Two Notions of Universality Considered for Bounded Lattices and Kleene Algebras

presented by

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A class $\mathbf{V}$ of algebras is a *variety* (*equationally defined class*) providing $\mathbf{V} = \mathbf{HSP}(\mathbf{V})$ (equivalently it is defined by a set of equations of the form $p = q$.)

Notion I:

$\mathbf{V}$ is *universal* (in the categorical sense) if every category of algebras of finite type is isomorphic to a full subcategory of it. As shown by Hedrlín, Pultr, and Vopěnka in 1960's, this is equivalent to the category $\mathbf{G}$ of all directed graphs together with all their compatible mappings being isomorphic to a full subcategory.

If the functor $\Phi : \mathbf{G} \longrightarrow \mathbf{V}$ may be chosen to assign a finite algebra to each finite graph, then $\mathbf{V}$ is *finite-to-finite universal.*

Consequence:

For every monoid M, there exists a proper class of non-isomorphic algebras belonging to $\mathbf{V}$ each of which has an endomorphism monoid isomorphic to M.

Further, if $\mathbf{V}$ is finite-to-finite universal, for each finite monoid M, there exist infinitely many non-isomorphic finite algebras with the preceding property.

Examples:

- Unary algebras, i.e. $(A; f_1, \ldots, f_n)$ of type $\langle 1, \ldots, 1\rangle$ are universal iff $n \geq 2$ (and then they are finite-to-finite universal). Hedrlín and Pultr

- Bounded lattices $(L; \vee, \wedge, 0, 1)$ of type $\langle 2, 2, 0, 0 \rangle$ where $(L; \vee, \wedge)$ is a lattice with minimum element 0 and greatest element 1 (homomorphisms preserve 0 and 1).

Grätzer and Sichler

Finite-to-finite. $\sim$ and Sichler

Notion II:

A class $\mathbf{V}$ of algebras is a *quasivariety* providing $\mathbf{V} = \mathbf{ISPP_u}(\mathbf{V})$ (equivalently it is defined by a set of quasi-identities of the form $(p_0 = q_0 \wedge \ldots \wedge p_n = q_n) \longrightarrow p = q$.)

Note: Every variety is a quasivariety.

For a variety $\mathbf{V}$, let $L(\mathbf{V})$ denote the lattice (ordered by inclusion) of all quasivarieties contained in $\mathbf{V}$.

$\mathbf{V}$ is *Q-universal* providing, for any quasivariety $\mathbf{K}$ of finite type, $L(\mathbf{K})$ is a homomorphic image of a sublattice of $L(\mathbf{V})$ (i.e., $L(\mathbf{K})$ is a *divisor* of $L(\mathbf{V})$.)

Consequences:

- $|L(\mathbf{V})| = 2^\omega$.
- A free lattice on ω generators is embeddable in $L(\mathbf{V})$ and, hence, $L(\mathbf{V})$ fails every non-trivial lattice identity.

Examples:

- (Commutative 3-nilpotent) semigroups. Sapir

- Unary algebras for $n \geq 1$. Kartashov and Gorbunov
- Bounded lattices.

Dziobiak and Tumanov, independently

Connection:

If $\mathbf{V}$ is finite-to-finite universal, then it is Q-universal.

$\sim$ and Dziobiak

Examples:

- Unary algebras. By Hedrlín and Pultr, for $n \geq 2$, finite-to-finite universal, so by the *Connection*, Q-universal. However for $n = 1$, not universal, but by Kartashov and Gorbunov, are Q-universal.

  (i.e., the *Connection* is a strict implication.)

- Bounded lattices:

  A discussion of the *Lattice of all varieties of bounded lattices ordered by inclusion*

  Goralčík, Koubek, and Sichler have shown: a variety $\mathbf{V}$ of bounded lattices is (finite-to-finite) universal iff it contains a (finite) simple non-distributive bounded lattice.

- There are 2^{ω} varieties of bounded lattices which are Q-universal but not universal. $\sim$ and Dziobiak

Problem: Characterize Q-universal varieties of bounded lattices.

Kleene Algebras:

A *de Morgan algebra* $(L; \vee, \wedge, \sim, 0, 1)$ of type $\langle 2, 2, 1, 0, 0\rangle$ where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and

$$\sim (x \vee y) \ = \sim x \wedge \sim y,$$

$$\sim (x \wedge y) \ = \sim x \vee \sim y,$$

$$\sim 0 = 1, \quad \sim 1 = 0$$

$$\sim\sim x = x.$$

It is a *Kleene algebra* providing

$$x \wedge \sim x \ \leq \ y \vee \sim y.$$

The lattice of subvarieties of de Morgan algebras is a 4-element chain ($\mathbf{T}, \mathbf{B}, \mathbf{K}$ and $\mathbf{M}$): Kalman

A discussion of the *Lattice of all subvarieties of de Morgan algebras ordered by inclusion*

Relevant Facts:

- De Morgan algebras are finite-to-finite universal and, hence Q-universal. $\sim$ and Priestley
- Kleene algebras are not universal.

Reason: The only *rigid* (i.e., the identity is the only endomorphism) Kleene algeras are the 2-element and 3-element chains, but to be universal requires a proper class of non-isomorphic algebras which are rigid.

- Kleene algebras are *almost universal* (in the sense that there is an embedding $\Phi : \mathbf{G} \longrightarrow \mathbf{K}$ which is almost full meaning that every non-constant homomorphism between objects of $\Phi(\mathbf{G})$ is in the image of Φ, where a homomorphism is called a *constant* whenever its image consists only of the constants 0 and 1.) $\sim$ and Priestley

- Kleene algebras are not finite-to-finite almost universal.

- Kleene algebras are Q-universal. $\sim$ and Dziobiak

Consequence: Since $\mathbf{K}$ is generated by a 3-element chain, there exists a quasivariety generated by a 3-element algebra with 2^ω subquasivarieties. Sapir and Schein (independently) had examples of varieties generated by 3-element semigroups with 2^ω subquasivarieties.

Note: Shafaat showed that any quasivariety generated by a 2-element algebra has only 2 subquasivarieties.

Aside Note: By adding a binary operation $+$ (one of 130 possible) to the 3-element Kleene algebra, the resulting algebra K^+ is such that $L(K^+) \cong \omega + 3$, where, for x and $y \in \{0, a, 1\}$, $x + y = a$ if $x = a$ or $y = a$ but not both, and $x + y = 1$ otherwise.

Problem: Does finite-to-finite almost universal imply Q-universal?