Convex geometric hypergraphs

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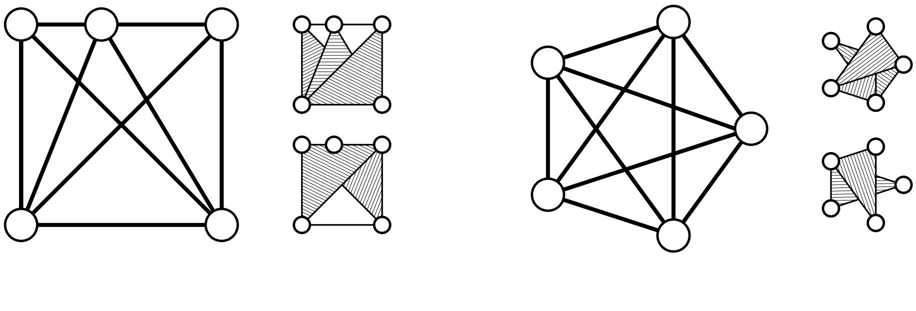
Convex Geometric (Hyper)Graphs

A convex geometric graph is a graph with a cyclic ordering of the vertices.

A convex geometric graph gives the combinatorial structure of a set of diagonals and sides selected from a convex polygon.

A convex geometric hypergraph is a hypergraph with a cyclic ordering of the vertices.

A convex geometric hypergraph gives the combinatorial structure of a set of k-gons selected from a convex polygon.



Forbidden substructure extremal theories

Using the substructure relation of these combinatorial objects, one can study forbidden-substructure theories for them. For graphs this is an old subject.

For convex geometric graphs:

- Turán-type theories:

  How many edges can a convex geometric graph on n vertices have without containing a given convex geometric subgraph?

  - Has been studied by Brass, Capoyleas, Károlyi, Kupitz, Pach, Perles, Valtr

- Ramsey-type theories:

  How many colors are needed to color the complete convex geometric graph on n vertices such that each color class does not contain a given convex geometric subgraph?

  - Has been studied by Bialostocki, Harborth, Jaywardene, Károlyi, Lefmann, Pach, Rousseau, Tóth, Valtr.

Turán-type problems for convex geometric hypergraphs

We will study the same type of problem for convex geometric hypergraphs.

> How many hyperedges can a k-uniform hypergraph on n vertices with a cyclic order have that does not contain a given cyclically ordered subhypergraph?

Several known theorems have such bounds as their combinatorial core.

Much less is known about hypergraph Turán theory than about graph Turán theory.

Difference between hypergraphs and convex geometric hypergraphs seems larger than between graphs and convex geometric graphs.

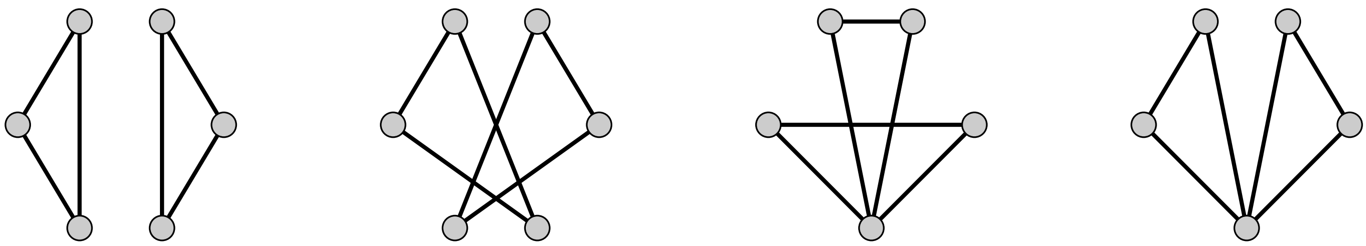
Example 1
Triangles of Maximum Area

How many triangles of maximum area can be spanned by n points?

Points that are not vertices of the convex hull are not part of any maximum-area triangle. How many triangles of maximum area can occur among the vertices of a convex n-gon?

If Δ_1, Δ_2 are two maximum-area triangles, then each side of Δ_1 has some common point with Δ_2.

Forbidden sub-arrangements of two triangles:



At most n triangles among n vertices possible without occurrence of these forbidden substructures.

At most n triangles of maximum area among n points in the plane.

Braß, Rote, Swanepoel 2001

Example 2
Congruent Equilateral Triangles

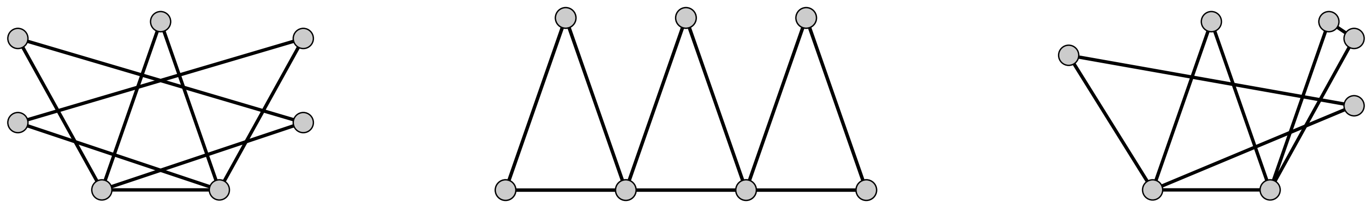
How many congruent equilateral triangles can occur among n points in convex position?

Simpler variant of number of unit distances among n points in convex position.

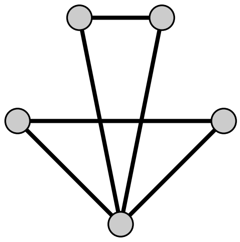
Congruent equilateral triangles among points in convex position have the following property:

If xyz is a triangle, xy left edge with respect to x, then xy is either rightmost left edge with respect to x or leftmost right edge with respect to y.

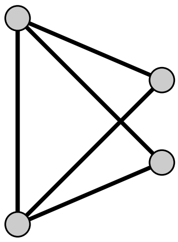
Forbidden subarrangements:



Also no triangle can at a vertex be completely contained in another triangle.



And no two triangles can share a side, and be in the same halfplane of that side.



At most $\left\lfloor \frac{2}{3}n \right\rfloor$ triangles among n vertices possible without occurrence of these forbidden substructures.

Using a slight bit more geometry, this number can be reduced by one, to give the exact maximum number of congruent equilateral triangles among points in convex position:

At most $\left\lfloor \frac{2}{3}(n-1) \right\rfloor$ congruent equilateral triangles among the vertices of a convex n-gon.

Pach, Pinchasi 2001

New Example: Axis-aligned Ellipses Touching Four Sides of a Polygon

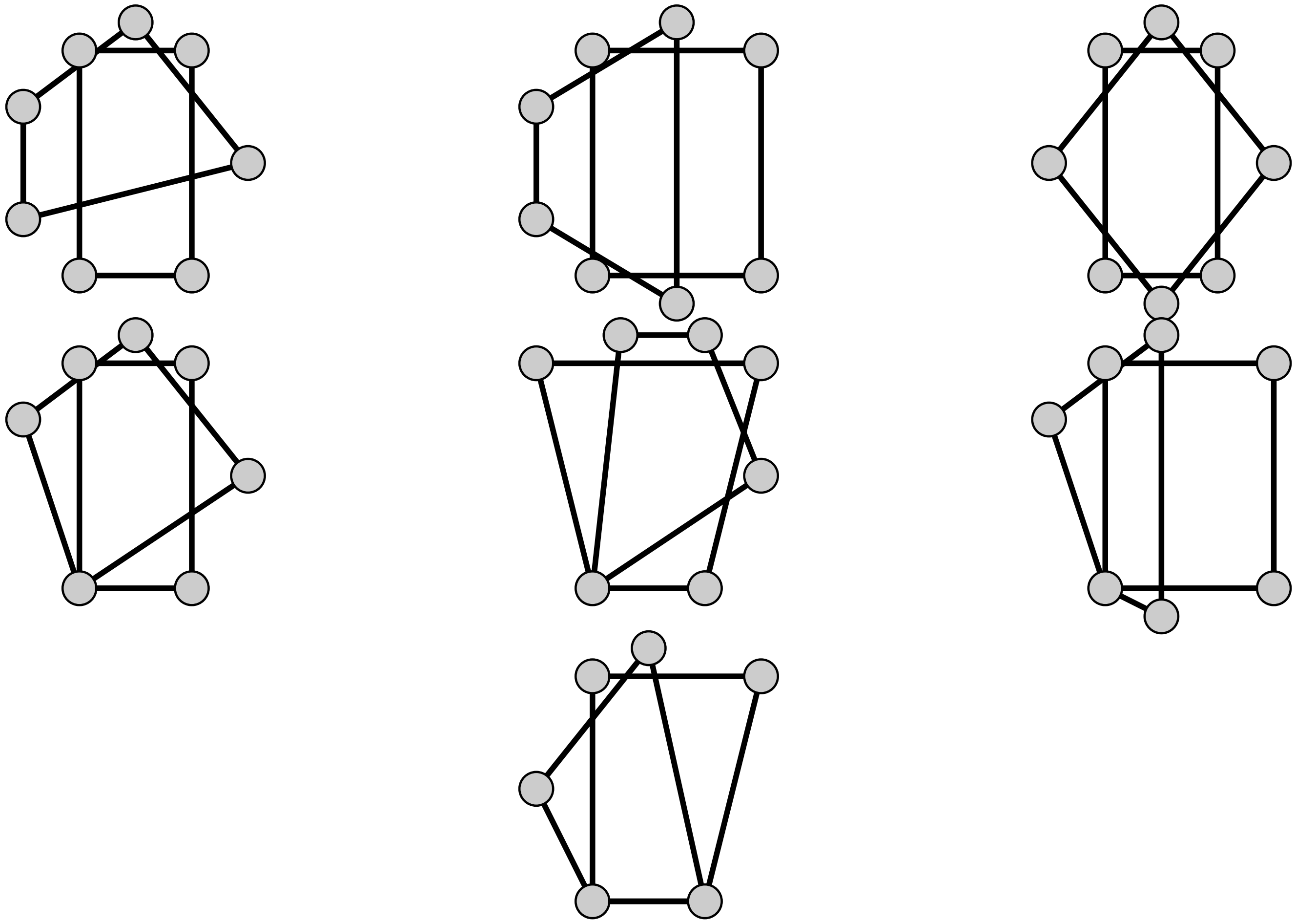
Given a simple n-gon in the plane. How many distinct four-tuples of sides can be touched by inscribed axis-aligned ellipses?

question by Efrat 2002

Model: sides of simple polygon as vertices, with their natural cyclic order. Four-tuples of sides touched by some ellipse as fourgons (hyperedges) in this convex geometric hypergraph.

Since any two ellipses intersect in at most four points, any two of these four-gons intersect in at most four points, counting common vertices as intersection points.

Forbidden subconfigurations



Contraction of two consecutive vertices cannot generate a new occurence of one of these forbidden subconfigurations.

When contracting to consecutive vertices of an n-vertex 4-uniform convex geometric hypergraph, we loose edges:

- all edges that contain both of the contracted vertices.
- one copy of each edge pair that differs only in the contracted vertices.

Of both types there are at most $O(n)$ edges in a convex geometric hypergraph that does not contain any of the above forbidden fourgon pairs.

So any convex geometric hypergraph (set of four-gons) selected from n vertices which does not contain two fourgons intersecting in more than four points contains at most $O(n^2)$ fourgons.

So there are at most $O(n^2)$ distinct types of axis-aligned ellipses inscribed in a simple n-gon.

General Theory
for Convex Geometric Hypergraphs

As with normal and convex geometric graphs there is a chromatic number defined on convex geometric hypergraphs that determines the Turán function if it is big enough.

Define the chromatic number of a convex geometric hypergraph as the smallest number of colors necessary in a coloring where

- no edge contains two vertices of the same color, and
- the color classes form intervals in the cyclic ordering.

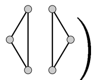
Then for a given excluded k-uniform hypergraph the Turán function is $\Theta(n^k)$ iff the chromatic number is at least $k+1$, otherwise it is $O(n^{k-\epsilon_k})$.

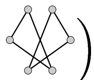
Small Cases

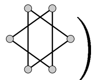
There are three possible 3-uniform hypergraphs (without cyclic ordering) with two edges:

- the two edges can be disjoint: this gives a hypergraph Turán function of order $\Theta(n^2)$
- the two edges can have one common vertex: this gives a hypergraph Turán function of order $\Theta(n)$
- the two edges can have two common vertices: this gives a hypergraph Turán function of order $\Theta(n^2)$

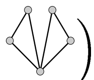
By introducing the cyclic ordering, this splits up into eight possible convex geometric hypergraphs.

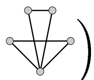
$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^3)$ (chromatic number)

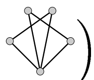
$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^2)$

$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^2)$

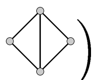
$$\mathrm{ex}_{\text{hypergraph}}(n, \{\{a, b, c\}, \{d, e, f\}\}) = \Theta(n^2)$$

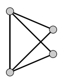
$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^3)$ (chromatic number)

$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^2)$

$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^2)$

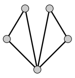
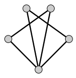
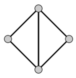
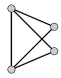
$$\mathrm{ex}_{\text{hypergraph}}(n, \{\{a, b, c\}, \{a, d, e\}\}) = \Theta(n)$$

$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^3)$ (chromatic number)

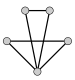
$\mathrm{ex}_{\circlearrowleft}(n,$  $) = \Theta(n^2)$

$$\mathrm{ex}_{\text{hypergraph}}(n, \{\{a, b, c\}, \{a, b, d\}\}) = \Theta(n^2)$$

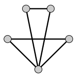
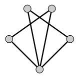
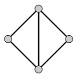
Construction 1

Convex geometric hypergraph which contains neither  nor , nor , nor 

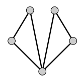
Number the vertices by 1 to $3k$. Take all triples $\{a, b, c\}$ as hyperedges (triangles) where $a+b+c = 4k$.

$n = 3k$ vertices, $\Theta(n^2)$ triangles, and any pair of triangles which shares shares at least one vertex forms .

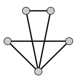
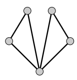
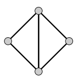
Construction 2

Convex geometric hypergraph which contains no  nor , nor .

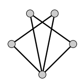
Number the vertices by $1, \ldots, 3k$. Take all triples $\{2i-1, 2i, j\}$ as hyperedges (triangles) with $1 \leq i \leq k$, $2k+1 \leq j \leq 3k$.

$n = 3k$ vertices, $\Theta(n^2)$ triangles, and all pairs of triangles with one common vertex form .

Construction 3

Convex geometric hypergraph which contains no  nor , nor .

Number the vertices by $1 \ldots, 3k$. Take all triples $\{i, i+k, j\}$ as hyperedges (triangles) with $1 \leq i \leq k$, $2k+1 \leq j \leq 3k$.

$n = 3k$ vertices, $\Theta(n^2)$ triangles, and all pairs of triangles with one common vertex form .