On the maximum multiplicity of some extreme geometric configurations in the plane

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- Geometric matching
- Geometric spanning tree
- Geometric tour

- Minimum
- Maximum

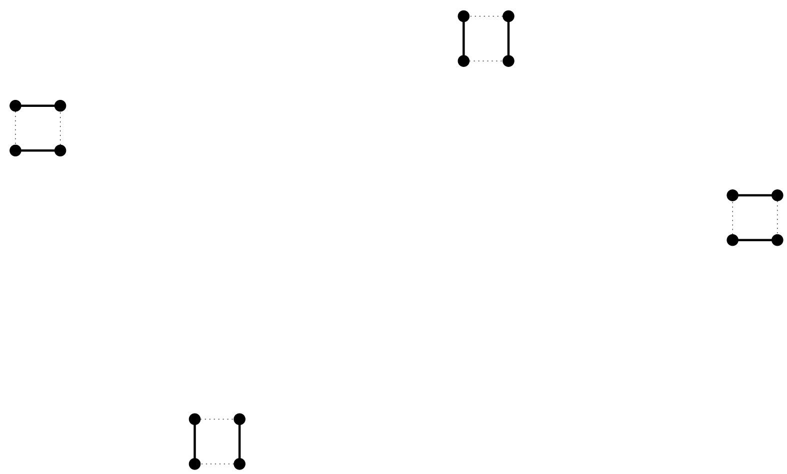
Minimum matching

Let $m(n)$ denote the maximum number of times the minimum perfect matching can occur in an n-element point set in the plane. Similarly, let $m^{gen}(n)$ (resp. $m^{conv}(n)$) denote the maximum number of times the minimum perfect matching can occur in an n-element point set in the plane whose points are in general position (resp. in convex position). Then

(i) $m^{conv}(n) = 2^{\Theta(n)}$.

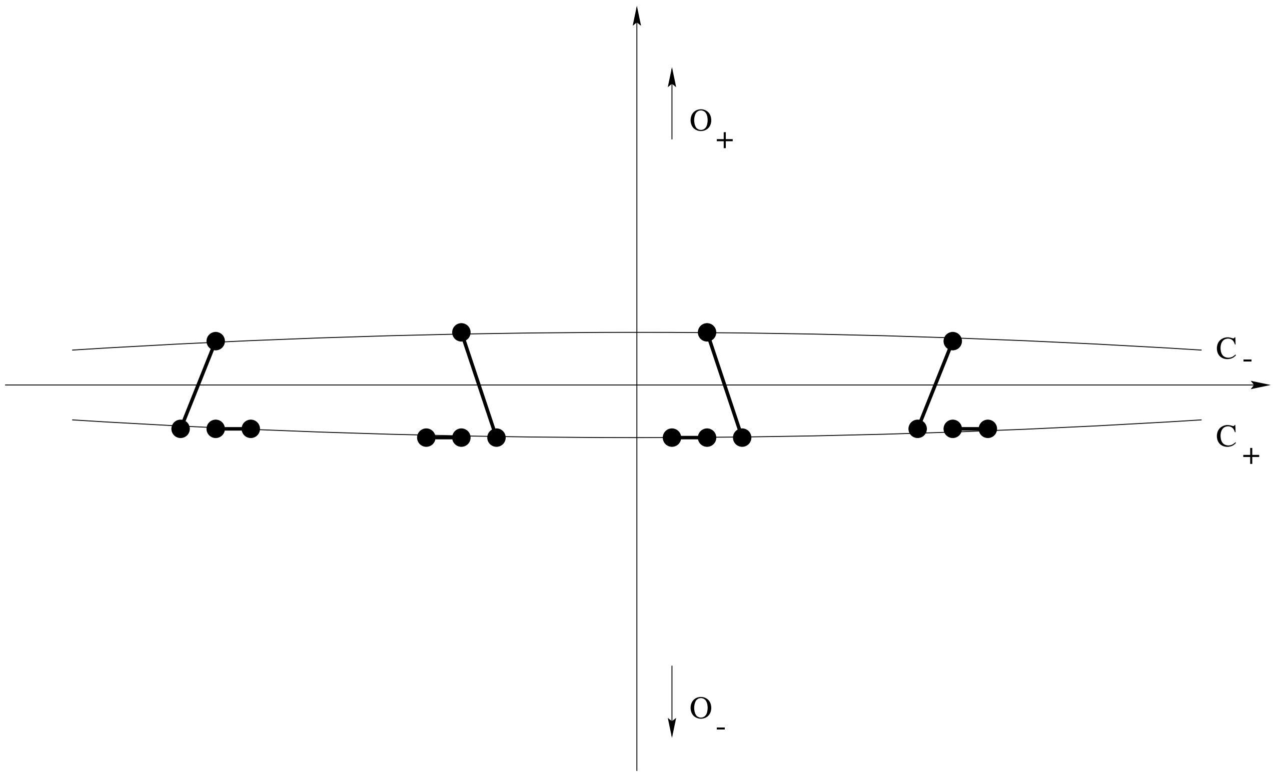
(ii) $m^{gen}(n) = 2^{\Theta(n)}$.

(ii) $m(n) = 2^{\Theta(n)}$.



Lower bound for minimum matchings (points in general position).

Each group can be matched independently in two ways. Minimum matching has multiplicity $2^{n/4}$ in this point set.



Lower bound for minimum matchings (points in convex position).

Each group can be matched independently in two ways. Minimum matching has multiplicity $2^{n/4}$ in this point set.

Upper bound on $m(n)$: Given a set S of n points in the plane, the complete geometric graph $G = G(S)$ has vertex set S and edge set consisting of all straight line segments between points of S.

A subgraph of G is *non-crossing* if its edges intersect only at common vertices. A result of Ajtai et. al. says that the number of non-crossing subgraphs of any drawing of K_n (even without the rectilinear restriction on edges) is bounded from above by $10^{13n} = 2^{O(n)}$. Any minimum matching is non-crossing, which implies the upper bound.

Maximum matching

Let $M(n)$ denote the maximum number of times the maximum perfect matching can occur in an n-element point set in the plane. Similarly, let $M^{gen}(n)$ (resp. $M^{conv}(n)$) denote the maximum number of times the maximum perfect matching can occur in an n-element point set in the plane whose points are in general position (resp. in convex position). Then

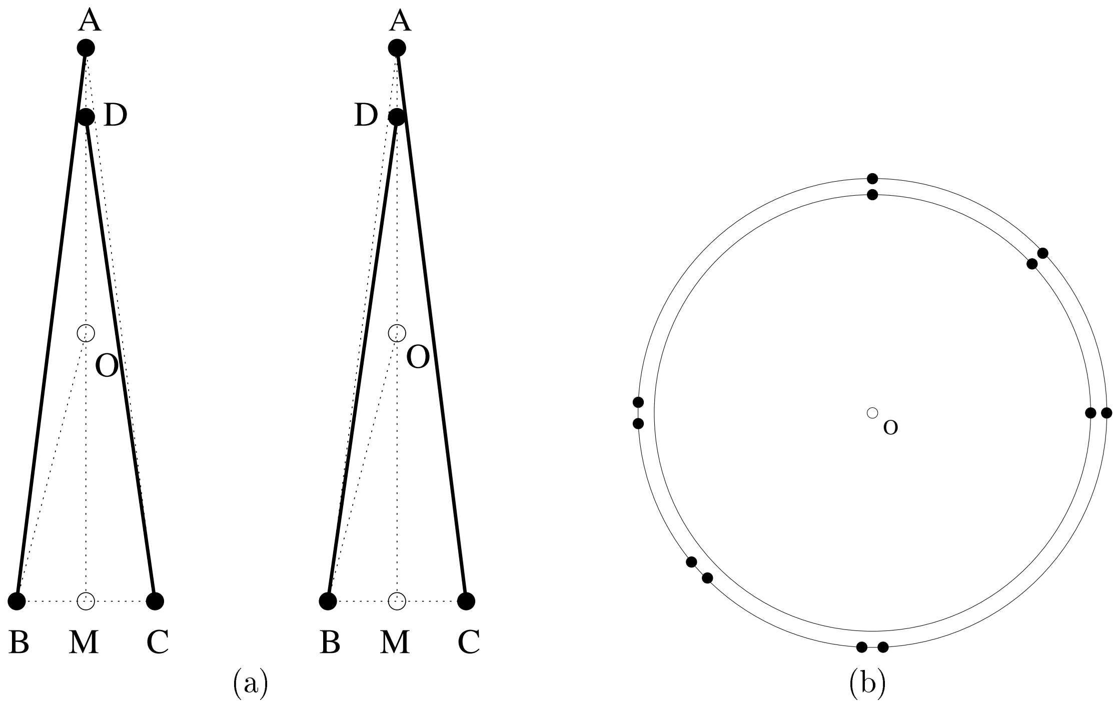
(i) $M(n) = 2^{\Theta(n \log n)}$.

(ii) $M^{gen}(n) = 2^{\Omega(n)}$.

(iii) $M^{conv}(n) = 1$.

$P = A \cup B$ where
$A = \{-1, -1+\epsilon, -1+2\epsilon, \ldots, -1+(n/2-1)\epsilon\}$,
$B = \{1-(n/2-1)\epsilon, 1-(n/2-2)\epsilon, \ldots, 1\}$.
Each maximum matching of P is a perfect matching between A and B, and each such perfect matching is maximum.
P admits $(n/2)!$ maximum matchings.

A maximum matching for points in convex position must consist of pairwise crossing segments, so it is unique.



(a). Two maximum matchings for points A, B, C, D.
(b). Lower bound for maximum matchings (points in general position).

$|AD| = |BM| = |MC| = \epsilon$.
A maximum matching for T has length $|AB| + |CD| = |AC| + |BD| \geq 4 - 3\epsilon$.
P admits $2^{n/4} = 2^{\Omega(n)}$ maximum matchings.

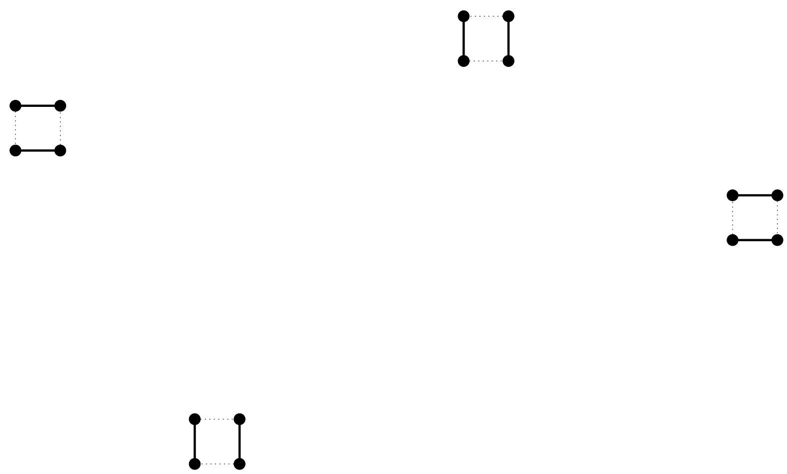
Minimum spanning tree

Let $t(n)$ denote the maximum number of minimum spanning trees that an n-element point set in the plane can have. Similarly, let $t^{gen}(n)$ (resp. $t^{conv}(n)$) denote the maximum number of minimum spanning trees in an n-element point set in the plane whose points are in general position (resp. in convex position). Then

(i) $t^{conv}(n) = 2^{\Theta(n)}$.

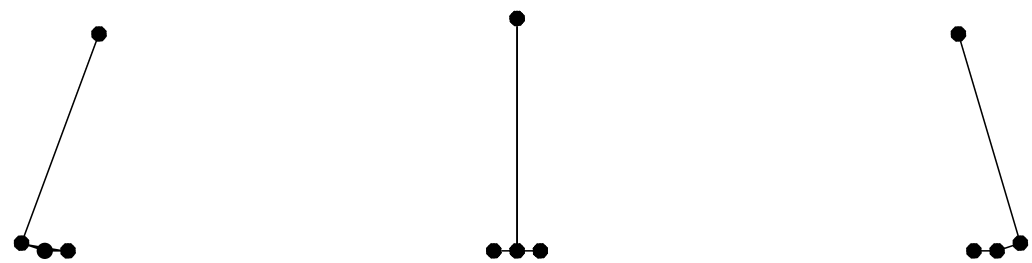
(ii) $t^{gen}(n) = 2^{\Theta(n)}$.

(iii) $t(n) = 2^{\Theta(n)}$.



Lower bound for minimum spanning trees (points in general position).

Kruskal's algorithm for computing a minimum spanning tree on P. The n (unit length) edges of the $n/4$ unit squares are considered first during the execution of the algorithm. Each unit square may generate four distinct ways of selecting three edges, which gives at least $4^{n/4} = 2^{\Omega(n)}$ distinct minimum spanning trees.



Lower bound for minimum spanning trees (points in convex position).

This point set has at least $3^{n/4} = 2^{\Omega(n)}$ distinct minimum spanning trees.

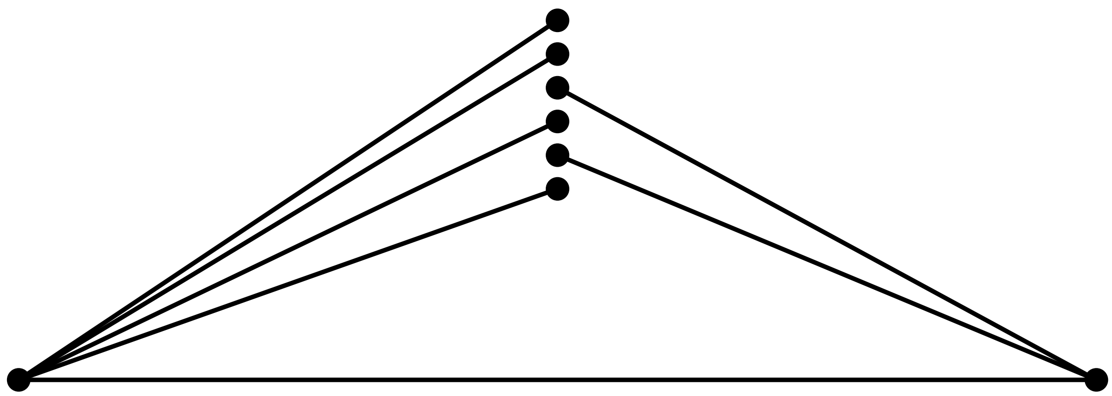
Maximum spanning tree

Let $T(n)$ denote the maximum number of maximum spanning trees that an n-element point set in the plane can have. Similarly, let $T^{gen}(n)$ (resp. $T^{conv}(n)$) denote the maximum number of maximum spanning trees in an n-element point set in the plane whose points are in general position (resp. in convex position). Then

(i) $T^{conv}(n) = 2^{\Omega(n)}$.

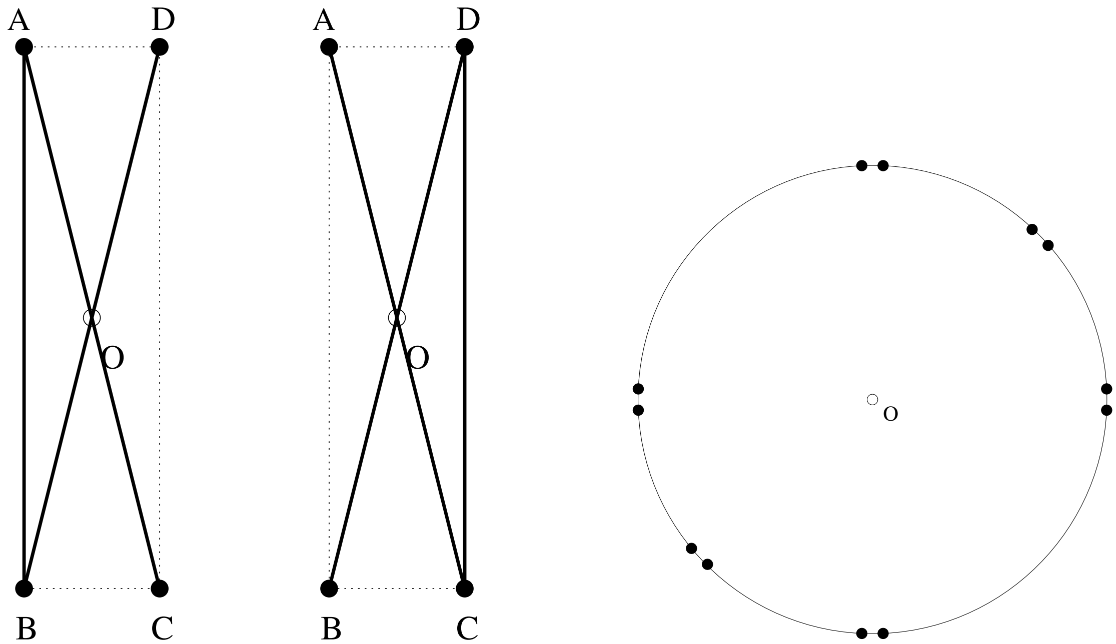
(ii) $T^{gen}(n) = 2^{\Omega(n)}$.

(iii) $T(n) = 2^{\Omega(n)}$.



Lower bound for maximum spanning trees.

This point set admits $2^{n-2} = 2^{\Omega(n)}$ maximum spanning trees.



(a) (b)

(a). Two maximum spanning trees on points A, B, C, D. (b). Lower bound for maximum spanning trees (points in convex position).

This point set admits $2^{n/4} = 2^{\Omega(n)}$ maximum spanning trees.

Conjecture 1 $M^{gen}(n) = 2^{\Theta(n)}$.

Conjecture 2 **(i)** $T^{conv}(n) = 2^{\Theta(n)}$.

(ii) $T^{gen}(n) = 2^{\Theta(n)}$.

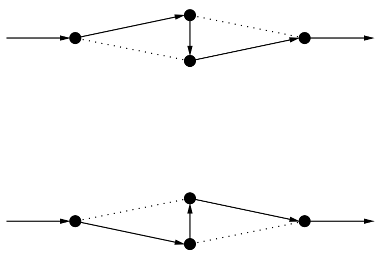
(iii) $T(n) = 2^{\Theta(n)}$.

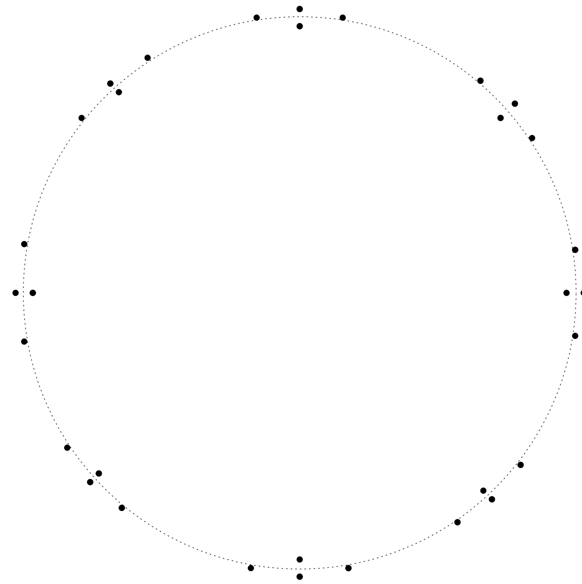
Shortest geometric tours

Let $t^{gen}(n)$ (resp. $t^{conv}(n)$) denote the maximum number of shortest tours that an n-element point set in the plane whose points are in general position (resp. in convex position) can have. Then

(i) $t^{conv}(n) = 1$.

(ii) $t^{gen}(n) = 2^{\Theta(n)}$.





Lower bound for shortest tours.

This point set admits $2^{n/4} = 2^{\Omega(n)}$ shortest tours.

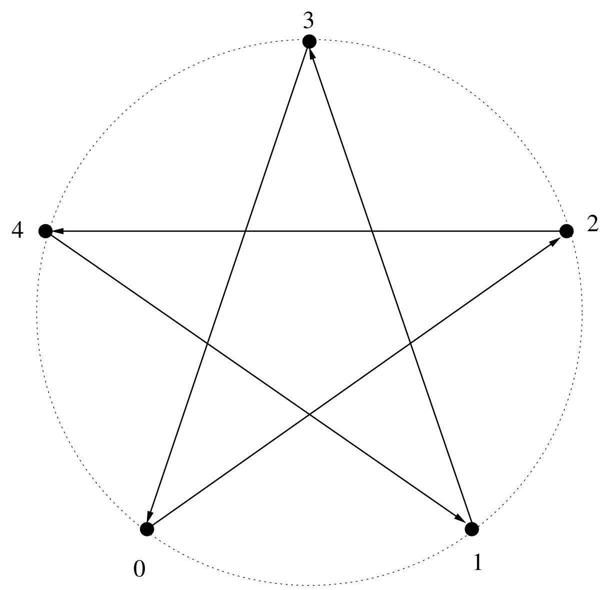
Longest geometric tours

Let $T^{gen}(n)$ (resp. $T^{conv}(n)$) denote the maximum number of longest tours that an n-element point set in the plane whose points are in general position (resp. in convex position) can have. Then

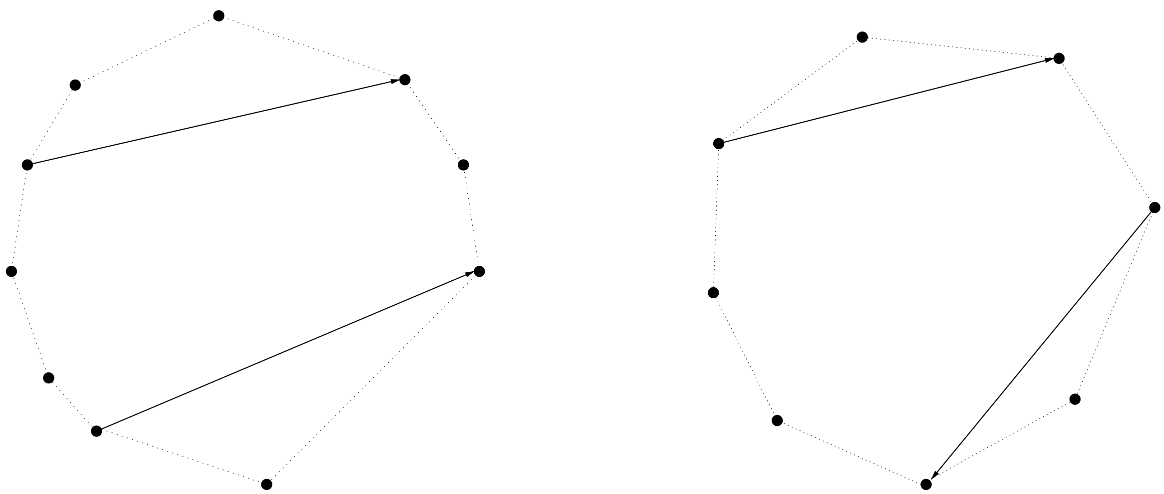
(i) For odd n, $T^{conv}(n) = 1$.

(ii) For even n, $T^{conv}(n) \geq 2$.

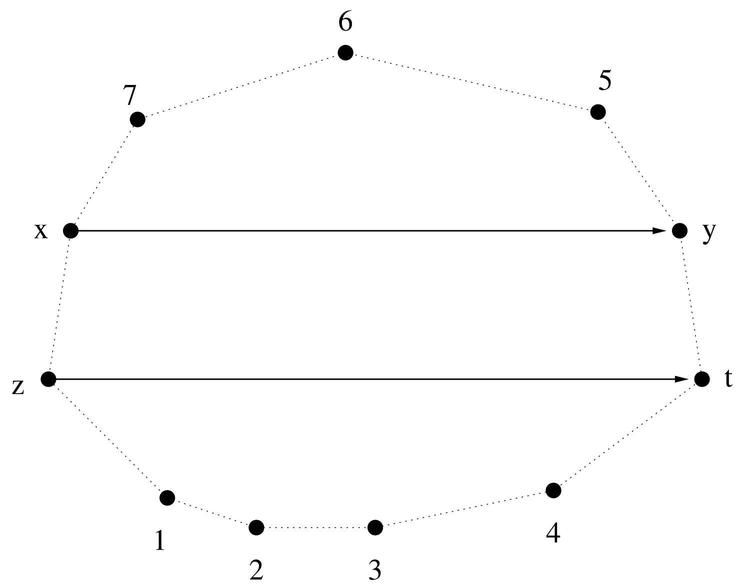
(iii) $T^{gen}(n) = \Omega(n)$.



A geometric thrackle on the point set of a regular pentagon.

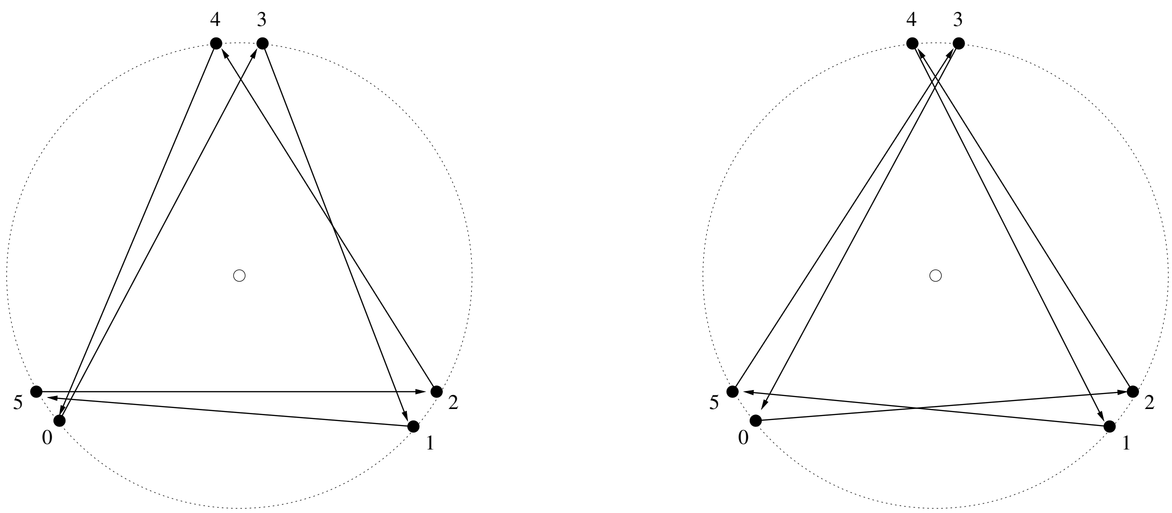


A pair of parallel edges (left), and a pair of antiparallel ones (right).

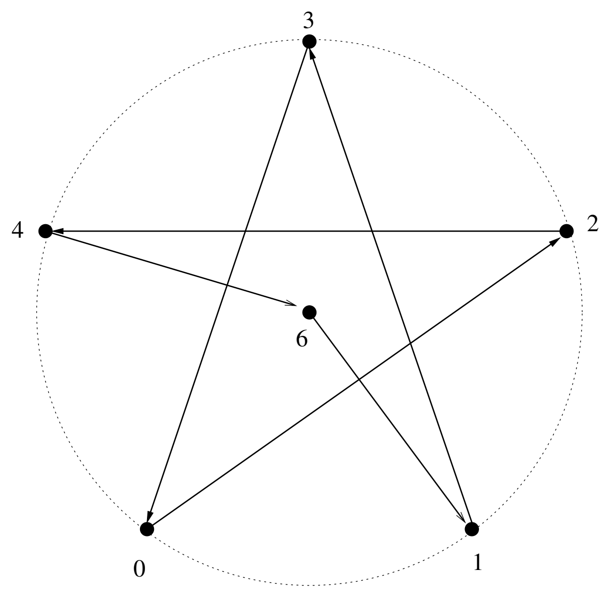


The uniqueness of a longest tour on an odd number of points in convex position.

Since n is odd, w.l.o.g. $|A| > |B|$.
$A = \{1, 2, 3, 4, t\}$,
$B = \{5, 6, 7, x\}$.
$f : S \to S$,
$f(u) = v$, if uv is an edge of the tour T.
f is injective and $f(A) \subseteq B$
$|f(A)| = |A|$,
$|A| \leq |B|$, contradiction.



A six point configuration which admits two longest tours: 031524 (left) and 024153 (right).



This point set admits $n-1=5$ longest tours.