Planar graphs without cycles of specific lengths

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Abstract

It is easy to see that planar graphs without 3-cycles are 3-degenerate. Recently, it was proved that planar graphs without 5-cycles are also 3-degenerate. In this paper it is shown, more surprisingly, that the same holds for planar graphs without 6-cycles.

1 Introduction

A graph G is *d-degenerate* if every subgraph H of G has a vertex of degree at most d in H. It is an easy consequence of Euler's formula that every triangle-free planar graph contains a vertex of degree at most 3. Therefore, triangle-free planar graphs are 3-degenerate.

Recently, Weifan and Lih [12] proved that planar graphs without 5-cycles are 3-degenerate. In this paper we study planar graphs without cycles of length 6. We show that every such graph is 3-degenerate, a result that was first conjectured by Weifan and Lih [13]. This implies:

Theorem 1.1 *If G is a planar graph of minimum degree ≥ 4, then G contains a 3-cycle, a 5-cycle, and a 6-cycle.*

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There exist planar graphs of minimum degree 4 without cycles of length 4. An example of such a graph is obtained by taking the line graph of a cubic planar graph of girth 5, e.g., the line graph of the dodecahedron. Also, for every $k \geq 7$, there is a planar graph of minimum degree 4 without k-cycles. Such an example is the octahedron graph. Hence, Theorem 1.1 is best possible.

One of the main motivations to study degenerate graphs is the theory of graph colorings. The concept of a list coloring is a generalization of ordinary colorings that attracted considerable attention in the last decade, cf. [1, 3, 4, 9].

A graph G is *k-choosable* if for every function $L: V(G) \to \mathcal{P}(\mathbb{N})$ with $|L(v)| \geq k$ for every $v \in V(G)$, there exists a list coloring $\lambda: V(G) \to \mathbb{N}$, where $\lambda(v) \in L(v)$ for every vertex $v \in V(G)$ and $\lambda(u) \neq \lambda(v)$ for every edge $uv \in E(G)$.

List colorings of planar graphs have been extensively studied. Thomassen [7] proved that every planar graph is 5-choosable. Examples of non-4-choosable planar graphs were constructed by Voigt [10], and later also by Gutner [2] and Mirzakhani [6]. The Grötzsch theorem states that $\chi(G) \leq 3$ for every planar graph G without triangles. This is not true for list colorings as shown in [11]; see also [2]. On the other hand, triangle-free planar graphs are 3-degenerate which implies that they are 4-choosable. Thomassen [8] also proved that every planar graph of girth at least 5 is 3-choosable.

Theorem 1.1 combined with a result of Lam, Xu, and Liu [5], who proved that every planar graph without 4-cycles is 4-choosable, implies:

Theorem 1.2 *Let G be a planar graph and k an integer, $3 \leq k \leq 6$. If G has no cycles of length k, then G is 4-choosable.*

We conjecture that Theorem 1.2 can be extended to $k = 7$. It would also be interesting to find the maximum integer κ such that every planar graph without κ-cycles is 4-choosable. An example of a non-4-choosable planar graph by Mirzakhani [6] shows that $\kappa \leq 63$.

2 Planar graphs without 6-cycles

We begin this section with a useful tool from elementary topology.

Interlacing Lemma. *Let D be a closed disc in the plane and let the points p_1, q_1, p_2, and q_2 appear in this order along the boundary of D. Then we* *cannot simultaneously join p_1 with p_2 and q_1 with q_2 with disjoint paths that are disjoint from the interior of D.*

Let G be a plane graph. A vertex of degree d is called a *d-vertex*. If f is a face of G, then $\deg(f)$ denotes the length of f and we say that f is a *$\deg(f)$-face*. A 3-face is also called a *triangle*. Two faces of G are said to be *adjacent* if their facial walks have an edge in common. A *cluster of triangles* is a subgraph of G which consists of a nonempty minimal set of 3-faces such that no other 3-face is adjacent to a member of this set. Let us remark that each cluster corresponds to a connected component of the subgraph of the dual graph of G induced by the degree 3 vertices in the dual graph of G.

In the sequel, we shall assume that G has no cycles of length 6, and that G has no vertices of degree ≤ 3. First, we shall describe possible clusters in G.

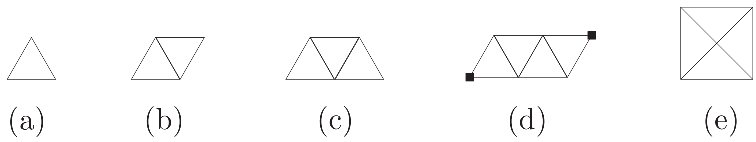

Figure 1: Possible clusters

Claim 1. *There are only five possible clusters of triangles in G. They are shown in Fig. 1 where the black squares in Fig. 1(d) represent the same vertex. In particular, every cluster of triangles contains at most 4 triangles.*

Proof. The claim is clear for clusters with at most three triangles. There are four nonisomorphic ways to increase the number of triangles in a cluster from three to four. Two of these clusters contain a 6-cycle. The cluster in Fig. 1(d) also contains a 6-cycle unless the two vertices, shown as black squares, are identified. Knowing possible clusters with four triangles, it is easy to see that adding a fifth triangle to a cluster of four triangles yields a forbidden subcluster on four triangles except in the case of the cluster which is obtained from Fig. 1(d) by adding a triangle at left side. This gives rise to a new vertex x. Excluding C_6, x must be identified with the lower right vertex of the cluster. By the Interlacing Lemma, this is not possible. $\square$

We say that a face f is *adjacent* to a cluster $\mathcal{C}$ if f is adjacent to a face in $\mathcal{C}$. The following claim describes faces of small length that can be adjacent to a given cluster.

Claim 2.

(1) *A cluster of two triangles has at most one adjacent 4-face and forces an identification as shown in Fig. 2(a).*

(2) *If a 4-face has two adjacent 3-faces, they are positioned as shown in Fig. 2(b).*

(3) *Two adjacent 4-faces force an identification as in Fig. 2(c), and there is only one way for them to be adjacent to a triangle as shown in Fig. 2(d).*

(4) *There can be only one triangle adjacent with a 5-face and it forces an identification as shown in Fig. 2(e). Two 5-faces cannot be adjacent to the same triangle, or even to a cluster of two triangles.*

(5) *A cluster of three triangles can be adjacent to a 4-face in a unique way, shown in Fig. 2(f).*

(6) *There is only one way for a cluster of three triangles to be adjacent to a 5-face, see Fig. 2(g).*

(7) *A cluster of two (or three) triangles cannot be simultaneously adjacent to a 4-face and a 5-face.*

(8) *A cluster of four triangles has no adjacent 4- and 5-faces.*

Proof. Since G does not contain a 6-cycle, the proofs of (1), (2), (3), and (4) are clear from Fig. 3(a)–(e). By (1) and (4), attaching a 4-face or a 5-face to a cluster of three triangles yields a 6-cycle (cf. Fig. 3(f)–(i)) or a configuration from (5) or (6), respectively. Claims (1), (4), and the Interlacing Lemma easily imply (7) for a cluster of two triangles. Similarly (5), (6), and the Interlacing Lemma imply (7) for clusters of three triangles. Finally, (8) follows by claims (1), (4), (5) and the Interlacing Lemma. $\square$

By Claim 1, no four consecutive faces around a vertex of degree ≥ 5 are triangles. This implies:

Claim 3. *Suppose that* $\deg(v) \geq 5$. *Then there are at most* $\lfloor \frac{3}{4}\deg(v) \rfloor$ *triangles containing* v.

Now, we are ready for the main result of this paper.

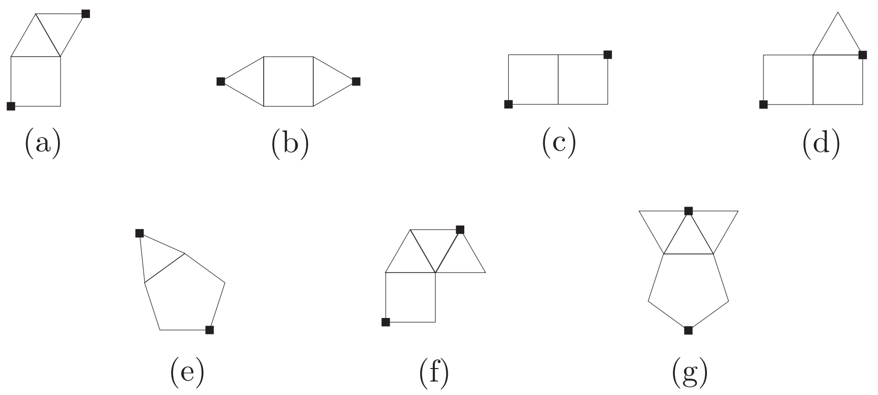

Figure 2: Clusters of triangles and adjacent small faces

Theorem 2.1 *Every planar graph without 6-cycles is 3-degenerate.*

Proof. Suppose that the theorem is false and let G be a counterexample. We may assume that G is connected. Suppose that G is not 2-connected. Take an end-block B of G. Let u be the corresponding cut-vertex in B, and let $v \neq u$ be a vertex of B lying on a common face with u. Denote by B^* the graph constructed from five copies $B_1, \ldots, B_5$ of B such that the copy u_i of u in B_i is identified with the copy v_{i+1} of v in B_{i+1} for $i = 1, \ldots, 4$. The vertices v_1 and u_5 are called the *connectors* of B^*. It is easy to see that B^* has an embedding in the plane such that the connectors are on the boundary of the outer face. Since G is not 2-connected, there is a face f which is incident with vertices x, y that are not contained in the same block of G. Now, we identify the connectors of B^* with x and y, respectively, and embed B^* into f. The resulting graph G' has fewer blocks than G. Clearly, G' has no 6-cycles and its minimum degree is ≥ 4. Therefore, G' is a counterexample to the theorem. By repeating the above construction sufficiently many times, we obtain a 2-connected counterexample.

Thus, we may assume that G is 2-connected and hence all its facial walks are cycles. In particular, G has no faces of length 6. In the rest of the proof, we shall apply the well-known discharging method.

Initial charge. Let $F(G)$ be the set of faces of G. We assign *charge* c to the vertices and faces of G as follows. For $v \in V(G)$, let $c(v) = 3\deg(v) - 12$

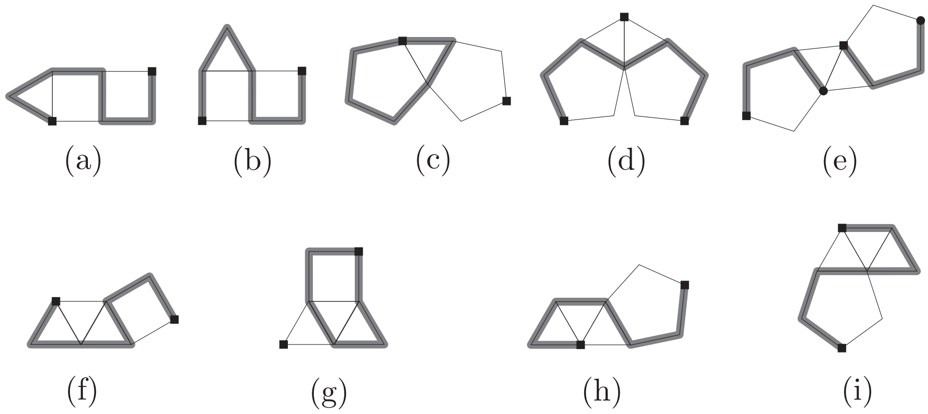

Figure 3: 6-cycles in some clusters

and for $f \in F(G)$, let $c(f) = 3\deg(f) - 12$. We can rewrite the Euler formula in the following form:

$$\sum_{v \in V(G)} (3\deg(v) - 12) + \sum_{f \in F(G)} (3\deg(f) - 12) = -24. \qquad (1)$$

This shows that the total charge of vertices and faces of G is negative. Next, we redistribute the charge of vertices and faces by applying the rules R1–R5 described below so that the total charge remains the same. The rules are such that the resulting charge of all vertices and of all faces of length $r \geq 4$ ($r \neq 7$) is clearly nonnegative. The same will be proved for the charge of 7-faces. Furthermore, we shall prove that in each cluster of triangles, the total charge is also nonnegative. This will contradict (1) and complete the proof.

After applying Rules R1–R3, which is called *Phase 1* of the discharging, only some 7-faces may have negative charge. Afterwards, we apply *Phase 2* (Rules R4 and R5) after which all vertices and faces (clusters) have nonnegative charge.

Discharging – Phase 1: In Phase 1, Rules R1–R3 described below are applied.

Rule R1. Let v be a vertex of degree ≥ 5 which is incident with $t \geq 1$ triangles.

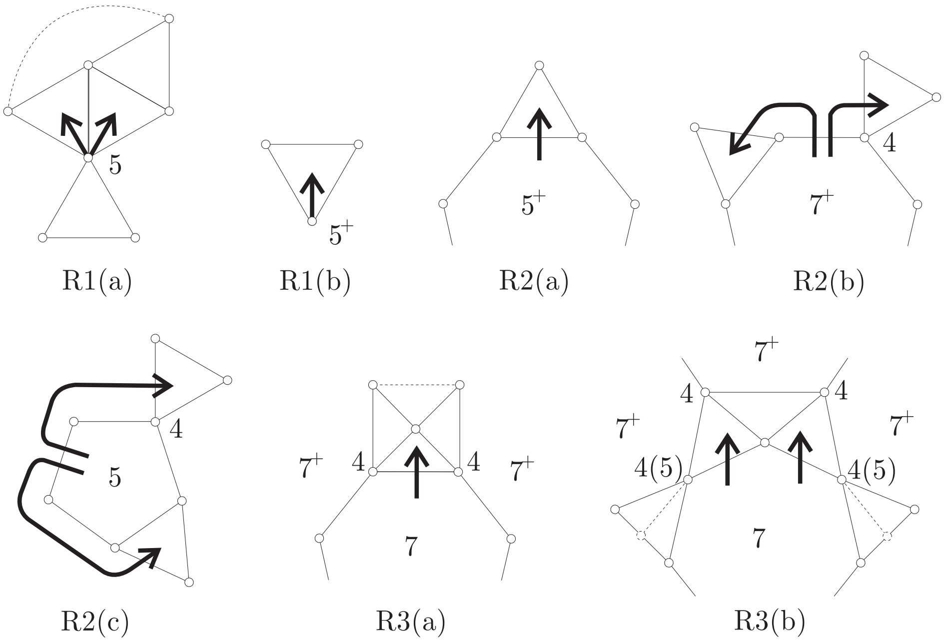

Figure 4: Discharging rules R1–R3.

(a) Suppose that $\deg(v) = 5$, $t = 3$, and v is a vertex of a cluster $\mathcal{C}$ from Fig. 1(c) or (e) that is contained in precisely two triangles of $\mathcal{C}$. Denote by f the 3-face incident with v which does not belong to $\mathcal{C}$. Then v sends $\frac{3}{2}$ to each incident triangle in $\mathcal{C}$ (and 0 to f). See Fig. 4.

(b) Otherwise, v sends charge $c(v)/t$ to each of the triangles incident with v.

Rule R2. Let f be a face of length ≥ 5. If e is an edge of f and the face $f' \neq f$ containing e is a 3-face, then f' is said to be a *sink* with respect to (f, e). Suppose that v is a 4-vertex on f and let f, f_1, f_2, f_3 be the faces incident with v in that (or reverse) order. Then f_2 is a *sink* with respect to (f, v) if $\deg(f_2) = 3$ and $\deg(f_1) \geq \deg(f_3) = 4$. Let $\bar{c} = 1$ if $\deg(f) = 7$, and let $\bar{c} = c(f)/\deg(f)$ otherwise.

(a) If f' is a sink with respect to (f, e), then f sends charge $\bar{c}$ to f' through e.

(b) If $\deg(f) \geq 7$ and $e = uv$ is an edge on f which is not incident with a 3-face, let e_1 and e_2 be the edges which precede and succeed e on the boundary of f, respectively. Then f sends through e charge $\bar{c}/2$ to each sink with respect to (f, e_1), (f, u), (f, v), and (f, e_2).

(c) Suppose that $\deg(f) = 5$ and that e_1 is an edge on f which is not incident with a 3-face and that $f = v_1 e_1 v_2 e_2 \cdots v_5 e_5 v_1$. Then f sends through e_1 charge $\bar{c}/2$ $(= 3/10)$ to the first existing sink with respect to (f, v_2), (f, e_2), (f, v_3), $(f, e_3), \ldots$, and sends $\bar{c}/2$ to the first existing sink with respect to (f, v_1), (f, e_5), (f, v_5), $(f, e_4), \ldots$ (assuming at least one sink exists).

Let us observe that in Rule R2(b), at most two of the four possible sinks exist.

Rule R3. Let f be a 7-face.

(a) Suppose that uv is an edge of f, and u, v are 4-vertices. Suppose also that u and v are contained in precisely two triangles of a cluster $\mathcal{C}$ from Fig. 1(c) or (e). Then f sends charge 1 to the adjacent triangle in $\mathcal{C}$ (in addition to the charge sent by Rule R2).

(b) Suppose that $u_1 v$ and $v u_2$ are edges of f, and v is a 4-vertex that is incident with three triangles of a cluster $\mathcal{C}$ from Fig. 1(c). Suppose also that u_i $(i = 1, 2)$ is of degree 4 or 5 and there are $\deg(u_i) - 2$ triangles incident with u_i and that the two vertices of $\mathcal{C}$ distinct from v, u_1, u_2 are both of degree 4 in G. Then f sends $\frac{1}{2}$ to each of the adjacent triangles in $\mathcal{C}$ (in addition to Rule R2).

Let v be a vertex and let f be a 3-face incident with v. If $\deg(v) = 5$, then v sends no charge to f only if f is the face in Rule R1(a) which does not belong to $\mathcal{C}$. Otherwise, v sends charge 1 to f if v is incident with three 3-faces. In all other cases, v sends to f at least charge $\frac{3}{2}$. If $\deg(v) \geq 6$, Claim 3 implies that v sends to f charge $\geq \widehat{c}$, where the values of $\widehat{c}$ are collected in Table 1.

Let f and f' be adjacent faces, where $\deg(f') = 3$. If $\deg(f) \geq 5$, then f sends to f' charge $\geq \widehat{c}$ where $\widehat{c}$ is given in Table 1. This is clear if $\deg(f) \geq 7$. If $\deg(f) = 5$, then f can be adjacent with only one 3-face by Claim 2(4). Therefore, f' receives $\frac{3}{5}$ from f through the common edge e by Rule R2(a), and receives twice $\frac{3}{10}$ through the edges which precede and succeed e in f, respectively, by Rule R2(c).

$\deg(v)$	6	7	≥ 8
$\widehat{c}$	$3/2$	$9/5$	$4 - 16/\deg(v) \geq 2$

$\deg(f)$	5	7	≥ 8
$\widehat{c}$	$6/5$	1	$3/2$

Table 1: The charge sent to a triangle from vertices and faces

Charge after Phase 1. It is easy to see that after Phase 1, the charge of every vertex and every face of length $r \geq 4$, $r \neq 7$, is nonnegative. Next, we shall prove that the same holds for every cluster $\mathcal{C}$ of triangles, i.e., the sum of the charges of triangles in $\mathcal{C}$ is nonnegative. In other words, if $\mathcal{C}$ contains k triangles, we will prove that the total charge c^* which the cluster receives by Rules R1–R3 is at least $3k$.

We split the analysis into five cases, depending on the type of the cluster, cf. Claim 1.

Case 1: $\mathcal{C}$ is a cluster consisting of one triangle f.

We say that a vertex v of f is *important*, if $\deg(v) \geq 5$ and Rule R1(a) does not apply to v. Let $U \subseteq V(f)$ be the set of important vertices of f. Let $U_1 \subseteq V(f)$ be the set of those vertices of degree 4 that are incident with exactly one 4-face adjacent to f, and $U_2 \subseteq V(f)$ the set of those vertices of degree 4 that are incident with exactly two 4-faces adjacent to f. Moreover, let F be the set of those faces adjacent to f that are of length ≥ 5. If every vertex of f is either important or of degree 4, it is easy to check that

$$|U| + \frac{1}{2}|U_1| + |U_2| + |F| \geq 3\,. \tag{2}$$

If a 5-vertex $v \in V(f)$ is not important, then both faces incident with v and adjacent to f are in the set F by Claim 2(2). This implies that (2) also holds if f contains at least two such vertices. If f has precisely one such vertex v, then either the third face adjacent to f is also in F, a vertex of f is in U, or both vertices of f distinct from v are in U_1. Therefore, (2) always holds.

We will show (with one possible exception) that each element from $U \cup U_2 \cup F$ contributes charge ≥ 1 to f and each element from U_1 contributes charge $\geq \frac{1}{2}$ to f. (As a *contribution* of a vertex $v \in U_1 \cup U_2$ we consider the charge sent to f from the face incident with v and not adjacent to f.)

If $v \in U$, it obviously sends charge ≥ 1 to f. The same also holds for the faces in F (cf. Table 1). Let us now consider a vertex $v \in U_1 \cup U_2$; see

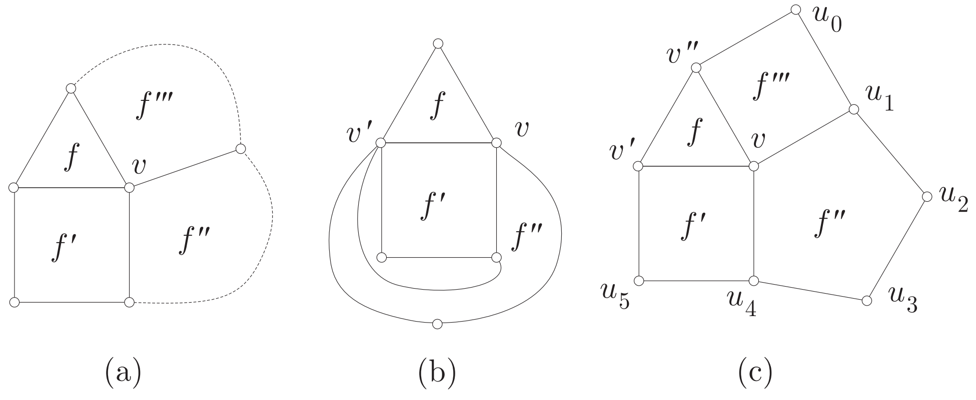

Figure 5: The case when $v \in U_1 \cup U_2$

Fig. 5(a). By Claim 2(2), $\deg(f'') \geq 4$. If $\deg(f'') = 4$, we have the situation in Fig. 5(b) by Claim 2(3). This situation is the exception and is treated separately as follows. By Claim 2(2) and the Interlacing Lemma, f is the only 3-face that is adjacent to f'. Since also f is adjacent to no other 3-face, it follows that v' always sends charge ≥ 3 to f.

If $\deg(f'') \geq 7$, then f'' sends by R2(b) charge $\geq 2 \cdot \frac{1}{2} = 1$ to f (since f is a sink for f''). Therefore, v contributes ≥ 1. It remains to consider the case when $\deg(f'') = 5$. If $v \in U_1$, we are done since f'' sends to f charge $\geq 2 \cdot \frac{1}{2} \cdot \frac{3}{5} > \frac{1}{2}$. Otherwise we have the situation in Fig. 5(c) where some identifications are possible. The only possible identifications are $v' = u_2$, $v'' = u_3$, or $u_0 = u_5$. Each of these identifications excludes the other two. All other identifications give rise to a 6-cycle in G: if $v' = u_3$, we have $v'u_5u_4vu_1u_2u_3$ (and similarly if $v'' = u_2$); if $u_0 = u_3$, we have $u_0v''vv'u_5u_4u_3$ (and similarly if $u_5 = u_2$); if $u_0 = u_4$, we have $u_0v''vu_1u_2u_3u_4$ (and similarly if $u_5 = u_1$). Let $f_1 \neq f''$ be the other face containing u_1u_2 in its boundary. We claim that $\deg(f_1) > 3$. If $\deg(f_1) = 3$, the third vertex of f_1 must be u_4 by Claim 2(4). But then $v' \neq u_2$, $v'' \neq u_3$, and $u_1vv'u_5u_4u_2u_1$ is a 6-cycle, a contradiction. The same arguments show that also u_3u_4 does not belong to a 3-face. We also claim that the face $f_2 \neq f'''$ containing u_0u_1 in its boundary is not a 3-face. If $\deg(f_2) = 3$, then the third vertex of f_2 must be v' by Claim 2(2). Then, again, $v' \neq u_2$, $v'' \neq u_3$, and $v'u_1u_2u_3u_4vv'$ is a 6-cycle, a contradiction. By symmetry, u_4u_5 does not belong to a 3-face. The preceding discussion implies that f receives half of the charge sent out of f'' through edges vu_1, vu_4, u_1u_2, and u_3u_4 by Rule R2(c). Therefore v contributes to f charge at least $4 \cdot \frac{1}{2} \cdot \frac{3}{5} > 1$.

Case 2: $\mathcal{C}$ is a cluster of two triangles, $f_1 = v_1v_2v_3v_1$ and $f_2 = v_1v_3v_4v_1$.

Suppose first that there is a 4-face adjacent to the cluster $\mathcal{C}$. By Claim 2(1), we may assume that the 4-face is $f' = v_1v_2v_4xv_1$. By Claim 2(1) and (7), other faces adjacent to $\mathcal{C}$ are of length ≥ 7. So, they send ≥ 3 to $\mathcal{C}$. If $\deg(v_1) \geq 5$ or $\deg(v_3) \geq 5$, then that vertex sends ≥ 1 to each of f_1 and f_2. Therefore, $c^* \geq 6$ if $\deg(v_1) \geq 5$ and $\deg(v_3) \geq 5$. If $\deg(v_1) = 4$, then v_4 is of degree ≥ 5; otherwise x is a cut-vertex. The face f'' containing the edge xv_4 ($f'' \neq f'$) is not a 3-face by Claim 2(2) and the Interlacing Lemma. Since x is not a cut-vertex, f'' does not contain the edge v_1v_4. This implies that v_4 is incident with at least four nontriangular faces. Therefore v_4 sends charge $\geq (3\deg(v_4) - 12)/(\deg(v_4) - 4) = 3$ to f_2. Consequently, $c^* \geq 6$. Finally, suppose that $\deg(v_1) \geq 5$ and $\deg(v_3) = 4$. Then $\mathcal{C}$ receives charge ≥ 3 from adjacent faces by R2(a), ≥ 2 from v_1, and another ≥ 1 from adjacent faces at v_3 by R2(b).

Now, assume that no 4-face is adjacent to $\mathcal{C}$. If there is an adjacent 5-face, it follows by Claim 2(4) that v_1 or v_3 is of degree ≥ 5, and so it sends ≥ 1 to each of f_1 and f_2. Thus, $c^* \geq 4 \cdot 1 + 2 \cdot 1 = 6$.

Finally, assume that all four faces adjacent to $\mathcal{C}$ are of length ≥ 7. If $\deg(v_1) \geq 5$ or $\deg(v_3) \geq 5$, then $c^* \geq 4 \cdot 1 + 2 = 6$. So, assume that $\deg(v_1) = \deg(v_3) = 4$. In this case, by Rule R2(b), each adjacent face contributes at least an additional $\frac{1}{2}$ to $\mathcal{C}$. Thus, $c^* \geq 4 \cdot 1 + 4 \cdot \frac{1}{2} = 6$.

Case 3: $\mathcal{C}$ is a cluster of three triangles, $f_1 = v_1v_2v_5v_1$, $f_2 = v_2v_3v_5v_2$, and $f_3 = v_3v_4v_5v_3$.

Suppose first that there is a 4-face adjacent to $\mathcal{C}$. Then, by Claim 2(5), we may assume that it is $v_1v_5xv_3v_1$. By Claim 2(7), it follows that other adjacent faces are of length ≥ 7. So, by Rule R2(a), they send ≥ 4 to $\mathcal{C}$. Vertex v_5 is of degree ≥ 5, so it sends ≥ 1 to each of f_1, f_2, f_3. Note also that v_3 is of degree ≥ 5, so it sends ≥ 1 to each of f_2, f_3. Thus, $c^* \geq 4 + 3 \cdot 1 + 2 \cdot 1 = 9$.

Now, we assume that there are no adjacent 4-faces. Suppose that there is a 5-face adjacent to $\mathcal{C}$. By Claim 2(6), we may assume that this 5-face is $v_2xv_5yv_3v_2$ and that all other faces adjacent to $\mathcal{C}$ are of length ≥ 7. Note that $\deg(v_5) \geq 6$. So, v_5 sends $\geq \frac{3}{2}$ to each of f_1, f_2, f_3. Hence, $c^* \geq 5 \cdot 1 + 3 \cdot \frac{3}{2} > 9$.

Finally, we may assume that all faces adjacent to $\mathcal{C}$ are of length ≥ 7.

Hence, they send ≥ 5 to $\mathcal{C}$. If v_2 is of degree ≥ 5, then it sends $\geq \frac{3}{2}$ to each of f_1, f_2. Otherwise, $\deg(v_2) = 4$. Then, by Rule R2(b), the face adjacent at v_1v_2 sends at least an additional $\frac{1}{2}$ to f_1, and the face adjacent at v_2v_3 sends at least an additional $\frac{1}{2}$ to f_2 by Rule R2(b). The same conclusions hold at v_3. Hence, if either $\deg(v_2) \geq 5$ or $\deg(v_3) \geq 5$, then $c^* \geq 5 + 2 \cdot \frac{3}{2} + 1 = 9$.

Suppose now that $\deg(v_2) = \deg(v_3) = 4$. Let $g_i \not\subseteq \mathcal{C}$ be the face which is adjacent to $\mathcal{C}$ and contains the edge v_iv_{i+1} (indices modulo 5). By R2(a) and R2(b), g_1 and g_3 each sends $\geq \frac{3}{2}$ to $\mathcal{C}$. Similarly, g_2 sends ≥ 3 to $\mathcal{C}$ if $\deg(g_2) \geq 8$. If $\deg(g_2) = 7$, then g_2 sends 2 to $\mathcal{C}$ by R2(a) and R2(b), and another 1 by Rule R3(a). In all cases, $\mathcal{C}$ receives ≥ 6 from g_1, g_2, g_3. If $\deg(v_5) \geq 5$, then $\mathcal{C}$ receives ≥ 3 from v_5. Hence, we may assume that $\deg(v_5) = 4$. Then $g_4 = g_5$. If $\deg(g_4) \geq 8$, then g_4 sends to $\mathcal{C}$ at least 3 by R2(a) through the edges v_4v_5 and v_5v_1. So, it remains to consider the case when $\deg(g_4) = 7$. Then $\mathcal{C}$ receives at least 6 from g_1, g_2, g_3, and 2 from g_4 by R2(a). We shall prove that additional charge 1 is sent to $\mathcal{C}$ either from v_1, from v_4, by R2(b) from g_1, g_3, and g_4, or by R3(b) from g_4. If $\deg(v_1) \geq 6$, then v_1 sends more than 1 to $\mathcal{C}$. Suppose now that $\deg(v_1) \leq 5$. Let g' (and g'') be the face(s) incident with v_1 and distinct from g_1, f_1, g_5. If $\deg(v_1) = 5$ and g', g'' are not both triangles, then v_1 sends more than 1 to $\mathcal{C}$. If $\deg(v_1) = 4$ and $\deg(g') \geq 4$, then g_1 and g_5 send to $\mathcal{C}$ at least 1 by R2(b). By symmetry, the same conclusion can be made at v_4. Now, the only remaining case is when $\deg(v_1)$ and $\deg(v_4)$ are equal to 4 or 5 and their incident faces satisfy the requirements in Rule R3(b). Therefore, the face g_4 sends an additional charge 1 to $\mathcal{C}$ by R3(b).

Case 4: $\mathcal{C}$ is a cluster of four triangles, $f_1 = v_1v_2v_6v_1$, $f_2 = v_2v_5v_6v_2$, $f_3 = v_2v_3v_5v_2$, $f_4 = v_3v_4v_5v_4$, and $v_1 = v_4$.

By Claim 2(8), all faces adjacent to $\mathcal{C}$ are of length ≥ 7. So, they send ≥ 6 to $\mathcal{C}$. Since G is 2-connected, if $\deg(v_2) = 4$, then $\deg(v_1) \geq 5$. In this case, v_1 sends $\geq \frac{3}{2}$ to each of f_1, f_4. And, if $\deg(v_2) > 4$, then v_2 sends ≥ 3 to the cluster. The same conclusion holds at v_5. Therefore, if either $\deg(v_2) \geq 5$ or $\deg(v_5) \geq 5$, then $c^* \geq 6 + 3 + 3 = 12$. Suppose now that $\deg(v_2) = \deg(v_5) = 4$. The triangle $v_1v_2v_3v_1$ is not facial. Hence, there are edges of G inside that triangle. Since $\deg(v_2) = 4$, they can be incident with v_1 and v_3 only. Since G is 2-connected, at least one such edge is incident with v_1. Similarly, there is an edge incident with v_1 inside the triangle $v_1v_5v_6v_1$. Therefore, $\deg(v_1) \geq 6$. Observe that at least four faces incident with v_1

are not triangles. Therefore, v_1 sends ≥ 3 to each of f_1 and f_4, and so $c^* \geq 6+3+3=12$.

Case 5: $\mathcal{C}$ is a cluster of four triangles, $f_1 = v_1v_2wv_1$, $f_2 = v_2v_3wv_2$, $f_3 = v_3v_4wv_3$, and $f_4 = v_4v_1wv_4$.

For $i = 1,2,3,4$, let g_i be the face which is adjacent to f_i and is not contained in $\mathcal{C}$. We say that g_i *contributes* to $\mathcal{C}$ the charge that is sent from g_i to $\mathcal{C}$ by Rules R2(a), R2(b), and R3(a), plus one half of the charge sent to $\mathcal{C}$ from v_i and v_{i+1} (indices modulo 4). It suffices to prove that g_i contributes to $\mathcal{C}$ at least 3. By Claim 2(8), $\deg(g_i) \geq 7$. Then g_i sends ≥ 1 to $\mathcal{C}$ through v_iv_{i+1}. If $\deg(v_i) \geq 5$, then v_i sends to $\mathcal{C}$ at least 3. If $\deg(v_i) = 4$, then g_i sends to $\mathcal{C}$ at least $\frac{1}{2}$ by R2(b) through the edge $g_i \cap g_{i-1}$. The same holds at v_{i+1}. This implies that g_i contributes at least 3 to $\mathcal{C}$ unless $\deg(v_i) = \deg(v_{i+1}) = 4$, which we assume henceforth. If $\deg(g_i) \geq 8$, then the charge sent to $\mathcal{C}$ by R2(a) and R2(b) is at least 3. If $\deg(g_i) = 7$, then the charge sent to $\mathcal{C}$ by R2(a) and R2(b) is equal to 2, and another 1 is sent by R3(a). This completes the proof of Case 5.

We shall need an extension of the analysis in Case 5 in the proof concerning Phase 2. Observe that for every index $i \in \{1,2,3,4\}$, g_i contributes to $\mathcal{C}$ at least 4 if $\deg(v_i) \geq 5$ and $\deg(v_{i+1}) \geq 5$.

Discharging – Phase 2: After Phase 1, all vertices, all faces of length $\neq 3, 7$, and all clusters of triangles have nonnegative charge.

A 7-face is *bad* if it is negatively charged after Phase 1 of the discharging process. As a bad 7-face f distributes charge ≤ 7 using Rule R2, it sends at least charge 3 to adjacent triangles by R3(a) and (b). Sending charge by R3(b) implies that four consecutive edges along f are incident with triangles, whereas using R3(a) implies that two nonconsecutive edges along f are incident with ≥ 7-faces only. This implies that in a bad 7-face either Rule R3(a) is applied three times, or Rule R3(b) is applied three times.

Suppose now that Rule R3(a) is applied three times in a 7-face f. We shall argue that f is not bad. Let $f = v_1e_1v_2e_2 \cdots e_6v_7e_7v_1$ and suppose that f sends 1 by Rule R3(a) through the edges e_2, e_4, and e_6. Note that faces adjacent to f at the edges e_1 and e_7 are of length ≥ 7. It is enough to see that f sends $\leq \frac{1}{2}$ through each of e_1 and e_7 by Rule R2. Suppose this is not the case and that f sends charge 1 through, say, e_1. Since e_1 and e_7 are not incident with 3-faces, charge $\frac{1}{2}$ is sent by R2(b) to the sink with respect to

(f, v_1). Hence, e_1 (or e_7) is incident with a 4-face, which is a contradiction.

Therefore, a bad face f sends charge 3 to adjacent clusters by using Rule R3(b) only. This implies that the neighborhood of f is as shown in Fig. 6 where $\mathcal{C}$ is a cluster of triangles and v_1, v_2 are of degree ≤ 5.

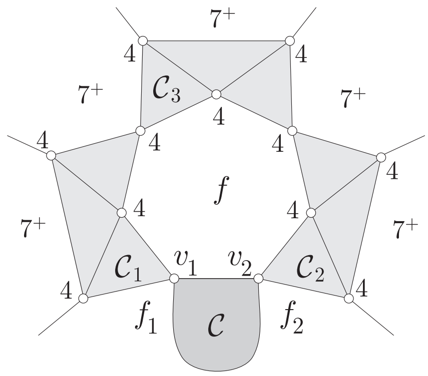

Figure 6: Neighborhood of a bad 7-face

In Phase 2, we send positive charge into bad 7-faces from their neighborhoods by using the following rules.

Rule R4. Let $\mathcal{C}$ be a cluster of triangles with positive charge c^* after Phase 1. If $\mathcal{C}$ is adjacent to $t \geq 1$ bad 7-faces, then $\mathcal{C}$ sends to each of them charge c^*/t.

Rule R5. Let f be a bad 7-face and let f' be a 7-face which has a 4-vertex v in common with f. (Since bad 7-faces are adjacent only to 3-faces, we have the situation shown in Fig. 7 where at least one of the edges e, e' is not incident with a face of length < 7.) We say that f' *touches* f at v. If f' has positive charge c^* after Phase 1 and touches bad 7-faces at t vertices, then f' sends charge c^*/t to f through v.

Let $f = v_1 v_2 \ldots v_7$ be a 7-face. As two bad 7-faces cannot have a vertex in common (see Fig. 6), f does not send charge by R5 through consecutive vertices v_i and v_{i+1}. If, say, f sends charge by R5 through v_3 and v_5, then both edges $v_1 v_2$ and $v_6 v_7$ are incident with faces of length ≥ 7 different from f. Therefore f can send charge by R5 to at most two bad 7-faces.

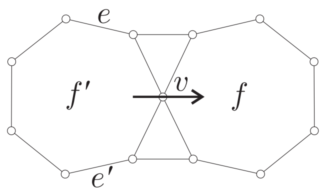

Figure 7: Rule R5

We will show that after Rules R4 and R5 have been applied, all vertices, clusters of triangles, and all faces of length ≥ 4 have nonnegative charge. This is clear in all cases except for bad 7-faces.

Let f be a bad 7-face. As shown in Fig. 6, f is adjacent to four clusters $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, where the notation is taken from the figure. We split the proof according to type of $\mathcal{C}$.

If $\mathcal{C}$ were as in Fig. 1(d), then either v_1 or v_2 would be of degree ≥ 6, which is not possible for a bad 7-face. Suppose that $\mathcal{C}$ is as in Fig. 1(b), where v_1 is incident with two triangles of $\mathcal{C}$. Let us observe that Rule R1(b) (and not Rule R1(a)) was applied at v_1. Then it is easy to see that $\mathcal{C}_1$ has charge ≥ 1 after Phase 1. Since the faces adjacent to $\mathcal{C}_1$ and distinct from f contain edges that are not incident with triangles, f is the only bad 7-face adjacent to $\mathcal{C}_1$. Therefore f receives charge ≥ 1 from $\mathcal{C}_1$ by R4. This proves that f has nonnegative charge after Phase 2.

Suppose now that $\mathcal{C}$ contains a single triangle v_1v_2w. If $\deg(f_1) \geq 8$, then it is easy to see that $\mathcal{C}_1$ has charge $\geq \frac{1}{2}$ and thus f receives $\geq \frac{1}{2}$ from $\mathcal{C}_1$ by R4. Suppose now that f_1 is a 7-face. Let e' be the edge on f_1 incident with w and distinct from wv_1. If e' is not contained in a 3-face, then $\mathcal{C}$ receives additional charge $\frac{1}{2}$ from f_1 through e' (by Rule R2(b)), and this $\frac{1}{2}$ contributes to the positive charge in $\mathcal{C}$ after Phase 1. Therefore, we may say that f *receives* charge $\frac{1}{2}$ from f_1 by R4. If, on the other hand, e' is contained in a 3-face, then it is easy to see that f_1 has sent charge ≤ 1 using Rules R3(a) and R3(b) in Phase 1. Therefore, f_1 has charge ≥ 1 after Phase 1. Since f_1 sends charge by Rule R5 to at most two bad 7-faces in Phase 2, f_1 sends $\geq \frac{1}{2}$ to f. The same analysis applied to f_2 shows that f receives $\geq \frac{1}{2}$ from $\mathcal{C}_2$ or from f_2. Therefore, the charge in f after Phase 2 is nonnegative.

Let us now consider the case when $\mathcal{C}$ is a cluster from Fig. 1(e). If $\mathcal{C}$ is adjacent to another bad 7-face, then that face is neither f_1 nor f_2. When $\mathcal{C}$ is adjacent to two bad 7-faces, all its exterior vertices have degree 5. The remark made after Case 5 in the analysis of Phase 1 implies that $\mathcal{C}$ has charge ≥ 2 after Phase 1, so it sends ≥ 1 to f by R4. If f is the only bad 7-face adjacent to $\mathcal{C}$, then $\mathcal{C}$ has two adjacent vertices v_1, v_2 of degree 5. The same remark implies that the charge at $\mathcal{C}$ after Phase 1 is ≥ 1, and this charge is transferred to f.

It remains to consider the case when $\mathcal{C}$ is a cluster shown in Fig. 1(c). If $\deg(v_1) = \deg(v_2) = 5$, then $\mathcal{C}$ received charge ≥ 5 from adjacent faces and charge 6 from v_1 and v_2 in Phase 1. Therefore, its charge after Phase 1 is ≥ 2. It is not hard to observe that f is the only bad 7-face adjacent to $\mathcal{C}$. Hence, $\mathcal{C}$ sends ≥ 2 to f by R4.

Since v_1 and v_2 have degree ≤ 5, we may henceforth assume that $\deg(v_1) = 4$ and $\deg(v_2) = 5$. If $\deg(f_1) \geq 8$, then after Phase 1, $\mathcal{C}_1$ has charge $\geq \frac{1}{2}$ and $\mathcal{C}$ has charge ≥ 1. Since $\mathcal{C}$ is adjacent to at most two bad 7-faces, f receives $\geq \frac{1}{2}$ from $\mathcal{C}$ and $\geq \frac{1}{2}$ from $\mathcal{C}_1$.

We may henceforth assume that $\deg(f_1) = 7$. For $i = 1, 2$, let v_i' be the vertex of $\mathcal{C}$ distinct from v_i which is contained in i triangles of $\mathcal{C}$. If $\deg(v_2') \geq 5$, then $\mathcal{C}$ has charge ≥ 2 after Phase 1 and is adjacent to at most two bad 7-faces. Therefore, $\mathcal{C}$ sends ≥ 1 to f. Otherwise, $\deg(v_2') = 4$. The edge incident with v_2' which is not contained in $\mathcal{C}$ is not incident with a 3-face. Therefore, f is the only bad 7-face adjacent to $\mathcal{C}$, and v_1' is not incident with a bad 7-face. Hence, Rule R5 does not send charge from f_1 through v_1'. Let t be the number of applications of Rule R5 in f_1. As in the case when $\mathcal{C}$ was a single triangle, we see that $t \leq 2$. If R3 was not used in f_1, then f_1 has charge ≥ 2 after Phase 1. Since $t \leq 2$, f_1 sends ≥ 1 to f. Suppose now that R3 was used in f_1. Since $\deg(v_2) = 5$, f_1 did not send charge to $\mathcal{C}$ by Rule R3(b). This implies that Rule R3 was used exactly once in f_1. Observe that f_1 does not touch a bad 7-face at vertices of the cluster in which f_1 sends charge by Rule R3. A short analysis shows that $t = 1$. The charge in f_1 after Phase 1 is ≥ 1. Hence, f_1 sends ≥ 1 to f by R5. This completes the proof. $\square$

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