i}\sum_{j\neq i} \bar{s}_i s_j\right) \\
&= \sum_{i=1}^{n} \underbrace{E(\bar{s}_i s_i)}_{1} + \sum_{i}\sum_{j\neq i} \underbrace{E(\bar{s}_i s_j)}_{0} \\
&= n
\end{aligned}
$$

$\boxed{\text{How to Get Unbiased Estimators?}}$

Application of random walks (to approximate permanents):

(a) Replace every entry 1 in the 0-1 matrix A by a random root of unity.
(Obtain matrix B.)

(b) (Every monomial of the permanent of B is a step of a random walk.)

(c) Every monomial of the **determinant** of B is a step of a random walk.

(d) The steps are pairwise independent.

Hence, $\mathrm{per} A = E(|\det B|^2)$.

$\boxed{\text{The Classical Unbiased Estimator}}$

Let ρ_{ij}^k be a uniformly random kth root of unity.

Algorithm $\mathcal{A}_k$ $(k \in \{2, 3, 4, \ldots, \infty\})$**:**
Replace every entry a_{ij} by $b_{ij} = \rho_{ij}^k \sqrt{a_{ij}}$
(obtaining matrix B)
Return $|\det B|^2$

This is the Godsil and Gutman estimator for $k = 2$, and the KKLLL estimator for $k = 3$.

An Other Class of Algorithms

Algorithm $\mathcal{B}_k(k \in \{1, 2, 3, \dots\})$**:**
Replace every entry a_{ij} by $b_{ij} = \rho_{ij}^k \sqrt[k]{a_{ij}}$
(obtaining matrix B)
Return $(\det B)^k$

Hence, b_{ij} is just a random kth root of a_{ij}. Only the case of even k is algorithmically interesting.

For k even, $\mathcal{B}_k$ is an unbiased estimator of per$A$. (For k odd, $\mathcal{B}_k$ is an unbiased estimator of $\det A$.)

Note: For non-negative matrices A, Algorithm $\mathcal{A}_2$ is equal to Algorithm $\mathcal{B}_2$.

$\boxed{\text{Dealing with Cancellation}}$

Definition: A sum $x_1 + x_2 + \cdots + x_n = y$ is computed with *weak relative accuracy* ϵ if a result z is produced with

$$|z - y| \leq \epsilon(|x_1| + |x_2| + \cdots + |x_n|)$$

We view the permanent as a sum of monomials.

$\boxed{\text{The Result}}$

Through repeated applications of the unbiased estimator $\mathcal{B}_2$, we achieve weak relative accuracy ϵ with probability $1 - \delta$ in time

$$\tilde{O}(3^{n/2}\epsilon^{-2}\log 1/\delta)$$

even for negative or arbitrary complex entries.

This is shown by applying Chebychev's inequality

$$\text{Prob}[|X - E(X)|^2 > c\text{Var}(X)] < 1/c$$

$\boxed{\textbf{Outlook}}$

There are several promising other approaches to approximating the permanent.

For random 0-1 matrices, Frieze and Jerrum [1995] have shown that the KKLLL estimator obtains an (ϵ, δ)-approximation in polynomial time.

Rasmussen [1994] has a very simple approach to approximate 0-1 matrices, based on a randomized counting technique. I can extend this to arbitrary matrices.

Jerrum and U. Vazirani [1996] have found a faster approximation for 0-1 matrices.

Linial, Samorodnitsky and Wigderson [1998] have found a deterministic polynomial time algorithm approximating permanents of non-negative matrices up to a factor of e^n in polynomial time. It is based on scaling and the theorem (previously) known as the van der Waerden Conjecture.

Barvinok [1999] has decreased the factor to 1.31^n using a randomized determinant based method.

It is still open whether an fpras exists for all 0-1 matrices.