ABSTRACT

Homotopy Methods for Computing Test Sets

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Let A be a real matrix with $m+1$ rows and n columns ($m \geq n$) such that $\pi A = 0$ for a strictly positive vector π. We consider the family of integer programs

$$\max \sum_j a_{0j} h_j subjectto$$
$$\sum_j a_{ij} h_j \geq b_i, for i = 1, \ldots, m,$$
$$h \in Z^n.$$

for all values of the right hand side b.

If A is in general position, it is associated with a unique, minimal test set $N(A)$ called the set of *neighbors of the origin*. $N(A)$ is a finite set of integer vectors in Z^n, typically taken to be symmetric about the origin. $N(A)$ is a *test set* in the sense that for any particular b, an integral, feasible solution to the constraints, h, is optimal if $h + k$ is infeasible for every $k \in N$ with $a_0 k > 0$. The test set is *minimal*: if any member of $N(A)$ is discarded, a specific right hand side b and feasible solution h can be selected, such that h is not optimal for that right hand side, but its lack of optimality cannot be detected using the smaller test set. Moreover any test set that detects optimality in this sense must be a superset of $N(A)$.

In general, the set of neighbors is robust to small changes in the entries of the matrix A. For each row i there is a polyhedral cone $C_i(A)$, such that $N(A)$ is constant as long as the ith row a_i remains in $C_i(A)$ for all i. If we select a particular row of A, say row 0, and imagine changing a_0 so that it passes through a facet of the cone $C_0(A)$ - into the adjacent cone on the opposite side of the facet - the neighbors will change in a specific, computable fashion. This information can be used to provide a homotopy algorithm for calculating test sets, by tracing the sets of neighbors from a matrix whose neighbors we know, to one whose neighbors we wish to find. The computational procedure has much in common with techniques used in algebraic geometry for computing Gröbner bases for certain polynomial ideals.