INTERVAL PACKING PROBLEMS IN TELECOM RESERVATION MODELS

Though existing/proposed multimedia networks may be big and robust, there are applications that will bring managers to their knees. Video teleconferencing, for example.

So reservation protocols are needed to avoid excessive service degradation.

There is a smallish engineering literature on design/analysis dating from the '70s. This literature seems to be ignorant of the highly relevant interval packing literature dating from the '50s.

A baseline single-resource model

- A process of arriving requests to use the resource is given (commonly assumed to be Poisson).
- Each request is a pair of independent random variables:
  - an advance notice, A
  - a service time, S
- On-line loss protocol: a reservation is made iff it does not conflict with (overlap) a prior reservation: if t is the request arrival time, then the interval $[t+A, t+A+S]$ does not overlap an already-reserved interval.
- The objectives are formulas for resource utilization, blocking probabilties, . . .

Subproblems: Interval packing

Advance notices make this model difficult as is, so instances of the following class of subproblems have been studied.

Consider just those arrivals before some T_0 that request an interval in $[T_0, T]$. Take $T_0 = 0$ for convenience and let n count arrivals in $(-\infty, 0)$ requesting intervals in $[0, T]$.

Compute expected number $\mathrm{E}N_T(n)$ of reservations made in $[0, T]$ by time 0, and resource utilization $\rho_T(n)$ (fraction of $[0, T]$ that the resource is in use).

First problem

Assume unit service times, starting times of requested intervals are i.i.d. $U(0, T-1)$, and we have the on-line loss protocol. Consider the limit

$$\mathrm{E}N_T := \lim_{n\to\infty} \mathrm{E}N_T(n)$$

where $[0, T]$ is "packed up" (all vacant intervals have sizes < 1).

Asymptotically,

$$\mathrm{E}N_T \sim \alpha T \quad as \quad T \to \infty$$

where

$$\alpha := \int_0^\infty e^{-2\int_0^v \frac{1-e^{-y}}{y} dy} dv \approx 0.748$$

Renyi's ('58) classical space filling (car parking) result. Mackenzie ('62) gets asymptotics of vacant-interval distribution.

Second problem

Discretize time into slots of duration $1/T$, constant service times of s slots, starting times still uniform, and same protocol. Take the limit $T \to \infty$, $n \to \infty$ with $\lambda := n/T$ held constant. Virtamo ('92) gets

$$\rho = 1 - \sum_{m \geq 1} m f_m$$

where

$$f_m = \begin{cases} x^2 e^{-\lambda(m-s+1)-2P(x)}, & m \geq s, \\ 2\int_0^x y(1-y)^m e^{-2P(y)} dy, & m < s \end{cases}$$

and where

$$x = x(\lambda) := 1 - e^{-\lambda},\ P(x) := \int_0^x \frac{dy}{y}\left[1-(1-y)^{s-1}\right]$$

Third problem

Off-line case: Hold the n requests until time 0, then pack optimally (find a maximum cardinality disjoint subset). Each service time is a *random subinterval* of $[0, T]$ (i.e., the interval between two independent uniform random draws from $[0, T]$).

Service times scale with T so Justicz, Scheinermann, and Winkler ('90) get

$$\mathrm{E}N(n) \sim \frac{2}{\sqrt{\pi}}\sqrt{n}, \quad n \to \infty$$

independent of T. Result continues to hold if uniform distribution is replaced by any other continuous distribution on $[0, T]$.

Also of interest: Find a disjoint subset that minimizes wasted space $W(n)$ (the unused portion of $[0, T]$). With the normalization $T = 1$, C., Poonen, and Winkler ('95) get

$$\mathrm{E}W(n) = \Theta(\frac{\log^2 n}{n})$$

Rhee and Talagrand ('96) improve a tail-probability bound used in the above result.

Fourth problem

Take the on-line loss version of the random-subinterval problem. C., Mallows, and Poonen ('94) find

$$\mathrm{E}N(n) \sim \beta n^{(\sqrt{17}-3)/4}, \quad n \to \infty$$

where $\beta \approx 1.84$ has an explicit (but complicated) formula.

Fifth problem

Stochastic extension of problems 1 and 2: Unit-length interval requests arrive by a Poisson process in two dimensions: an arrival during $[t, t+\Delta t]$ with starting time in $[y, y+\Delta y]$ has probability $\Delta t \Delta y + o(\Delta t \Delta y)$ (unit intensity process). With the on-line loss protocol, the number of reservations by time t has the mean

$$\mathrm{E} N_T(t) \sim \alpha(t) T, \quad T \to \infty$$

with the "incomplete Renyi's constant"

$$\alpha(t) := \int_0^t e^{-2 \int_0^u \frac{1-e^{-y}}{y} dy} du$$

as shown by C., Flatto, Jelenkovic, and Poonen ('98). (The first 3 of these authors also find asymptotics for the vacant interval distribution ('99)).

The result was used by C., Jelenkovic, and Poonen ('99) to prove a large-advance-notice limit law for the original baseline problem.

Extension of off-line problem to 2 dimensions

With $T = 1$ for convenience, the arena is $[0,1]^d$. A random box in d dimensions is the product of d independent random subintervals of $[0,1]$. C., Lueker, Spencer, and Winkler ('99) prove that there exist constants K, K' such that for all n sufficiently large

$$K\sqrt{n} < \mathrm{E}N(n) < K'\sqrt{n \ln^{d-1} n}$$

For $d = 2$, they show that the lower bound is tight,

$$\mathrm{E}N(n) = \Theta(\sqrt{n})$$

Proof of this last result studies a reduced Poissonized version with the same asymptotic upper bound.

Reduced problem, $d = 2$

A *canonical interval* has length 2^{-i} for some $i > 0$, and starts at a multiple of its length. A *canonical rectangle* is the product of two independent canonical intervals. The number having area a is Poisson distributed with mean λa^2. A count shows that this gives a Poisson distributed total with mean 4λ.

To prove $\mathrm{E}N(\lambda) = O(\lambda^{1/2})$, define

$Z_1(Z_2) :=$ no. in optimal packing disallowing rectangles spanning height (width) of $[0,1]^2$.

$Z_0 :=$ the indicator of event that there is a 1×1 rectangle among the given rectangles.

We find,

$$N(\lambda) \leq Z_0 + max(Z_1, Z_2) \tag{1}$$

with Z_1, Z_2 having the distribution of $N(\lambda/4)$. (To see this, argue that if a packing has no rectangles spanning the height (width) of $[0,1]^2$, then it can be broken down into two disjoint packings, one in the top (left) half and one in the bottom (right) half.)

Now work with the transform $Q(\lambda) := \mathrm{E}e^{\gamma N(\lambda)}$. From (1),

$$
\begin{aligned}
Q(\lambda) &\leq \mathrm{E}e^{\gamma Z_0}\mathrm{E}e^{\gamma \max(Z_1, Z_2)} \\
&\leq e^{\gamma}[\mathrm{E}e^{\gamma Z_1} + \mathrm{E}e^{\gamma Z_2}].
\end{aligned}
$$

Now, $\mathrm{E}e^{\gamma Z_i} = Q(\lambda/4),\ i = 1, 2$, so

$$Q(\lambda) \leq 2e^{\gamma}Q^2(\lambda/4) \tag{2}$$

Crude bounds give $Q(1) \leq e^{4(e^{\gamma}-1)}$ so with (2) and

$$\mathrm{E}N(\lambda) \leq \gamma^{-1} \ln \mathrm{E}e^{\gamma N(\lambda)} = \gamma^{-1} \ln Q(\lambda),$$

we get $\mathrm{E}N(\lambda) = O(\lambda^{1/2})$, as desired.