A General View of Approximation

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(presented by A. Pultr)

1. An *approximate equality* will be a relation $\doteq$ on a set X that is

reflexive and symmetric

but not necessarily transitive (typically not). Think of, e.g., reals with

$$a \doteq b \quad \text{iff} \quad |a-b| \leq 10^{-6},$$

or small open spots in the plane with

$$a \doteq b \quad \text{iff} \quad a \cap b \neq \emptyset$$

("both a and b can approximate the same point").

We will also use the symbol $\circeq$ when speaking of two approximate equalities in the same context.

2. Let us modify the notion of a *mapping* using the aproximate equality instead of equality. Thus we will have

$$\forall x \ \exists x' \doteq x \ \exists y \ x'fy \quad \text{instead of} \quad \forall x \ \exists y \ xfy$$

and

$$xfy \ \& \ x'fy' \ \& \ x \doteq x' \ \Rightarrow \ y \circeq y' \quad \text{instead of} \quad xfy \ \& \ xfy' \ \Rightarrow \ y = y'.$$

Now

(a) one might think, in the second case, of just $xfy \ \& \ xfy' \ \Rightarrow y \circeq y'$; this would not do, though: one should think of $x \doteq x'$ as undistinguishable in the given context. As it is, of course, it makes the map automatically "continuous" (in fact, uniformly continuous). Recall Mac Lane's remark.

(b) The first, however, was unnecessarily pedantic. The original "$\forall x \ \exists y$ such that xfy" can be easily achieved by taking the values in close x' for those in x, with the approximation only slightly rougher.

Thus, we will define an *approximation map* $f : (X, \doteq) \looparrowright (Y, \circeq)$ as a relation $f \subseteq X \times Y$ such that

$$\forall x \ \exists y \ xfy, \quad \text{and}$$
$$xfy \ \& \ x'fy' \ \& \ x \doteq x' \ \Rightarrow \ y \circeq y'.$$

3. Aside. Recall Dana Scott's *information systems* and *approximable maps.* Realize that the approximable maps constitute a special case of the approximation ones, with the approximate equality

$$x \doteq y \quad \text{iff} \quad x \cup y \text{ is consistent}$$

(the values can be distinct, but have to be compatible and hence predicates that can refer to the same point).

4. Localic homomorphisms as approximated maps. Recall that a *frame* is a complete lattice L satisfying the distribution law $(\bigvee a_i) \wedge b = \bigvee(a_i \wedge b)$ and that the *frame homomorphisms* $h : L \to M$ preserve all joins and finite meets. A frame can be viewed a space from which we see the "places of nontrivial extent" rather than points, and continuous maps (*"localic maps"*) are formally represented as frame homomorphisms with the arrows inverted. Instead of this formal reversal of the arrows we can view them as approximated maps as follows.

Set

$$\widetilde{L} = L \setminus \{0\}$$

(take just the spots that can approximate a point, however badly). Now for a frame homomorphism $h : M \to L$ define $\widetilde{h} : \widetilde{L} \looparrowright \widetilde{M}$ by setting

$$a\widetilde{h}b \quad \text{iff} \quad a \leq h(b).$$

This indeed defines an approximation map with respect to the approximate equality

$$x \doteq y \quad \text{iff} \quad x \wedge y \neq 0$$

(if $a \doteq a'$ we have $0 \neq a \wedge a' \leq h(b) \wedge h(b') = h(b \wedge b')$ and hence $b \wedge b' \neq 0$). Moreover it is easy to represent the original h as a well defined preimage function of $\widetilde{h}$.

5. One may see a sort of cheating in dismissing the first rule by $a\widetilde{h}1$. Now it is certainly not an appealing aspect of the approximate equation in the representation presented that

$$a \doteq 1 \quad \textit{for any} \quad a.$$

BUT : $\doteq$ is not transitive, and if a, a' are small (whatever this may so far mean) and $a \wedge a' \neq 0$ then a and a' are really close.

6. Nearness frames. As a better framework, to give a meaning to expressions like "small", or "arbitrarily small" we will use the

nearness frames

(think of uniform frames, the notion will be only slightly more general, not requiring the star refinements).

We will need some notions and notation.

First, a *cover* of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$. We say that A refines B and write $A \leq B$ if for every $a \in A$ there is a $b \in B$ such that $a \leq b$. For covers A, B set $A \wedge B = \{a \wedge b \mid a \in A,\ b \in B\}$ (the greatest common refinement of A and B).

For $x \in L$ and a cover A set $Ax = \bigvee\{a \in A \mid x \wedge a \neq 0\}$. Note that the corresondence $x \mapsto Ax$ is a left Galois adjoint to $y \mapsto y/A = \bigvee\{x \mid Ax \leq y\}$.

Finally, for a set of covers $\mathfrak{A}$ define a relation $\lhd_{\mathfrak{A}}$ on L by setting

$$a \lhd_{\mathfrak{A}} b \quad \text{iff} \quad \exists A \in \mathfrak{A},\ Aa \leq b.$$

A *nearness* on a frame L is a non-empty set $\mathfrak{A}$ of covers, none of them containing 0, such that

(1) if $A \in \mathfrak{A}$ and $A \leq B$ then $B \in \mathfrak{A}$,
(2) if $A, B \in \mathfrak{A}$ then $A \wedge B \in \mathfrak{A}$, and
(3) for every $x \in L$, $x = \bigvee\{y \mid y \lhd_{\mathfrak{A}} x\}(= \bigvee\{x/A \mid A \in \mathfrak{A}\})$.

(often one works just with *bases of nearness*, omitting (1) and replacing (2) with the requirement of the existence of a common refinement). A *nearness frame* is, of course, a couple $(L, \mathfrak{A})$ where L is a frame and $\mathfrak{A}$ a nearness on L.

Intuitively, one can think of the $A \in \mathfrak{A}$ as precisions to be chosen, or granulations in pictures of L; the condition (3) makes sure that we have enough of the (granulated) pictures that no information of the elements of L is lost.

A *uniform homomorphism* $h : (M, \mathfrak{B}) \to (L, \mathfrak{A})$ is a frame homomorphism such that for each $B \in \mathfrak{B}$, $h[B] \in \mathfrak{A}$ (if we work with the bases of uniformities, for each $B \in \mathfrak{B}$ there is an $a \in \mathfrak{A}$ such that $A \leq h[B]$.

7. Now let us look at the uniform frame homomorphisms similarly as we did at the plain ones in 4 above. The

$$h : (M, \mathfrak{B}) \to (L, \mathfrak{A})$$

represents approximations of maps in the L to M direction as follows:

(a) first, a suitable precision (granulation) $B \in \mathfrak{B}$ in the target space is chosen;
(b) then we chose a sufficiently fine granulation $A \in \mathfrak{A}$ of the source space
(c) and then we have an

$$f_{AB} : A \looparrowright B$$

defined by $af_{AB}b$ iff $a \leq h(b)$.

Note that now the values are sufficiently small and close to each other in the precision required ("the graph of the map is only as thick as the required precision").

8. Thus, nearness frames provide an easy model of approximation. The main point we want to communicate is that in fact

THIS MODEL IS, IN A WAY, UNIVERSAL.

That is, if we endeavour to formulate a general view of approximation, we will, after all, obtain a category equivalent with that of nearness frames.

9. An *approximation system* X is the following system of data.

First, we have a pre-order $\leq$ on X, $x \leq y$ interpreted as that x is a better approximation than y – of whatever we wish to approximate). The set X is stratified by a system $\mathfrak{N}X$ of subsets (*covers*), representing possible requirements on precision (a cover $C \subseteq X$ satisfies the condition that for every $x \in X$, $x = \sup(\downarrow C \cap \downarrow x)$, that is, if $y \geq u$ for every u less or equal than x and any $c \in C$, then $x \leq y$.

The covers in $\mathfrak{N}X$ are supposed to be sufficiently many (technically, for C and x set first $C_x = \{c \in C \mid \downarrow x \cap \downarrow c \neq \emptyset\}$, then define $x \lhd_C z$ as $C_x \subseteq \downarrow z$; it is required that for each $z \in X$, $z = \sup\{x \in X \mid x \lhd_C z$ for some $C \in \mathfrak{N}C\}$).

Now an approximated map $f : X \to Y$ is a relation $f \subseteq X \times Y$ such that

(AM1) $1f1$.

(AM2) If $x' \leq x$, xfy, and $y \leq y'$ then $x'fy'$.

(AM3) If $\downarrow x_1 \cap \cdots \cap \downarrow x_n \neq \emptyset$ in X and $x_i f y_i$ for $i = 1, \ldots, n$ then $\downarrow y_1 \cap \cdots \cap \downarrow y_n \neq \emptyset$.

(AM4) For any $C \in \mathfrak{N}Y$, $f^{-1}[C] = \{x \in X \mid xfy \in C\} \in \mathfrak{N}X$.

Our main result is that

the category of approximation systems and approximation maps is, up to the equivalence of morphisms given by common refinement, dually equivalent with the category of complete nearness frames.

10. Notes. 1. This can be easily modified to suitably stratified Scott information systems as an "intermediate generalization".

2. In the case of countable stratification one obtains complete metric spaces, if one so wishes – of course using choice principles.