Symmetries & Isometries

The distance

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Theorem ([Babai, 78]). *Let G be a multigraph having s distinct eigenvalues with respective multiplicities $m_1, m_2, \ldots, m_s$. Then, the abstract automorphism group γ of G is a subgroup of*

$$\boldsymbol{\omega}(m_1) \oplus \boldsymbol{\omega}(m_2) \oplus \cdots \oplus \boldsymbol{\omega}(m_s),$$

where $\boldsymbol{\omega}(m)$ denotes the real orthogonal group of dimension m.

References

[Babai, 78] Babai, L., *Automorphism group and category of cospectral graphs*, Acta Math. Acad. Sci. Hung. **31** (1978), 295–306

Lemma. *Let G be a multigraph and let $d : \mathbb{N} \to \mathbb{R}$ be injective. Assume $f : V(G) \to \mathbb{R}^k$ is an injective mapping such that*

- $\| f(\mu(x)) - f(\mu(y)) \| = \| f(x) - f(y) \|$ *for any automorphism μ of G,*
- $\{x, y\}$ *has multiplicity** $m \geq 0$ *if and only if* $\| x - y \| = d(m)$.

Then, the automorphism group of G is isomorphic to the group of the isometries of $\mathbb{R}^k$ mapping $f(V(G))$ into itself.

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*multiplicity 0 means $\{x, y\}$ is a non-edge

Lemma. *Let $\boldsymbol{D}$ be a $n \times n$ symmetric real matrix with null values on the diagonal and let k be an integer. Then, the following statements are equivalent:*

- *There exists a mapping $f : [\,1\,;\,n\,] \to \mathbb{R}^k$ such that $\left\| f(i) - f(j) \right\|^2 = \boldsymbol{D}_{i,j}$ (for any $(i,j) \in [\,1\,;\,n\,]^2$).*
- *The matrix $\boldsymbol{B} = -\frac{1}{2}(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J})\boldsymbol{D}(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J})$ has no strictly negative eigenvalue and $\operatorname{rank}(\boldsymbol{B})$ is at most k.*

Remark. The eigenspaces of $\boldsymbol{B}$ are stable under any isometry of $\mathbb{R}^k$ which maps $f([\,1\,;\,n\,])$ into itself.

Notation. $\boldsymbol{A}^\star = (\boldsymbol{I} - \frac{1}{n}\boldsymbol{J})\boldsymbol{A}(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J})$ has a basis $\mathcal{B}(\boldsymbol{A}^\star)$ of eigenvectors including $\mathbf{1}$. The eigenvectors of $\mathcal{B}(\boldsymbol{A}^\star) \setminus \{\mathbf{1}\}$ have eigenvalues $\lambda_1^\star > \cdots > \lambda_r^\star$ with respective multiplicities $m_1^\star, \ldots, m_r^\star$.

Remark. $\mathbf{1} \in \ker(\boldsymbol{A}^\star)$, as $(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J})\mathbf{1} = \mathbf{0}$

Definition. The *reduced distance matrix* of a multigraph G is

$$\boldsymbol{D} = -\lambda_r^\star(\boldsymbol{J} - \boldsymbol{I}) - \boldsymbol{A}.$$

Lemma. *Let* $\boldsymbol{B} = -\frac{1}{2}(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J})\boldsymbol{D}(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J})$. *Then,* $\mathcal{B}(\boldsymbol{A}^\star)$ *is a basis of eigenvectors of* $\boldsymbol{B}$, *and the eigenvalue of* $v \in \mathcal{B}(\boldsymbol{A}^\star)$ *is*

$$\begin{cases} 0 & \text{if } v = \mathbf{1} \\ \frac{\lambda^\star - \lambda_r^\star}{2} & \text{if } v \neq \mathbf{1} \text{ and } \boldsymbol{A}^\star v = \lambda^\star v \end{cases}$$

Thus, $\boldsymbol{B}$ *has no strictly negative eigenvalue and rank* $n - 1 - m_r^\star$.

Definition. A multigraph G with maximum multiplicity M on the edges is *reducible* if, for any edge $\{x, y\}$ having multiplicity M, any z adjacent to x is adjacent to y and the multiplicities of $\{x, z\}$ and $\{y, z\}$ are equal.

Lemma. *Let G be a multigraph, then $\boldsymbol{D}_{i,j} > 0$ for any $i \neq j$ if and only if G is non reducible.*

Theorem. *Let G be a non reducible multigraph. Then, there exists a set P of n points of $\mathbb{R}^{n-1-m_r^\star}$, such that $\mathrm{Aut}(G)$ is isomorphic to the group $\boldsymbol{\omega}(n-1-m_r^\star, P)$ of the isometries of $\mathbb{R}^{n-1-m_r^\star}$ mapping P to itself.*

Corollary. *Let G be a non reducible multigraph. Then, there exists a set P of n points of $\mathbb{R}^{n-1-m_r^\star}$, such that $\mathrm{Aut}(G)$ is a subgroup of*

$$\boldsymbol{\omega}(m_1^\star, P_1) \oplus \boldsymbol{\omega}(m_2^\star, P_2) \oplus \cdots \oplus \boldsymbol{\omega}(m_{r-1}^\star, P_{r-1})$$

where $P_1, \ldots, P_{r-1}$ are the projection of P into the proper spaces of $\boldsymbol{A}^\star$ corresponding to eigenvalues $\lambda_1^\star, \ldots, \lambda_{r-1}^\star$.

Proposition ([Biggs, 74]). *Let G be a d-regular graph. Then:*

- *d is an eigenvalue of G.*
- *If G is connected, then the multiplicity of d is one.*
- *For any eigenvalue λ of G, we have $\left|\lambda\right| \leq d$.*

References

[Biggs, 74] Biggs, N., *Algebraic Graph Theory*, ch. 3, p. 14, Cambridge Univeristy Press, 1974.

Lemma. *Let G be a connected d-regular multigraph. Then, $\mathcal{B}(\boldsymbol{A}^\star)$ is a basis of eigenvectors of $\boldsymbol{A}$ and, for any $v \in \mathcal{B}(\boldsymbol{A}^\star)$:*

$$\boldsymbol{A}v = \begin{cases} dv & \text{if } v = \mathbf{1} \quad \text{(but } \boldsymbol{A}^\star v = 0\text{)} \\ \lambda^\star v & \text{if } v \neq \mathbf{1} \text{ and } \boldsymbol{A}^\star v = \lambda^\star v \end{cases}$$

Thus, if $\lambda_1 > \lambda_2 > \dots \lambda_s$ are the eigenvalues of $\boldsymbol{A}$ with respective multiplicities $m_1, \dots, m_s$, then $s = r + 1$ and

$$\lambda_1 = d \qquad\qquad m_1 = 1$$

$$\lambda_{i+1} = \lambda_i^\star \qquad\qquad m_{i+1} = m_i^\star$$

Corollary. *Let G be a non reducible connected d-regular multigraph. Then, there exists a set P of n points of $\mathbb{R}^{n-1-m_s}$, such that* $\mathrm{Aut}(G)$ *is a subgroup of*

$$\boldsymbol{\omega}(m_2, P_2) \oplus \boldsymbol{\omega}(m_3, P_3) \oplus \cdots \oplus \boldsymbol{\omega}(m_{s-1}, P_{s-1})$$

where $P_2, \ldots, P_{s-1}$ are the projection of P into the proper spaces of $\boldsymbol{A}$ corresponding to eigenvalues $\lambda_2, \ldots, \lambda_{s-1}$.

Thus, the abstract automorphsim group of G is a subgroup of

$$\boldsymbol{\omega}(m_2) \oplus \boldsymbol{\omega}(m_3) \oplus \cdots \oplus \boldsymbol{\omega}(m_{s-1})$$

Theorem (A.J. Hoffman, [1]). *For any positive integer p, there is an integer N such that for any $n \geq N$ there exits p non-isomorphic cospectral connected regular graphs with n vertices each.*

References

[1] Mowshowitz, A., *The characteristic polynomial of a graph*, J. Combinatorial Theory (B) **12** (1972), 177–183