Disease Containment by Progressive Vaccination on Trees and Grids

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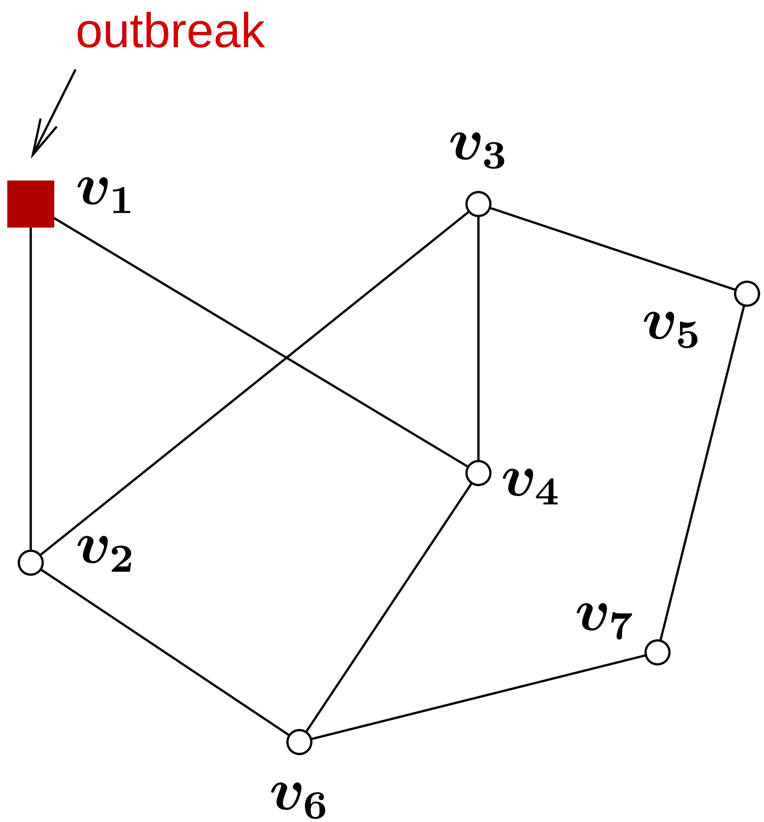
"Stirred pot" paradigm:

Traditional epidemiological models assume that every pair of individuals has the same likelihood of having contact, and thus transmitting the disease.

We would like to have a model that incorporates the structure of the different relationships between individuals.

A Simple Disease-Spread Model



Consider a disease spreading during discrete time: $t = 0, 1, \ldots$

The outbreak of disease starts at a root vertex.

If one of vertex v's neighbors is infected at time t, then v is infected at time $t + 1$.

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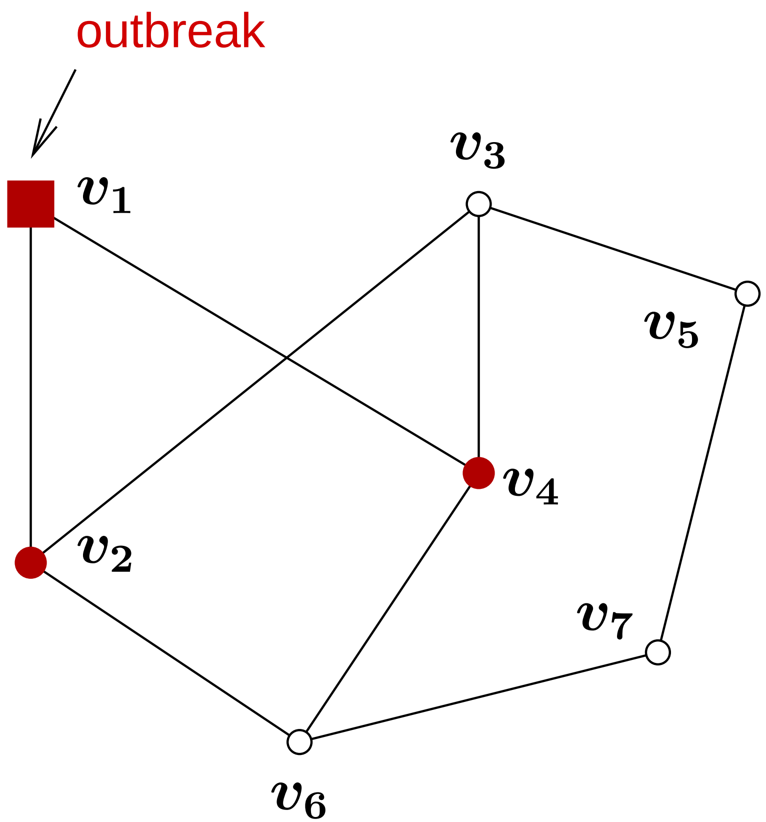

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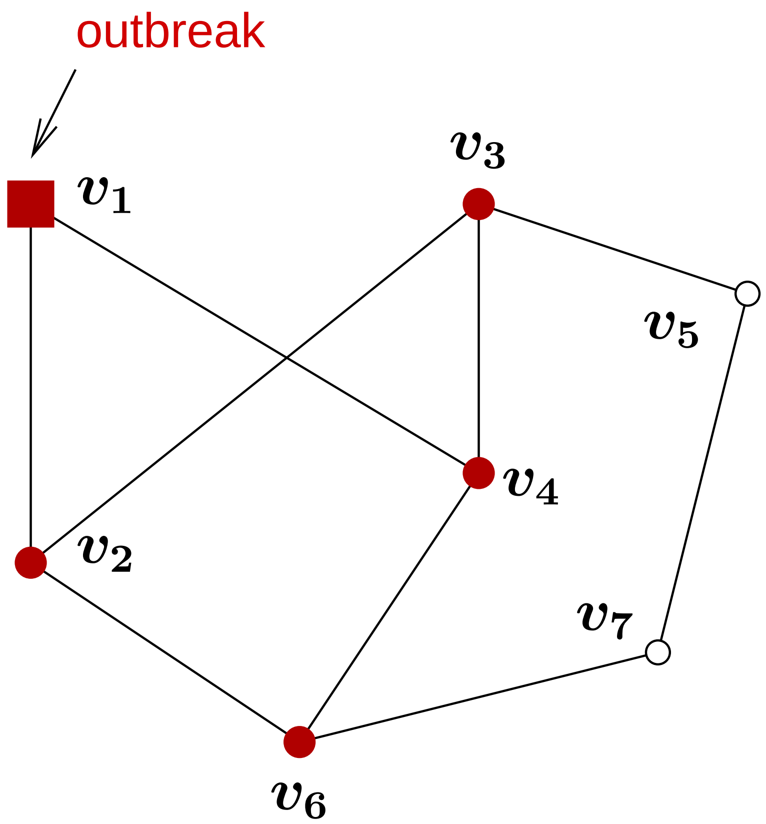

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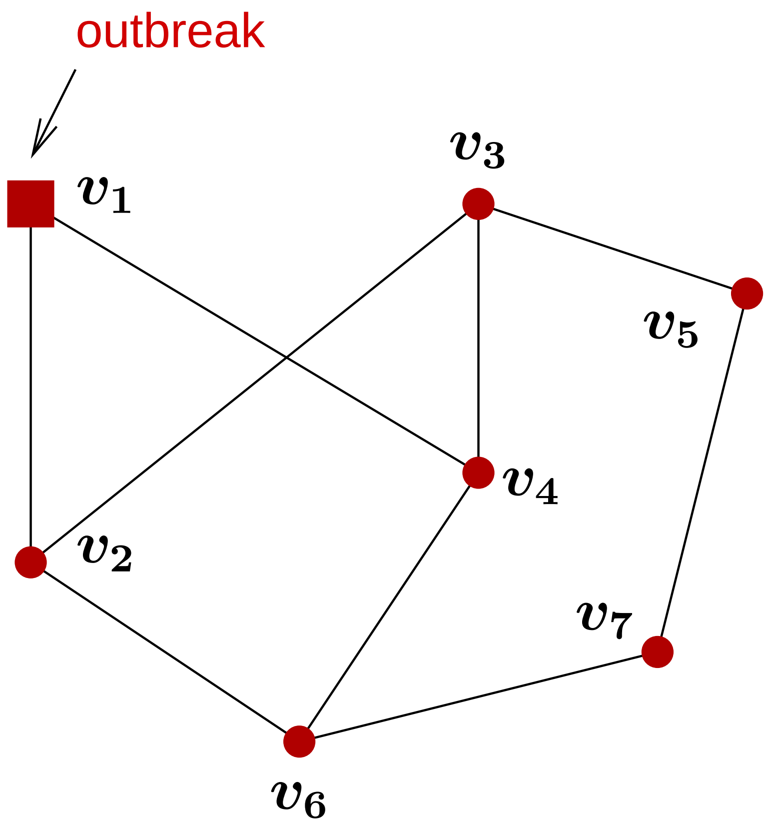

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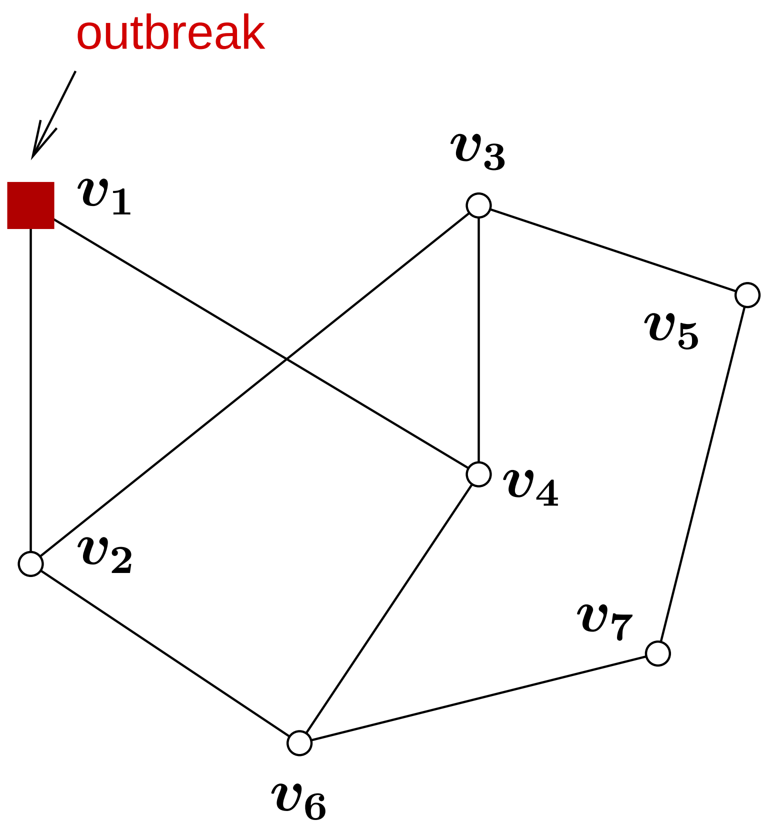
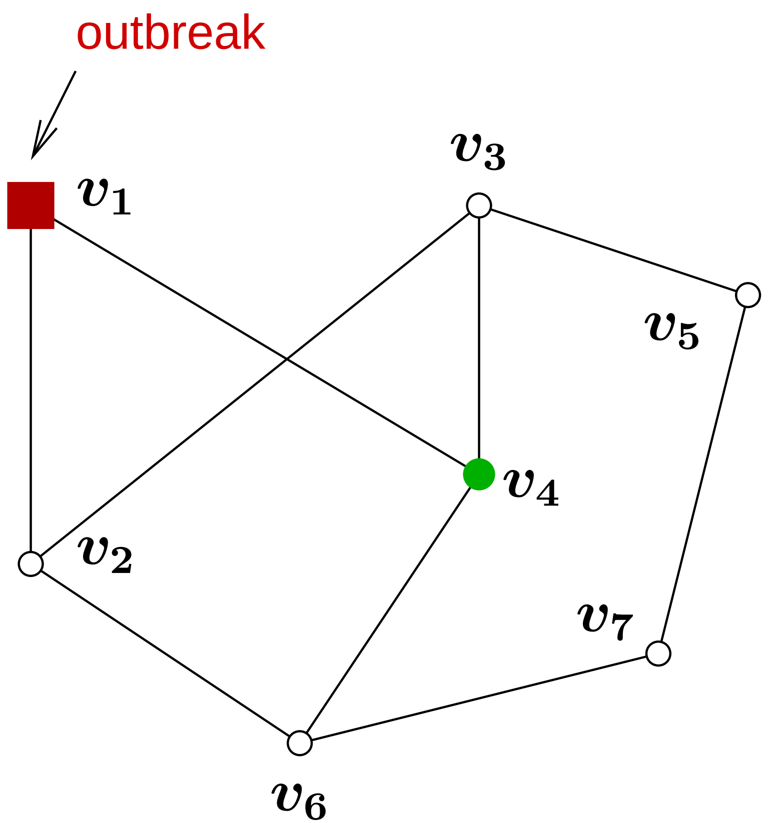


At each time step, at most f vertices can be vaccinated.



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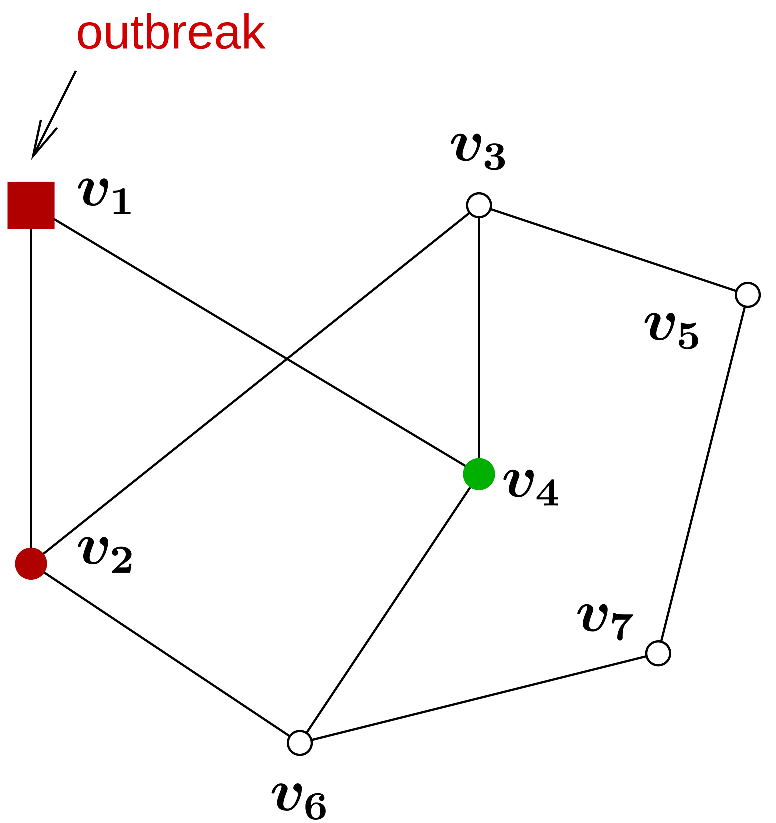

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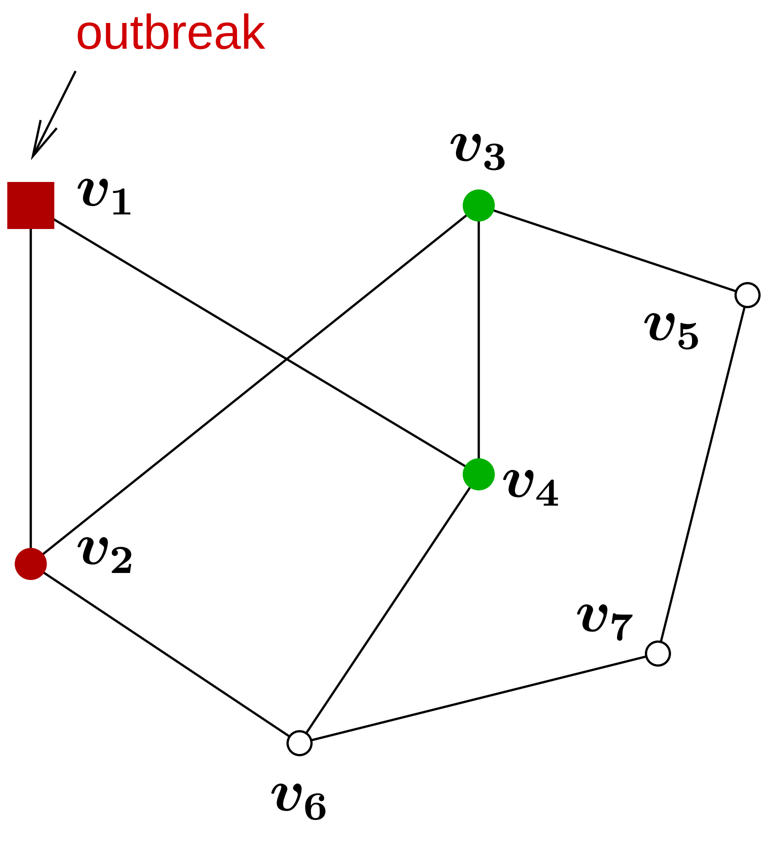

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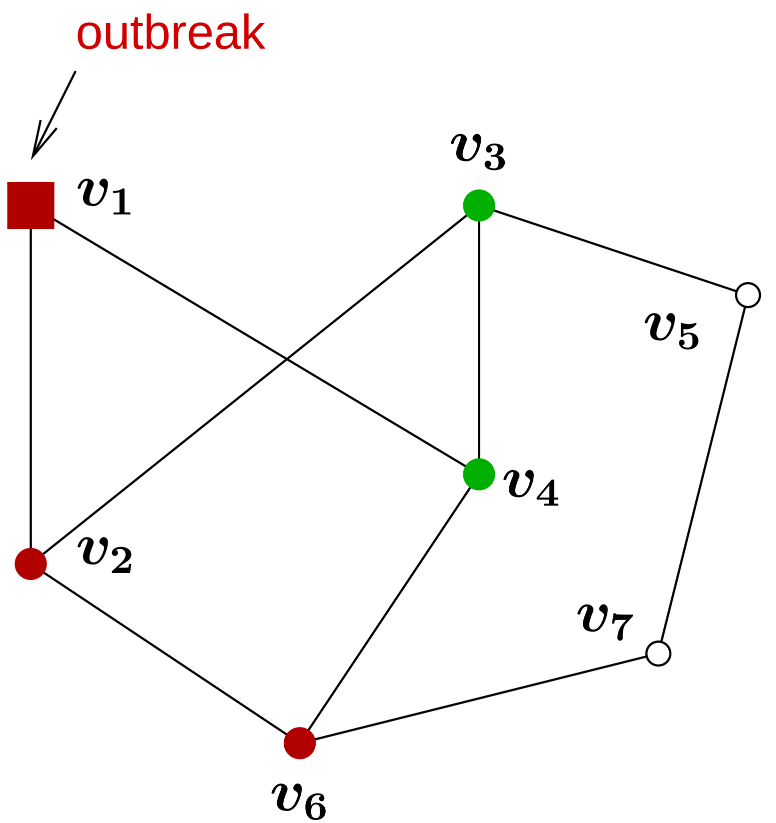

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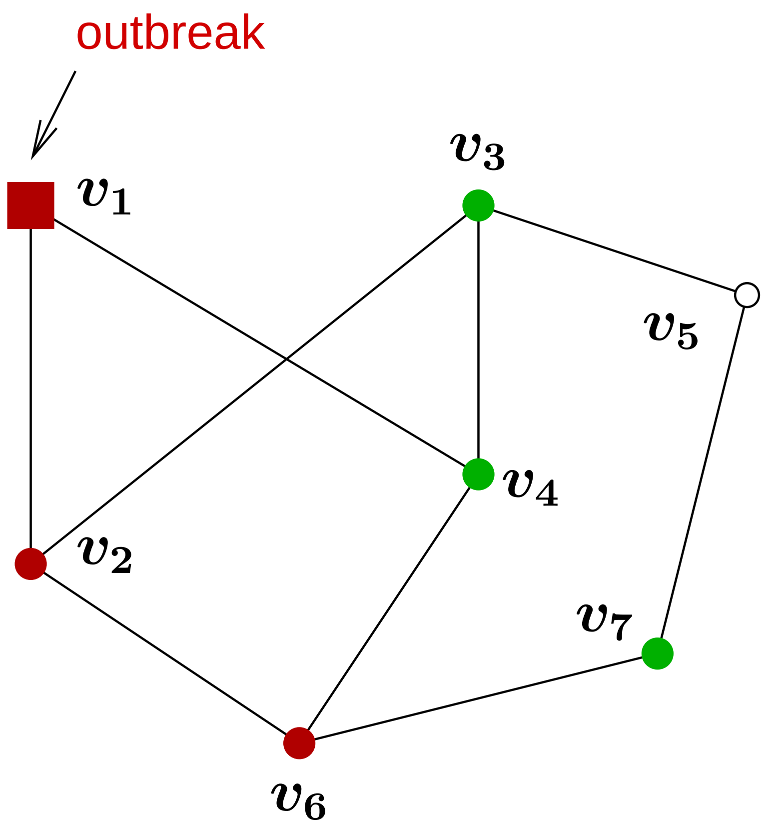

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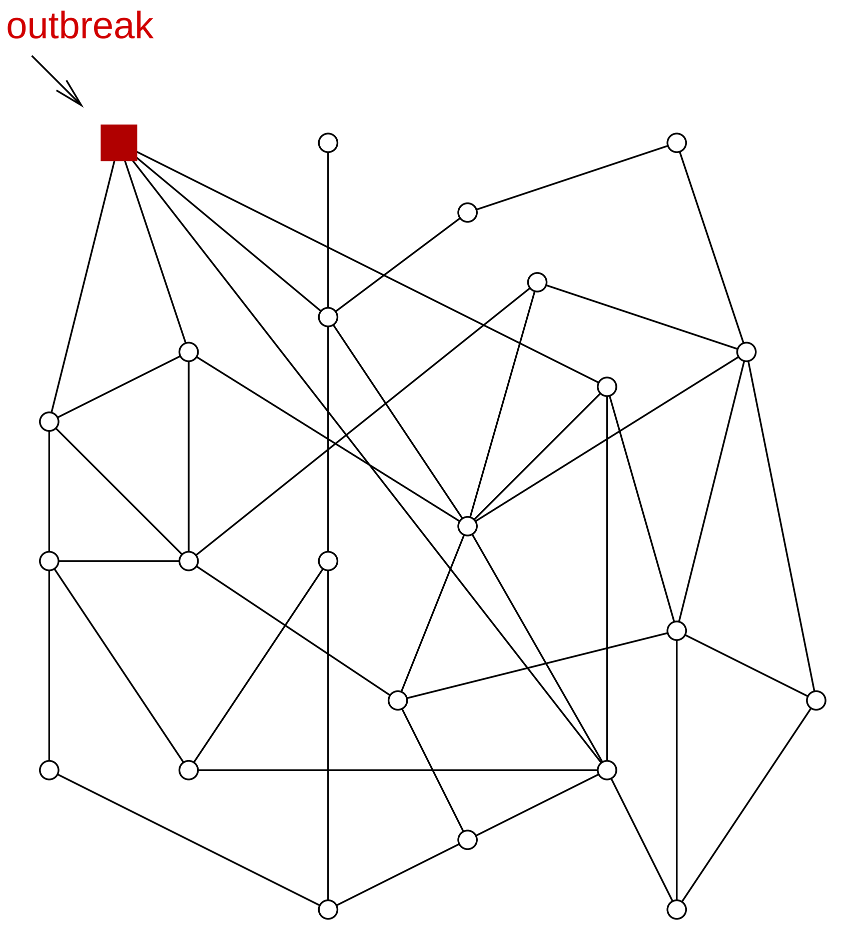
How many vertices can be saved?

The vaccination problem can be modeled by the firefighter model, which was introduced by Hartnell in 1995.

disease $\Longleftrightarrow$ fire

vaccination $\Longleftrightarrow$ firefighters defending a vertex

Response Strategy



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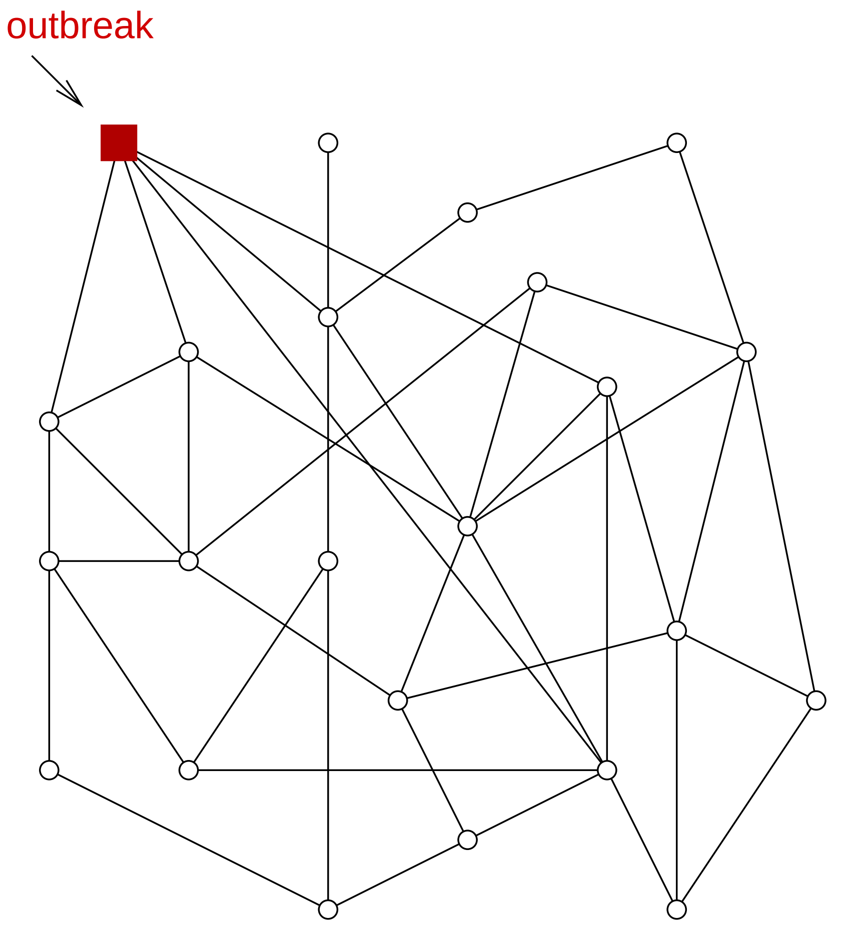
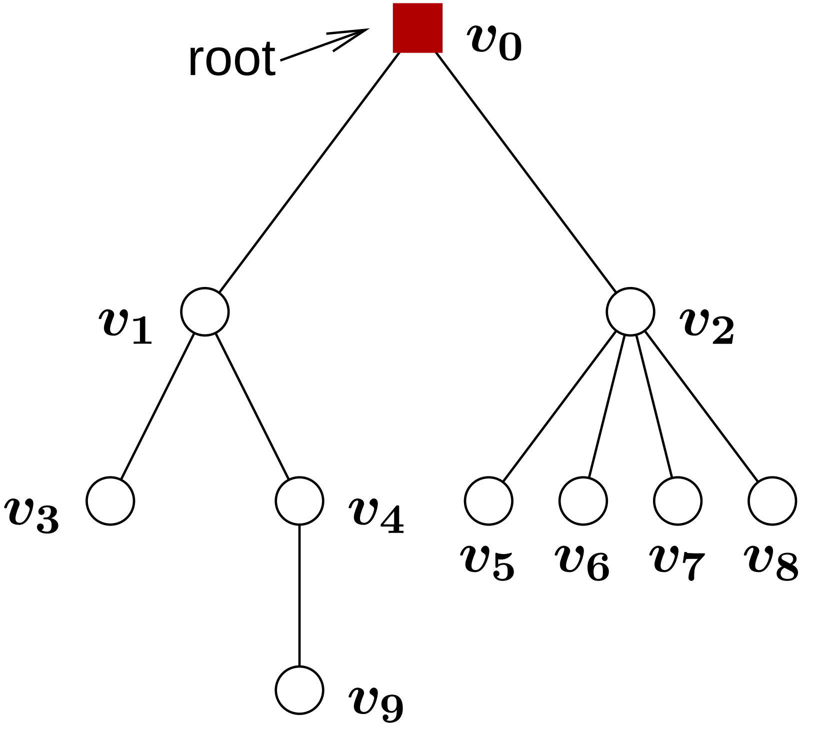


What is the optimal vaccination strategy?

Thm.
(MacGillivray-Wang 2003)
Finding the optimal strategy is NP-complete.

A tree is a finite connected graph with no cycles.
The root is the vertex at which the disease outbreak starts.



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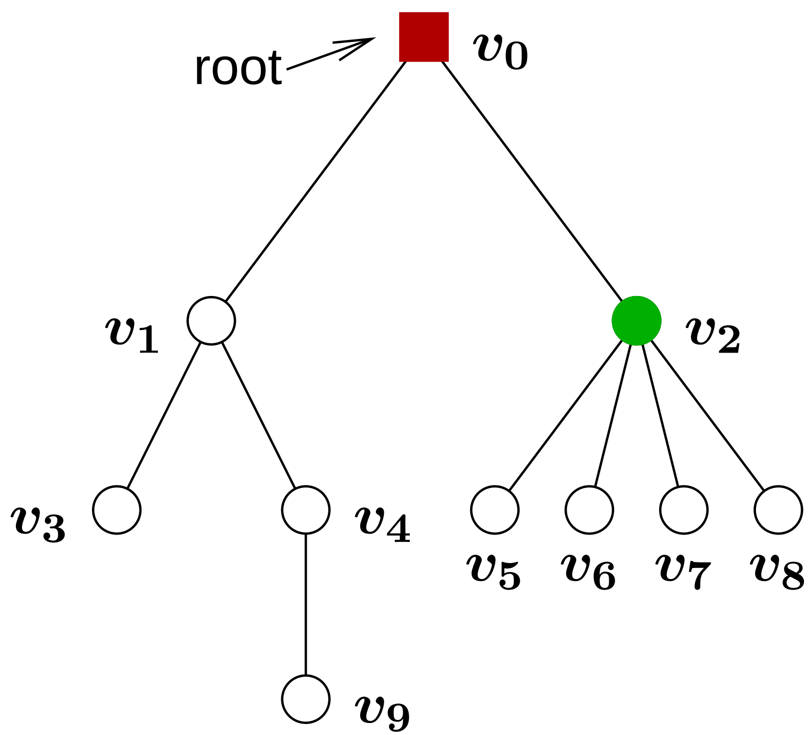

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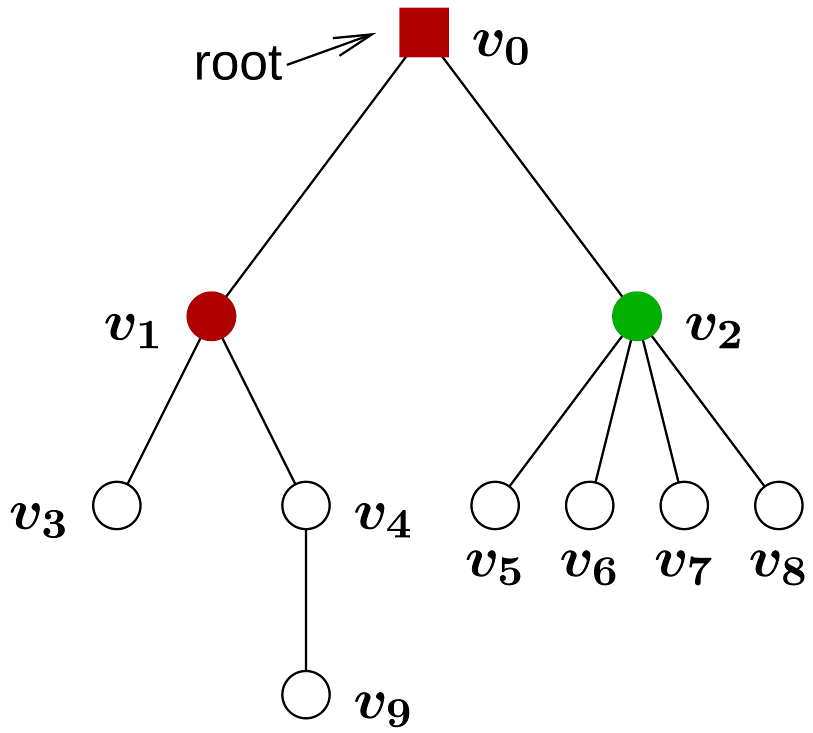

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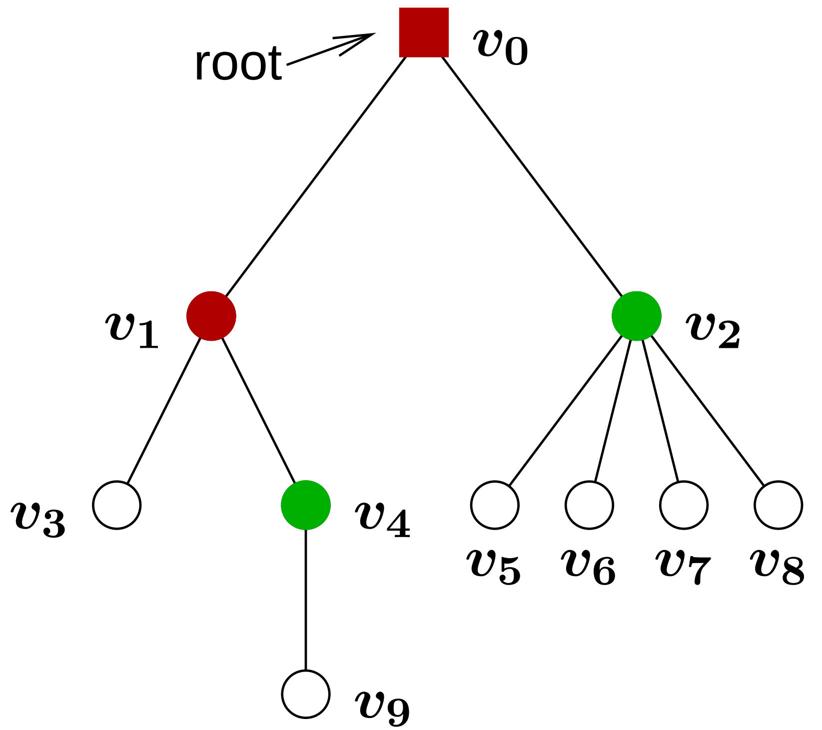

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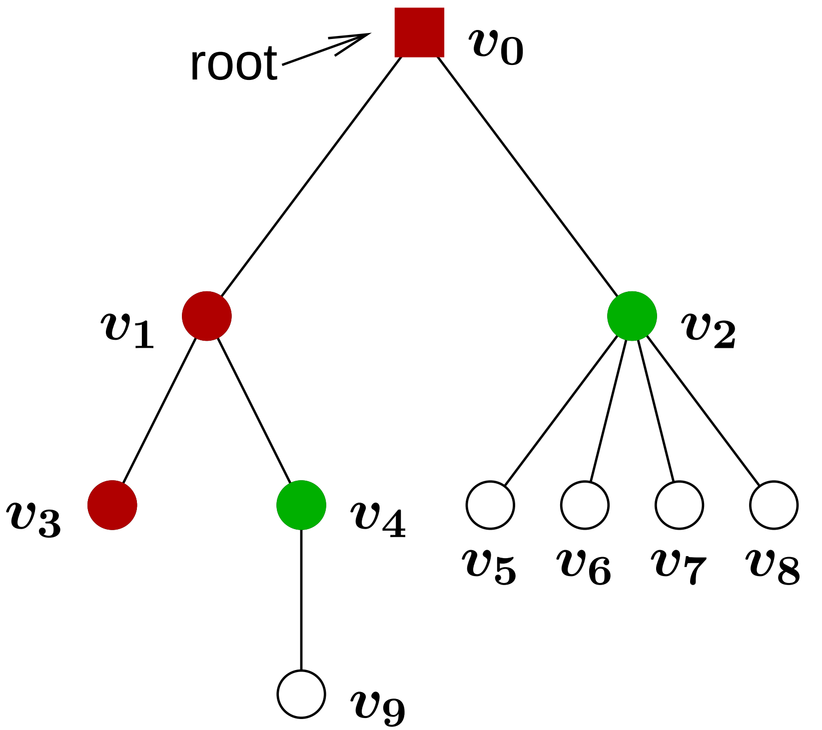
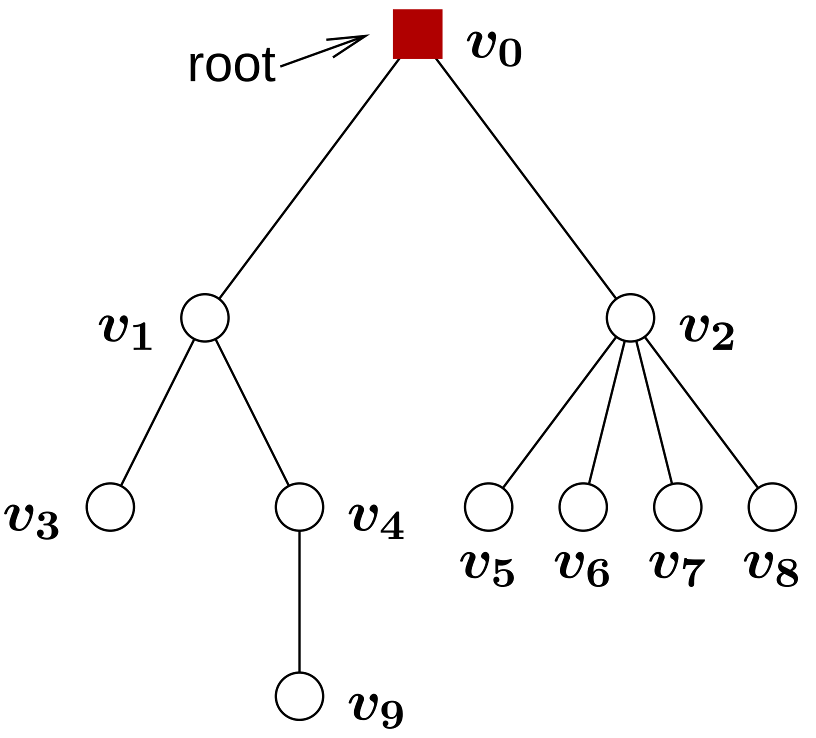


Thm. (Finbow, King, MacGillivray, and Rizzi 2004)
Finding the optimal strategy on trees is NP-complete.

We are interested in approximation algorithms.

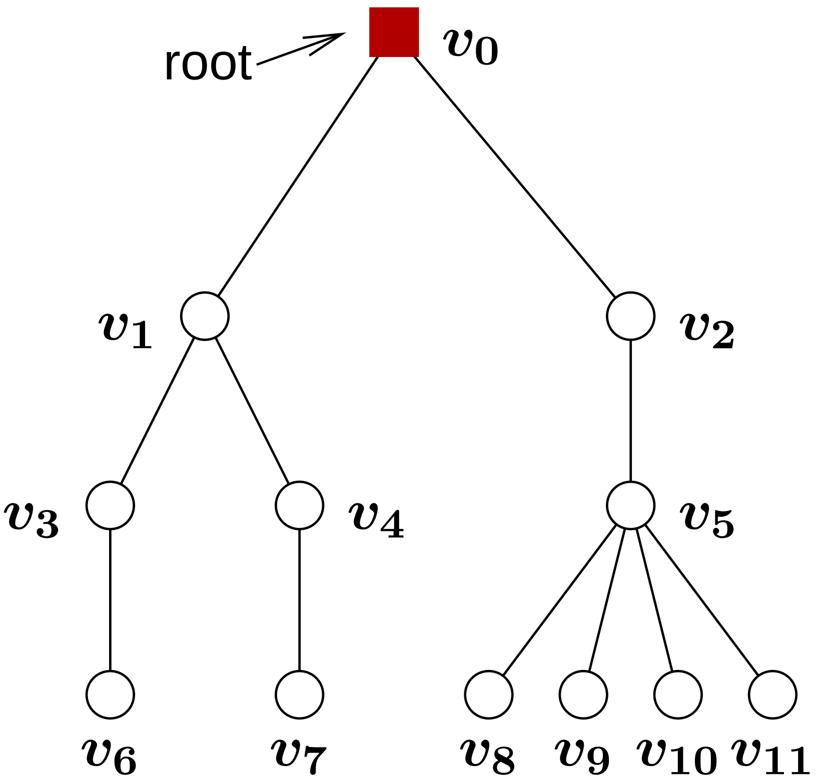
Obs. In an optimal strategy on trees, one vaccination is placed on each level.

If not, then move vaccination to parent, saving more vertices.



Greedy Algorithm:
On each level, defend the vertex that immediately saves the most vertices.

Is this optimal?



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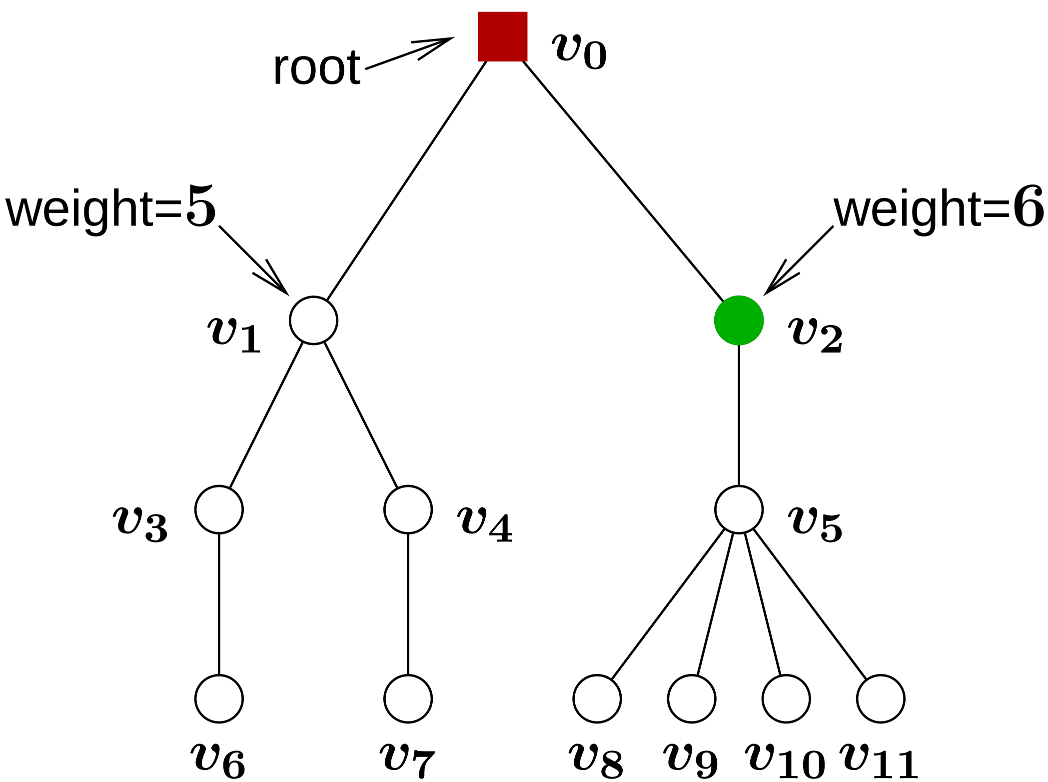

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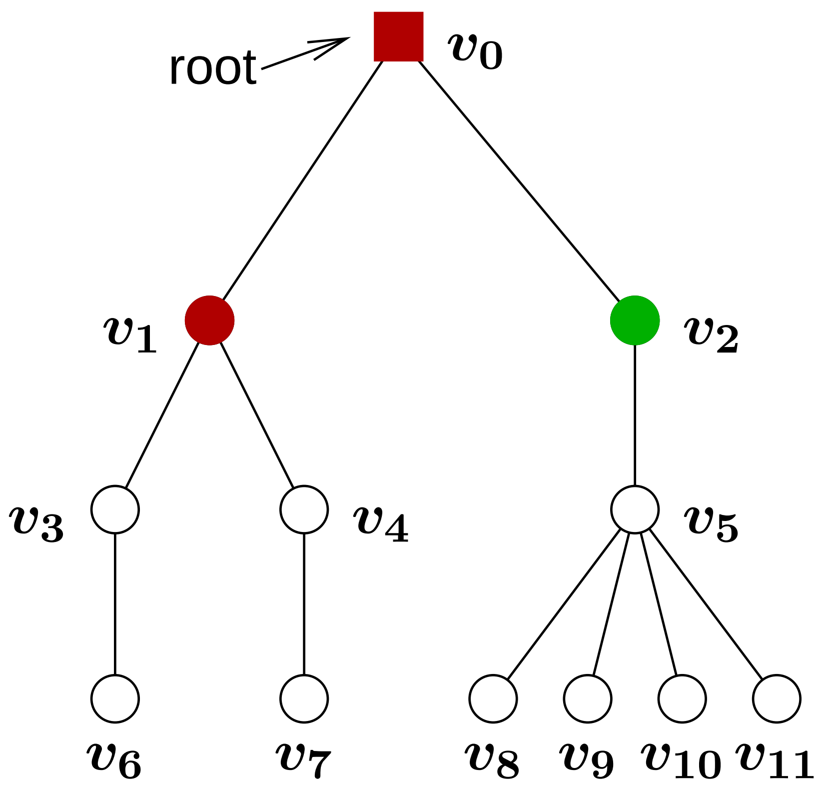

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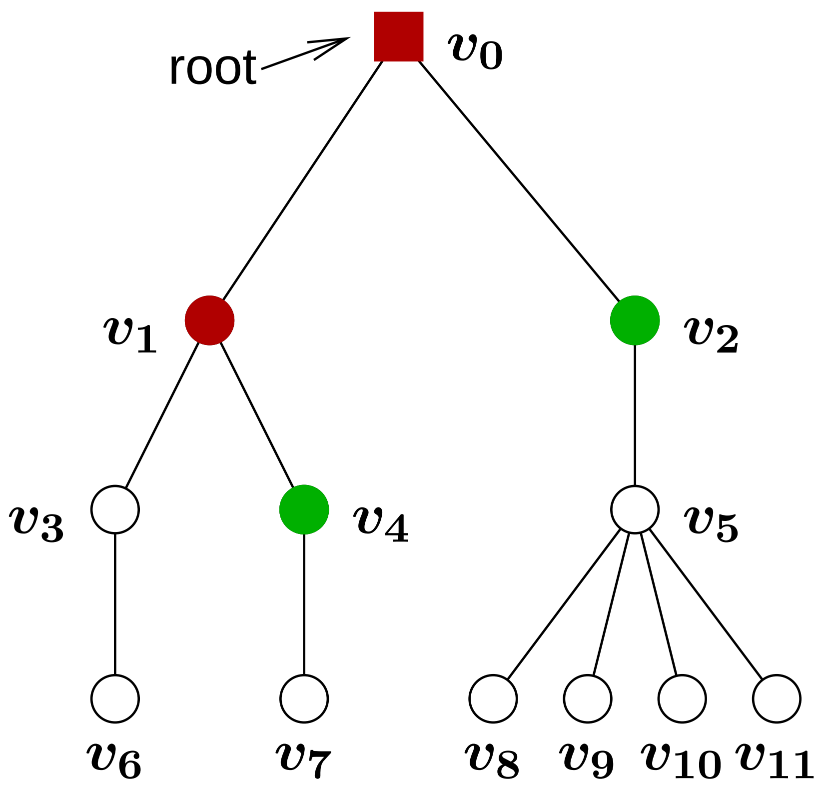

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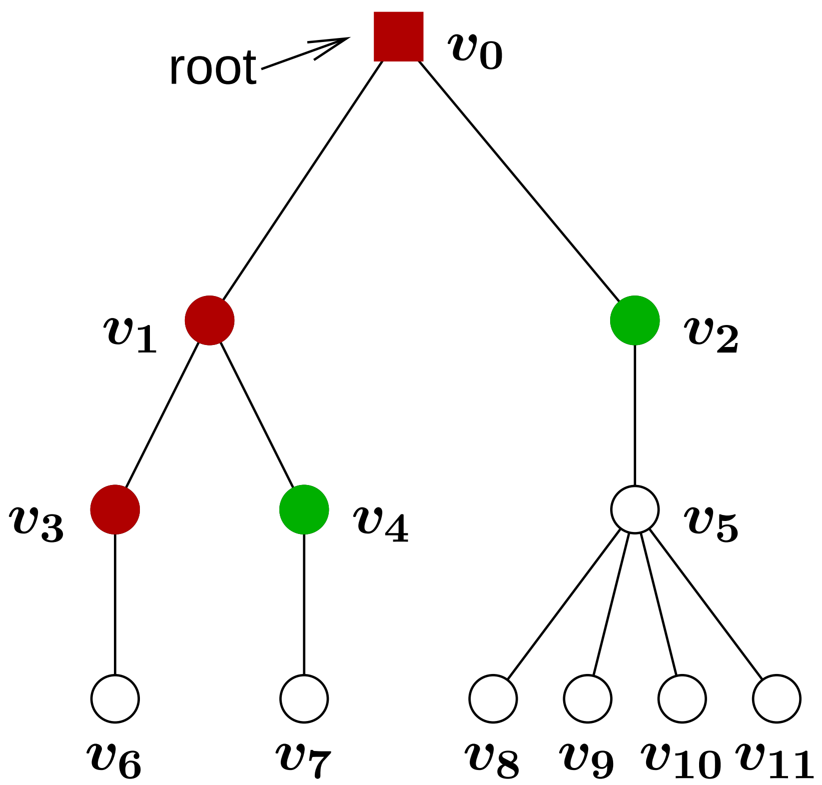

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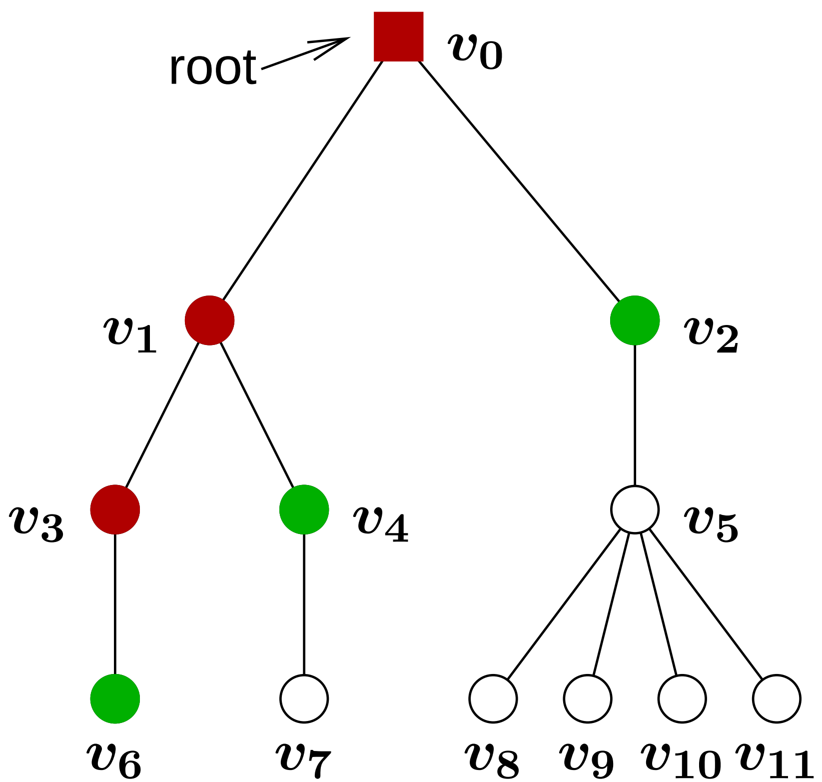
Is this optimal? Greedy saves 9 vertices,



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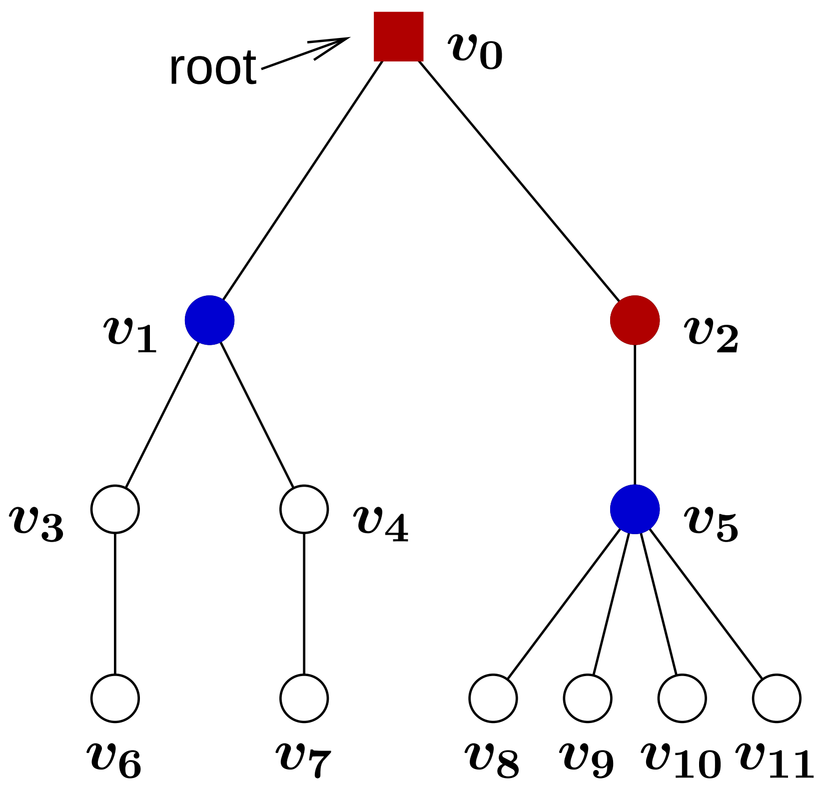
Is this optimal? Greedy saves 9 vertices, but the Optimal is 10.



Greedy Algorithm

Thm. (Hartnell and Li 2000) The greedy algorithm saves at least $\mathbf{1/2}$ as many vertices as an optimal strategy.

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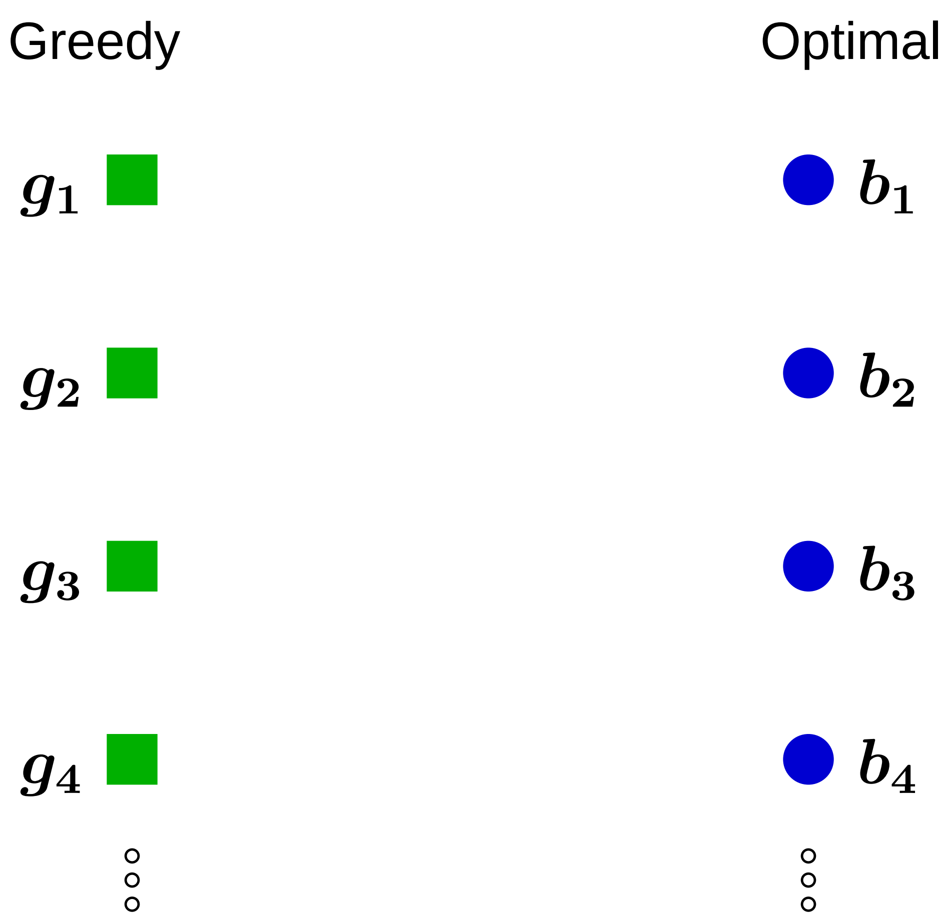
Let $g_1, g_2, \ldots, g_i, \ldots$ be the vertices defended by the greedy algorithm where g_i is on level i.

Let $b_1, b_2, \ldots, b_i, \ldots$ be the vertices defended in an optimal (or *best*) strategy on each level.

Let $wt(v)$ be the number of vertices saved by defending vertex v.

Analysis

Construct a bipartite graph. Draw one outgoing edge from each b_i.



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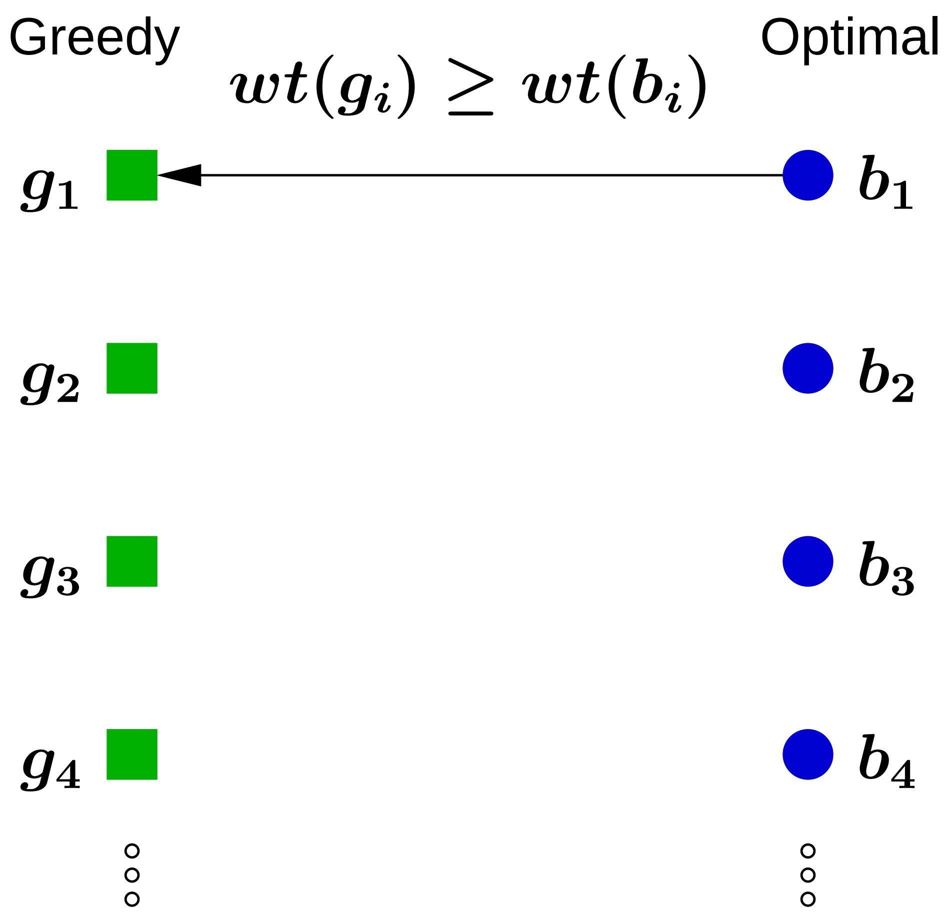

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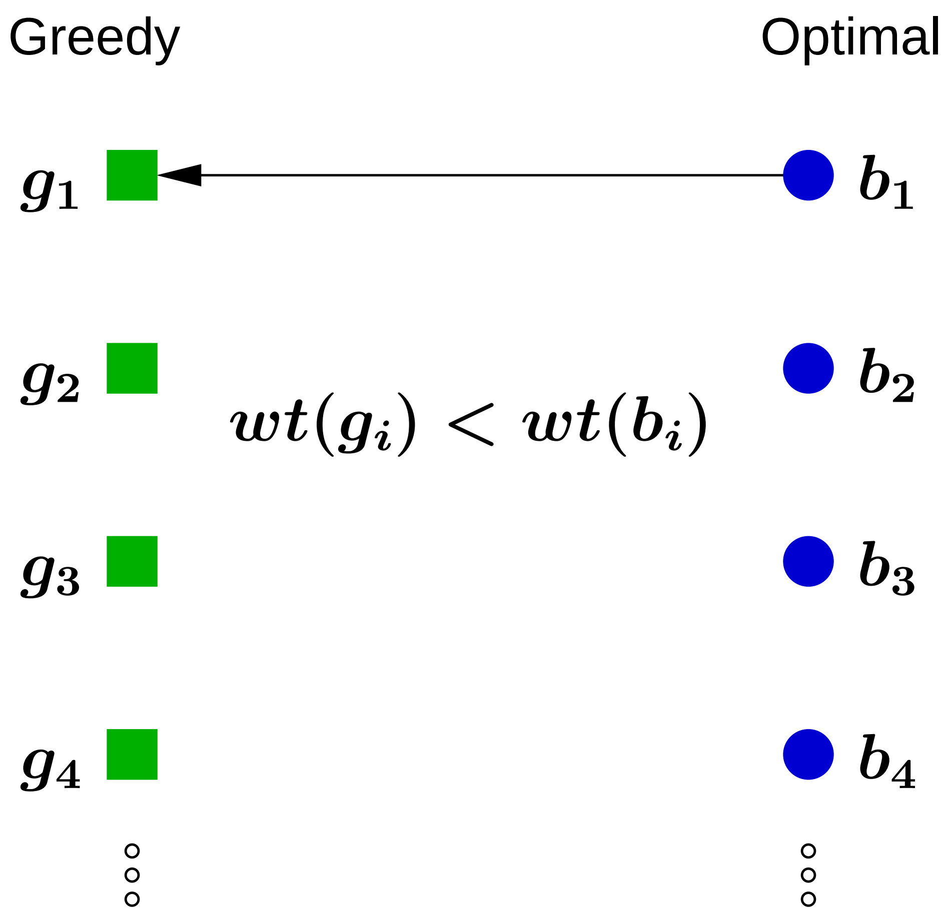

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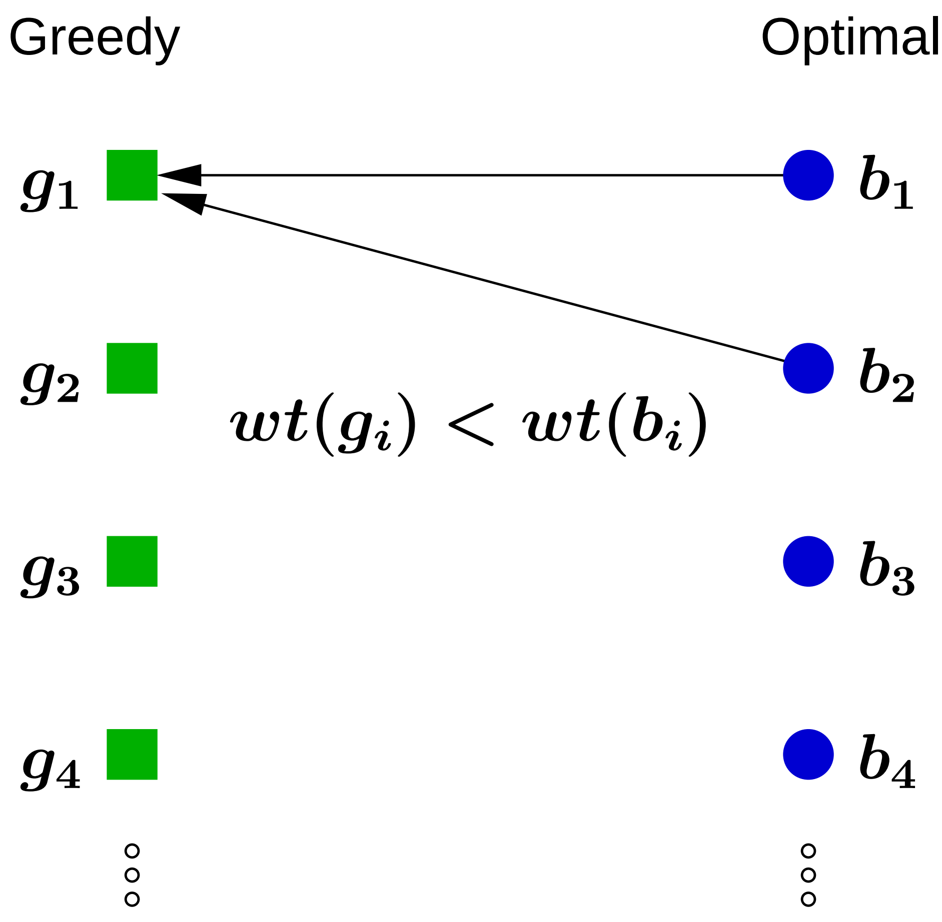

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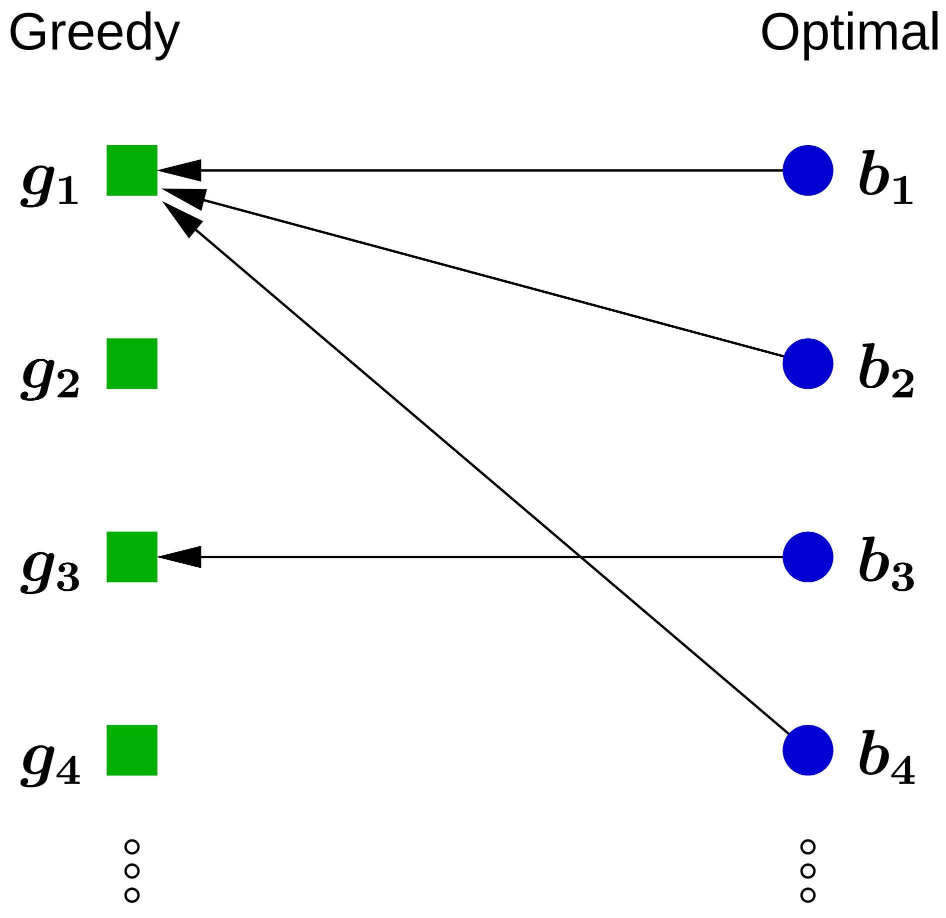

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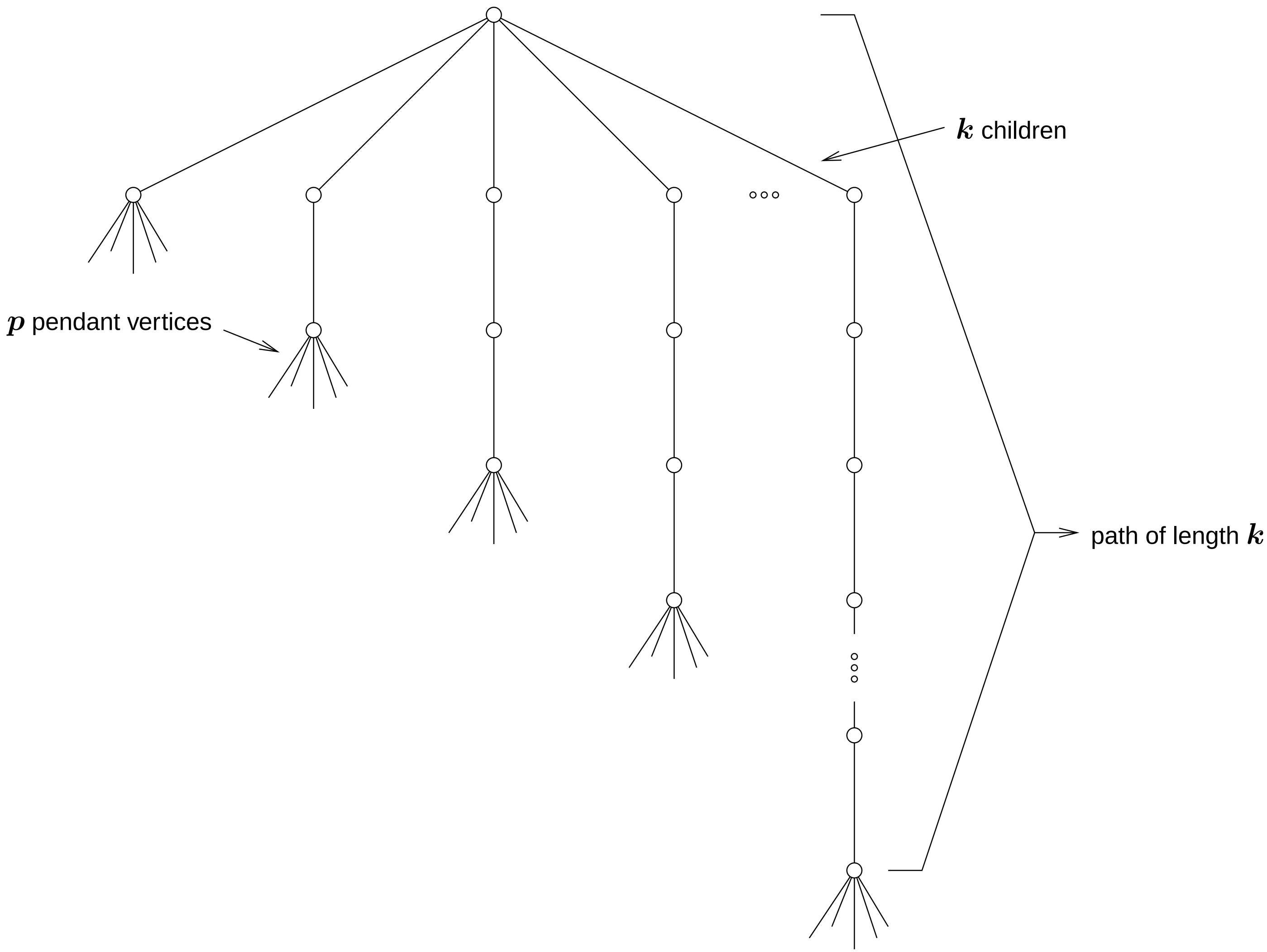
&= 2(\text{\# of vertices saved by greedy})
\end{aligned}
$$

$$\frac{1}{2}\text{optimal} \leq \text{greedy}$$

Sharpness

The value $\mathbf{1/2}$
is sharp:



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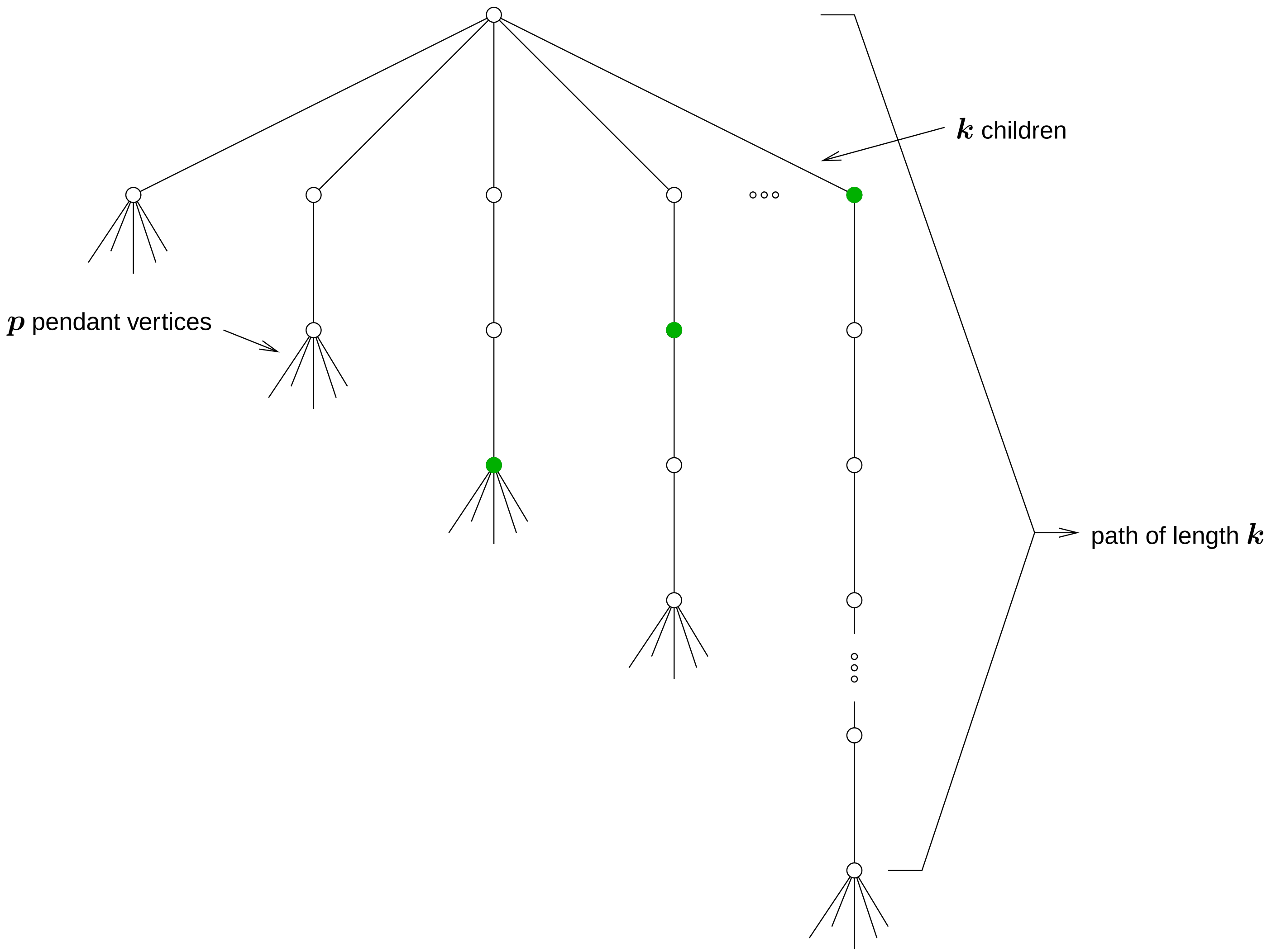

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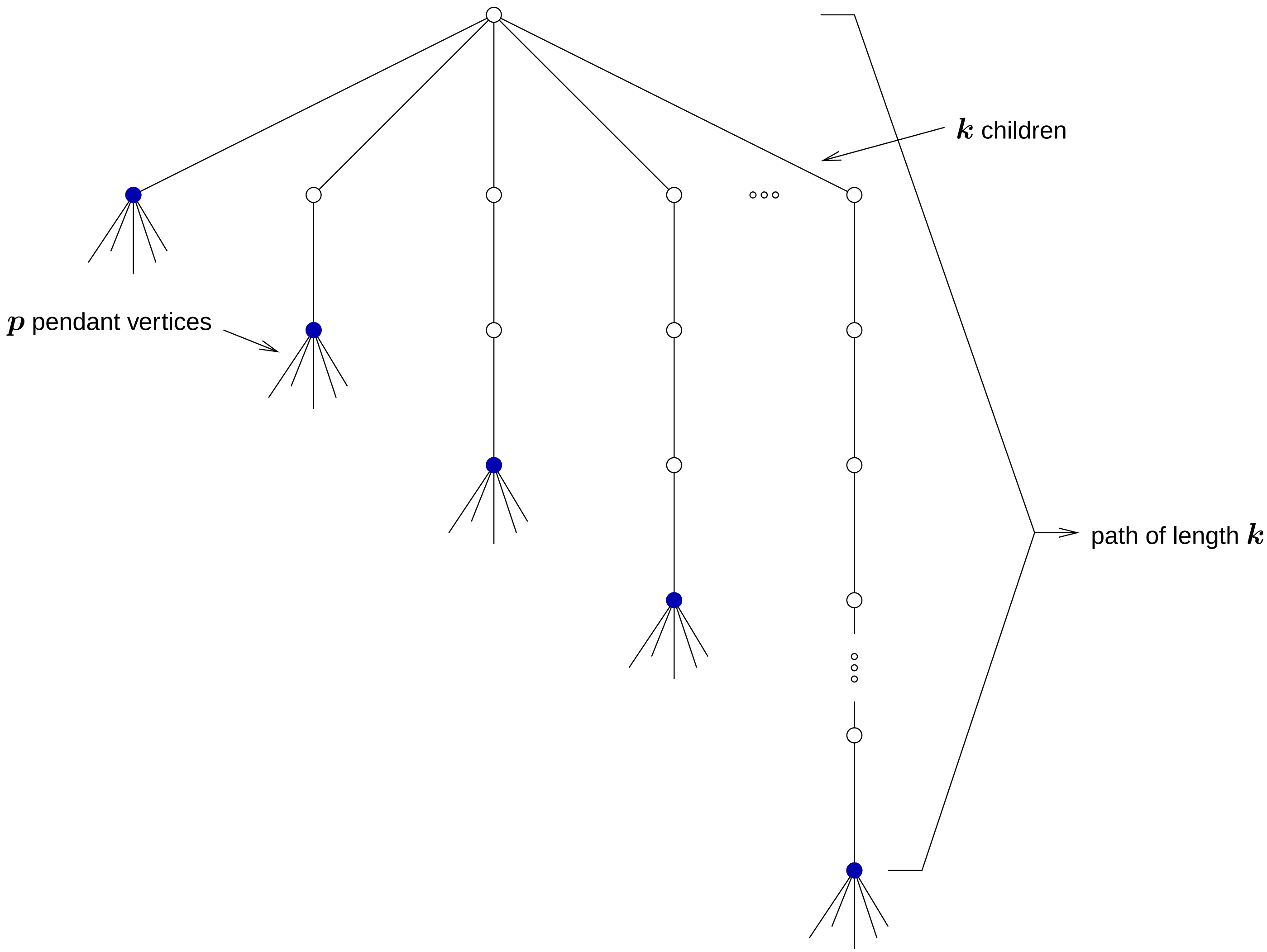


Questions

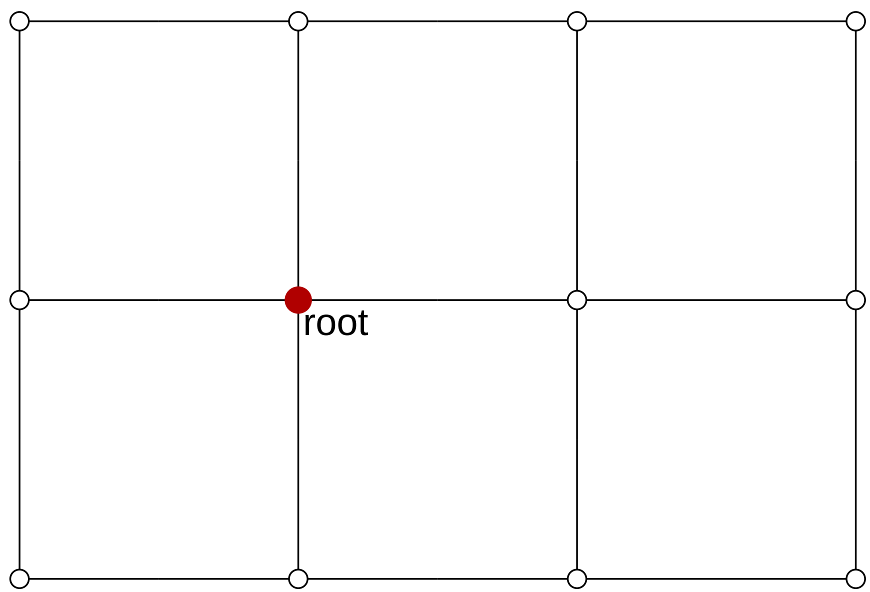
Can we construct better approximations?

Is it possible to approximate the optimal value in polynomial time to within $1 - \epsilon$?

Infinite Grids

Wang and Moeller (2002) showed $\boldsymbol{2d-1}$ vaccinations suffice for $\boldsymbol{L^d}$.

For $\boldsymbol{L^2}$, 3 vaccinations suffice.



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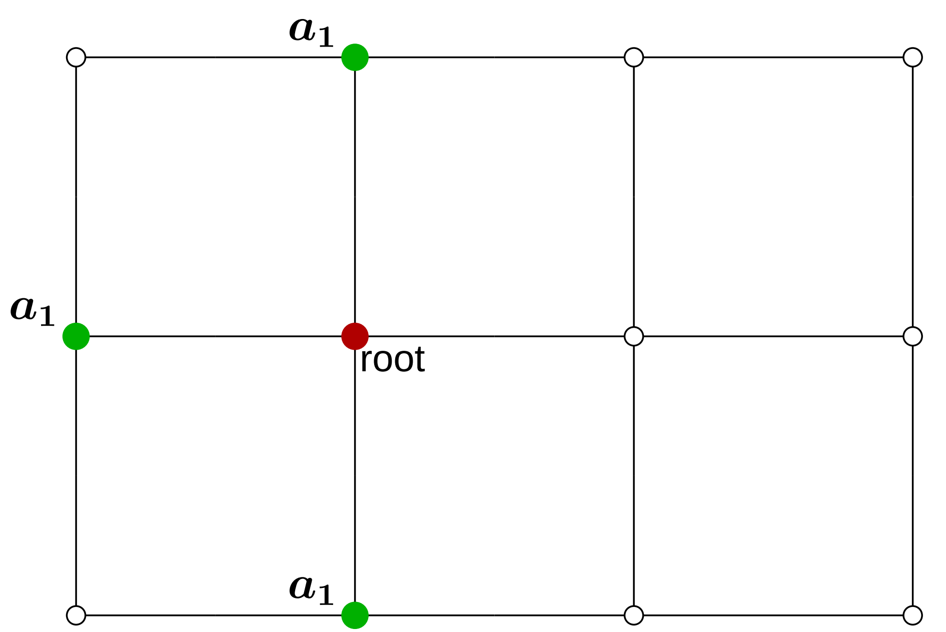

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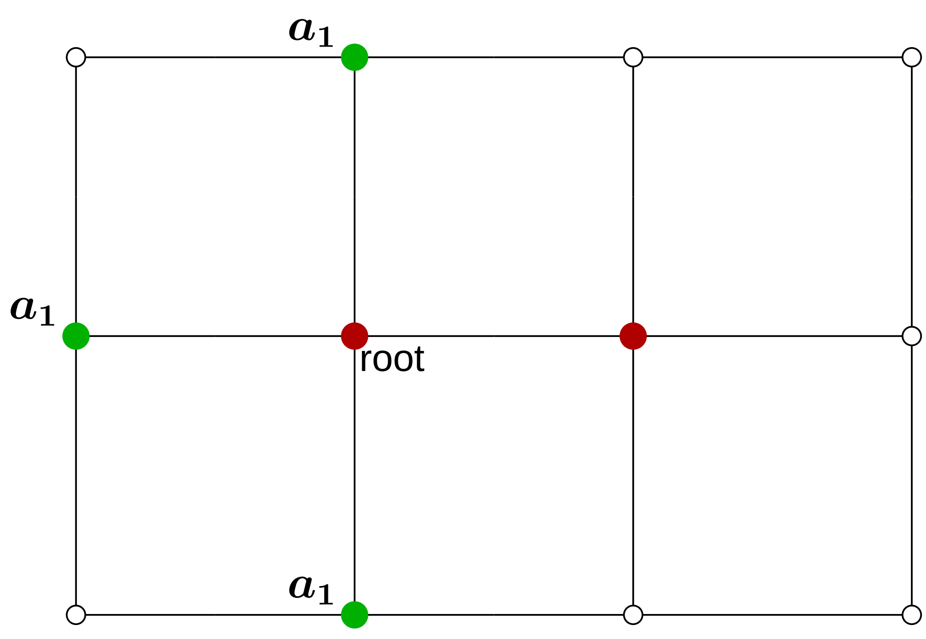

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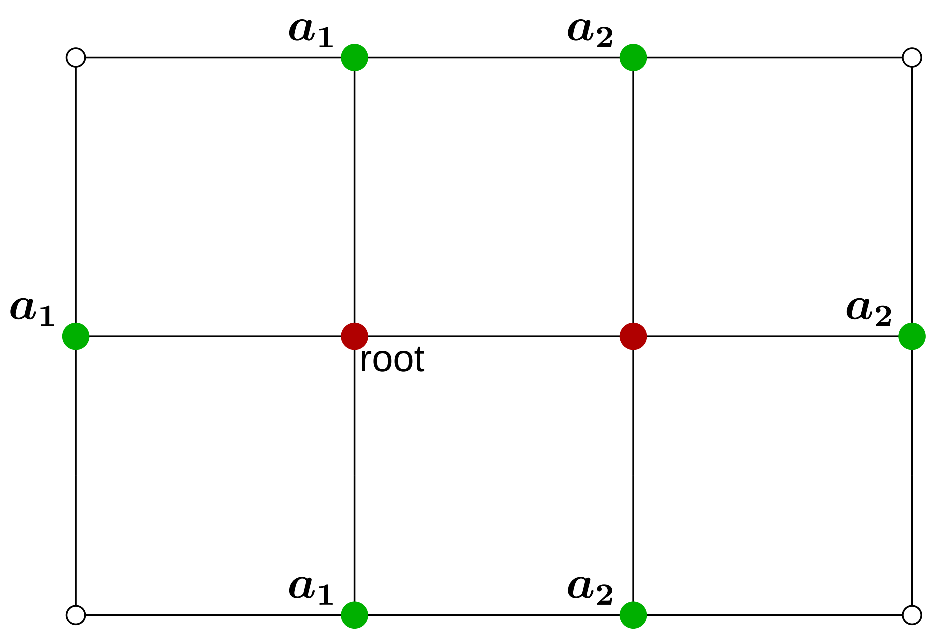
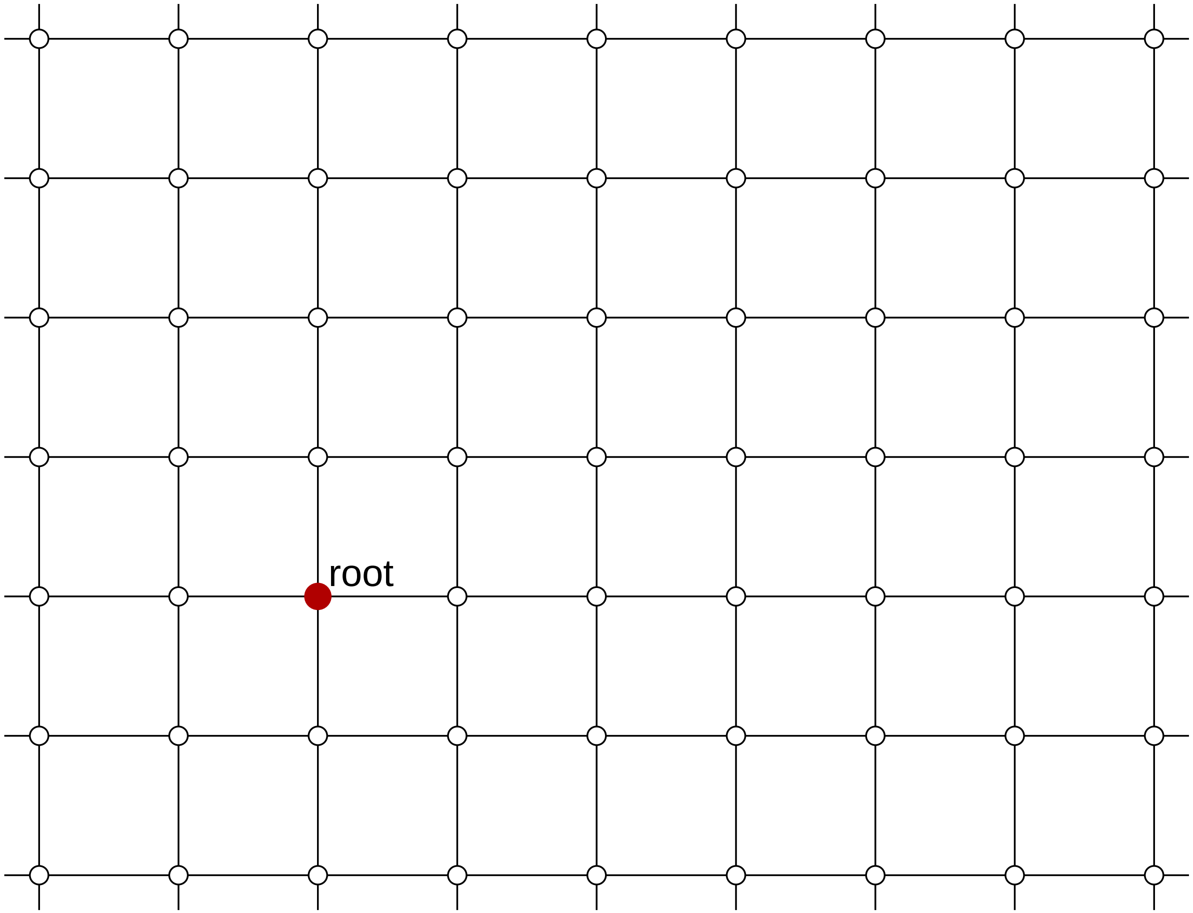


Two Dimensional Grid

How many vaccinations per time step are needed in L^2 to contain the outbreak?



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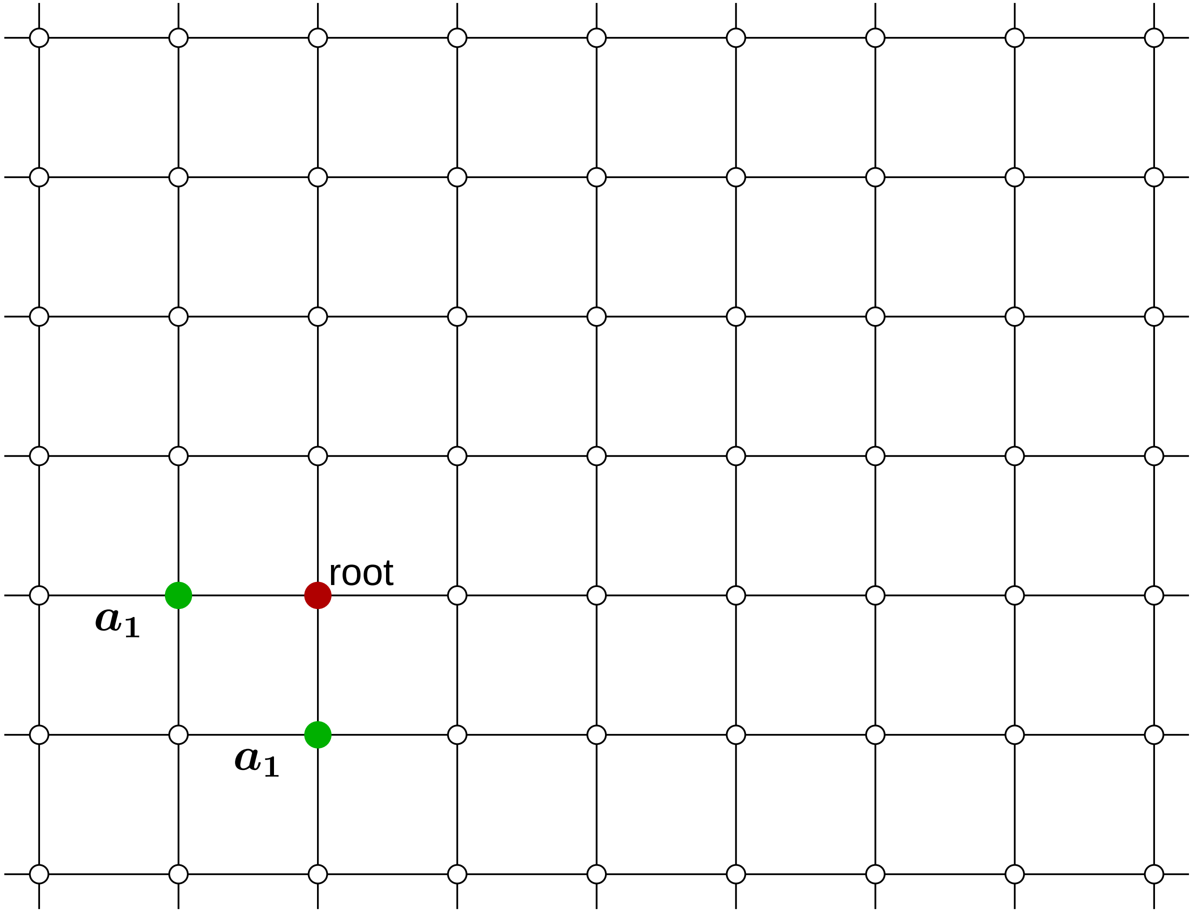

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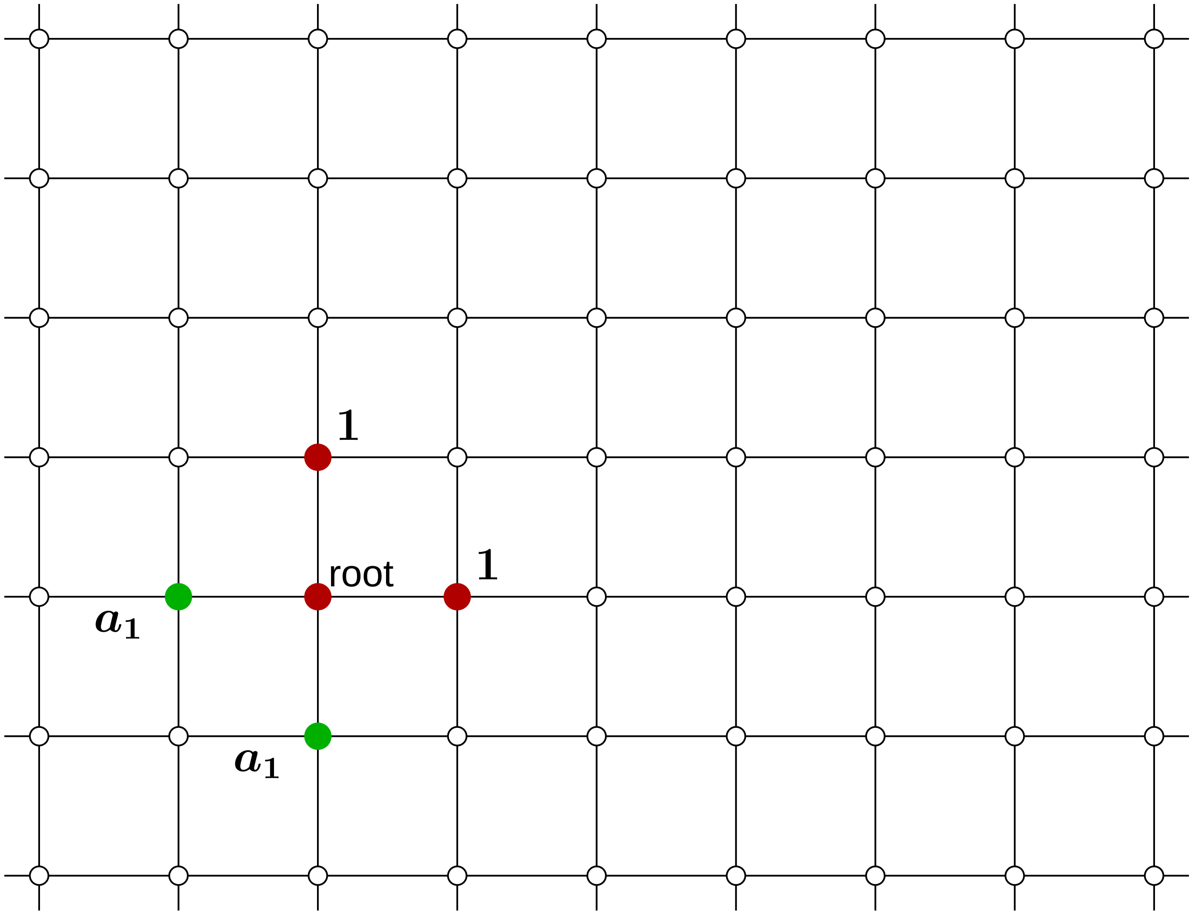

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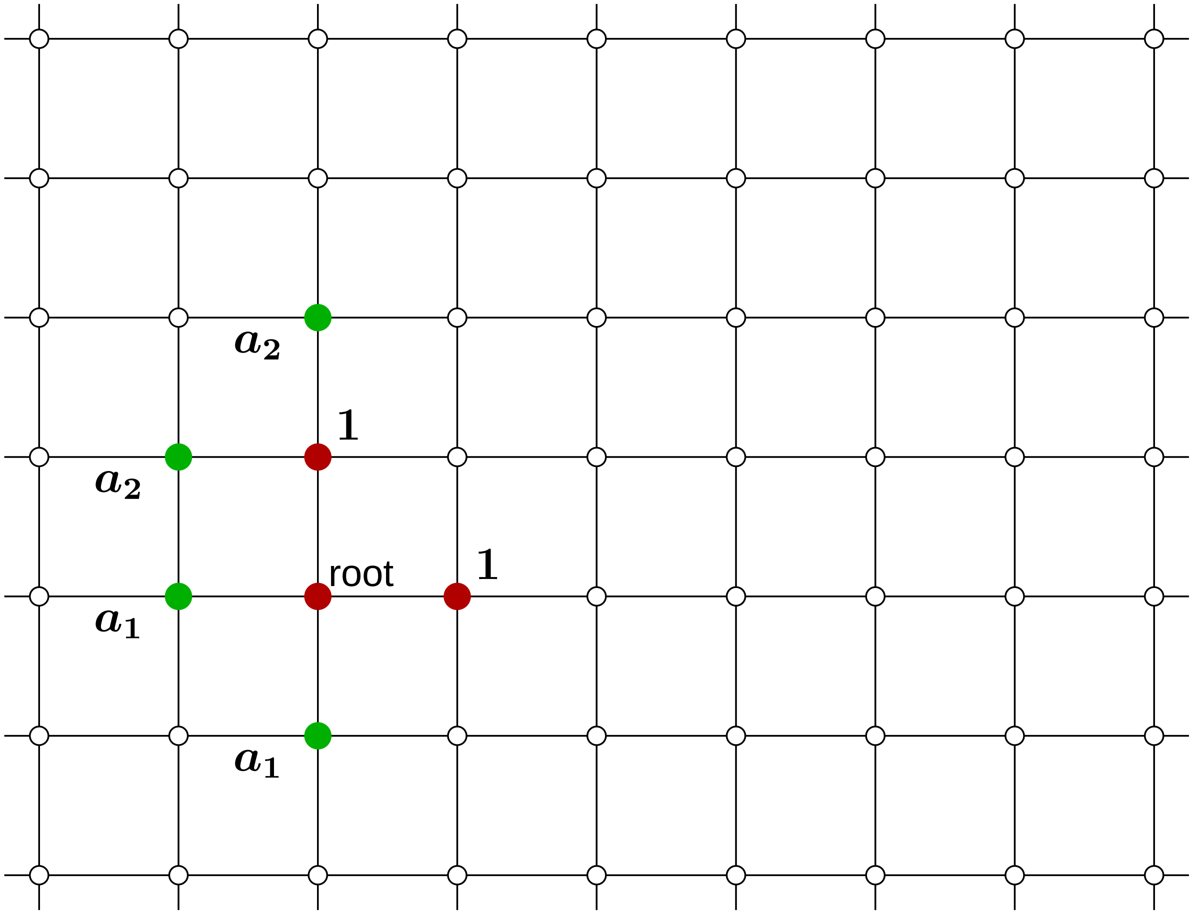

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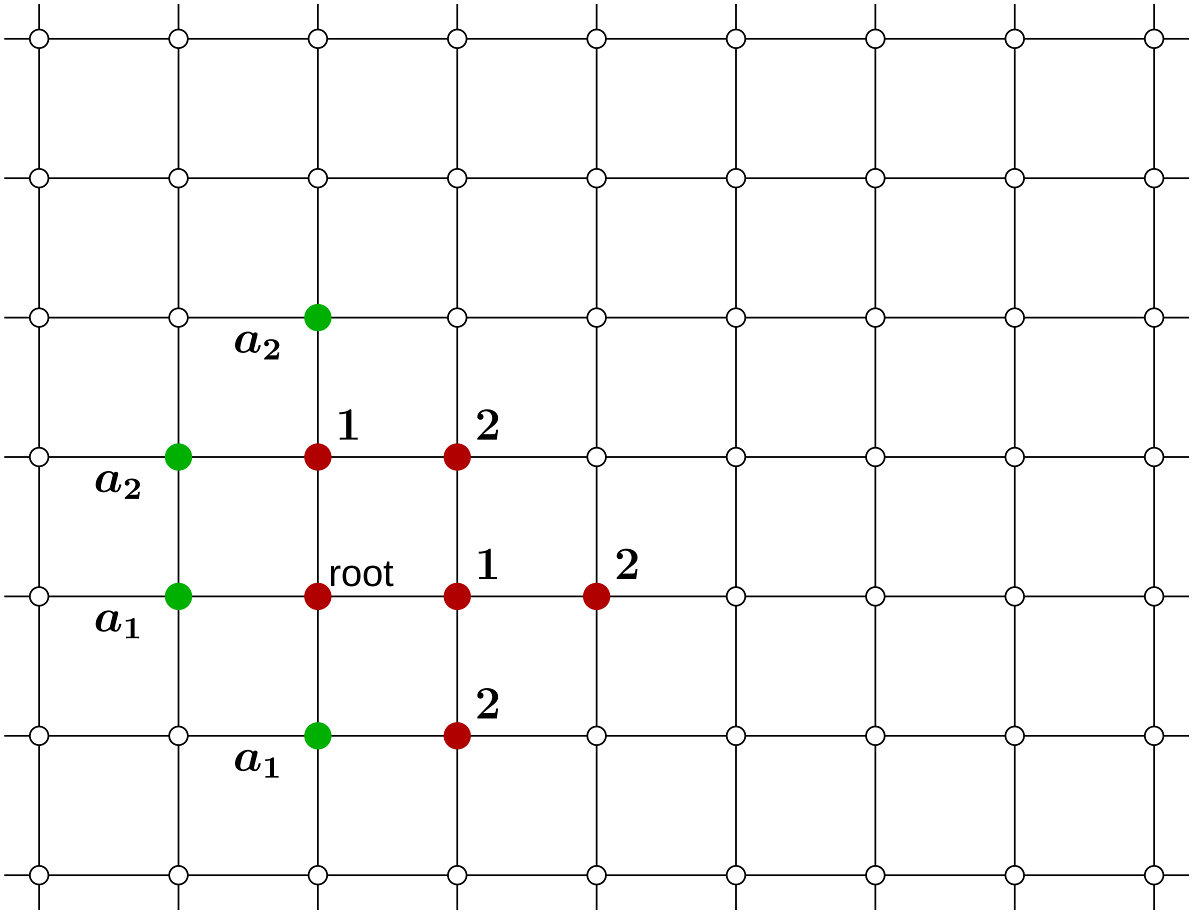

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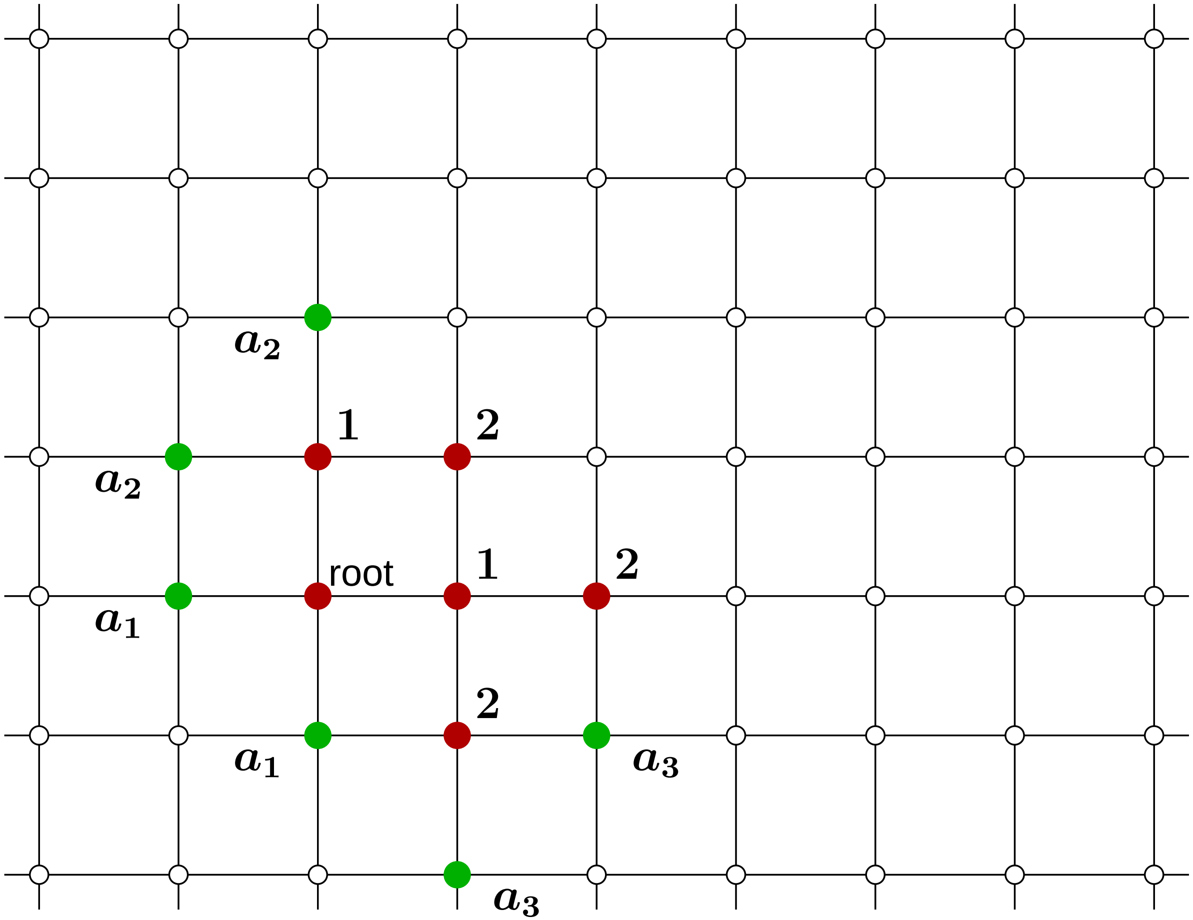

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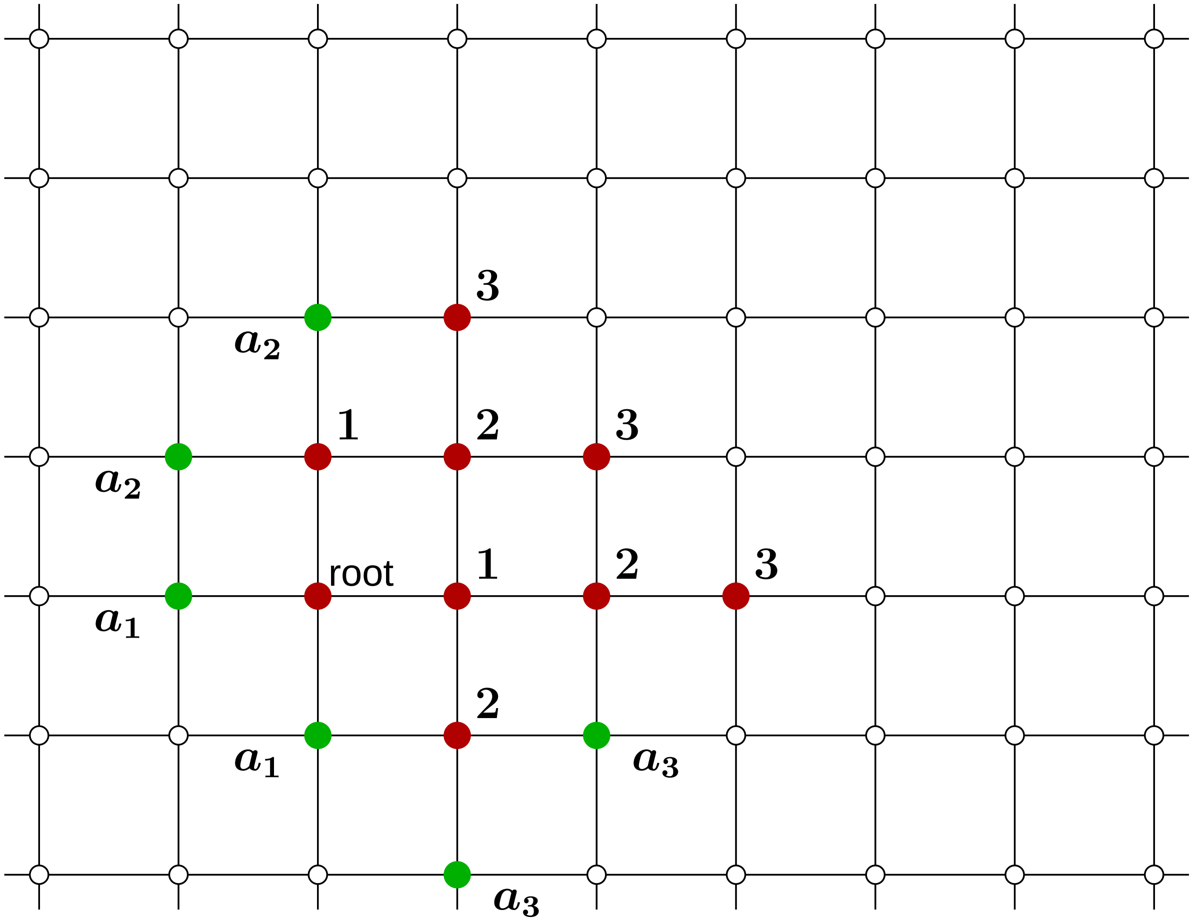

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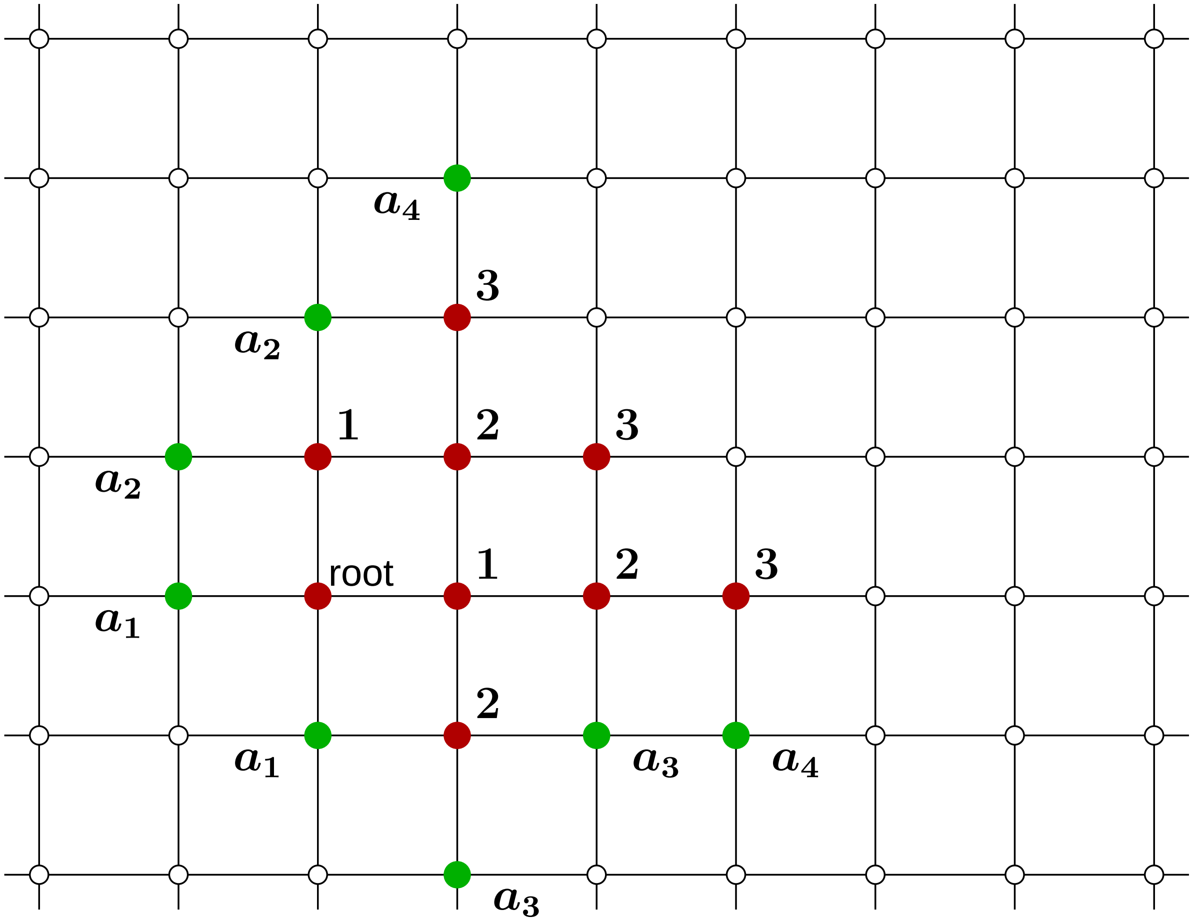

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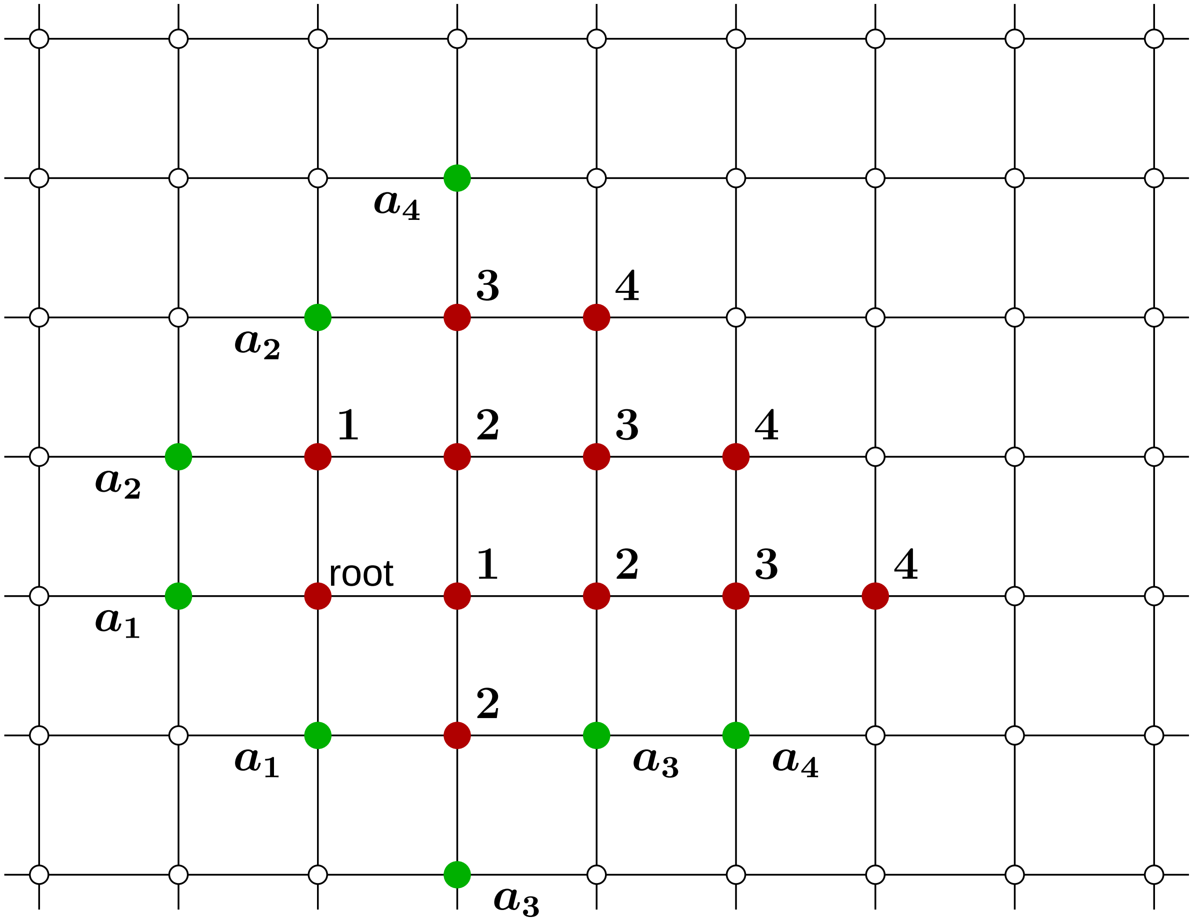

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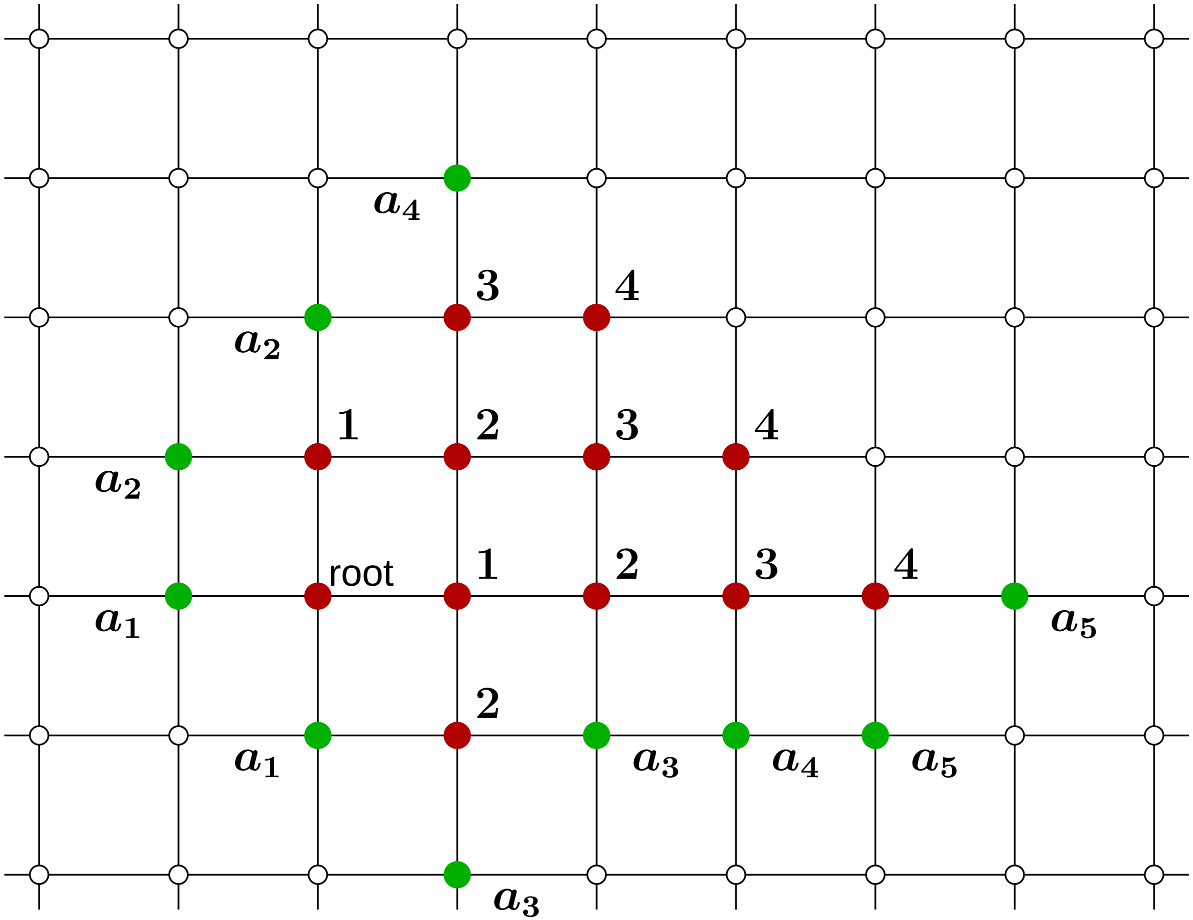

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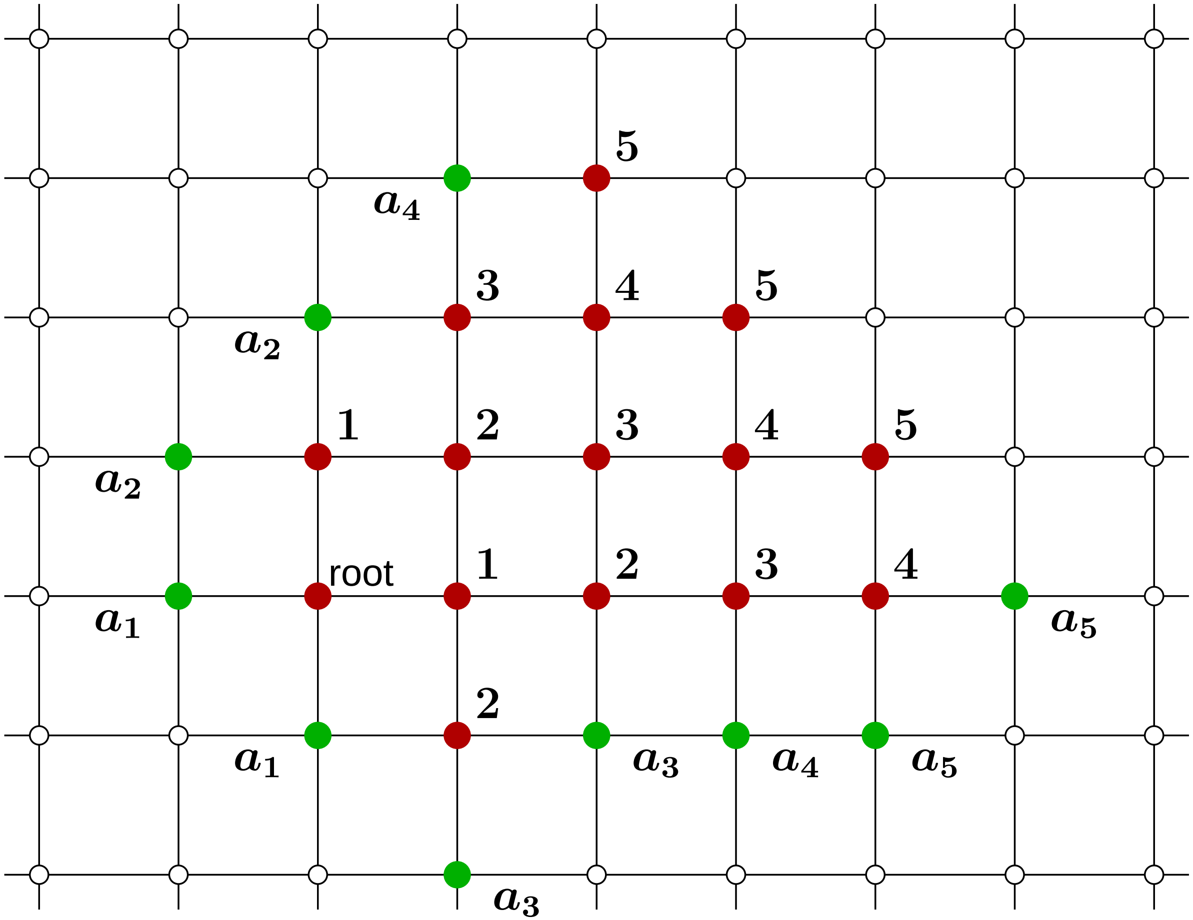

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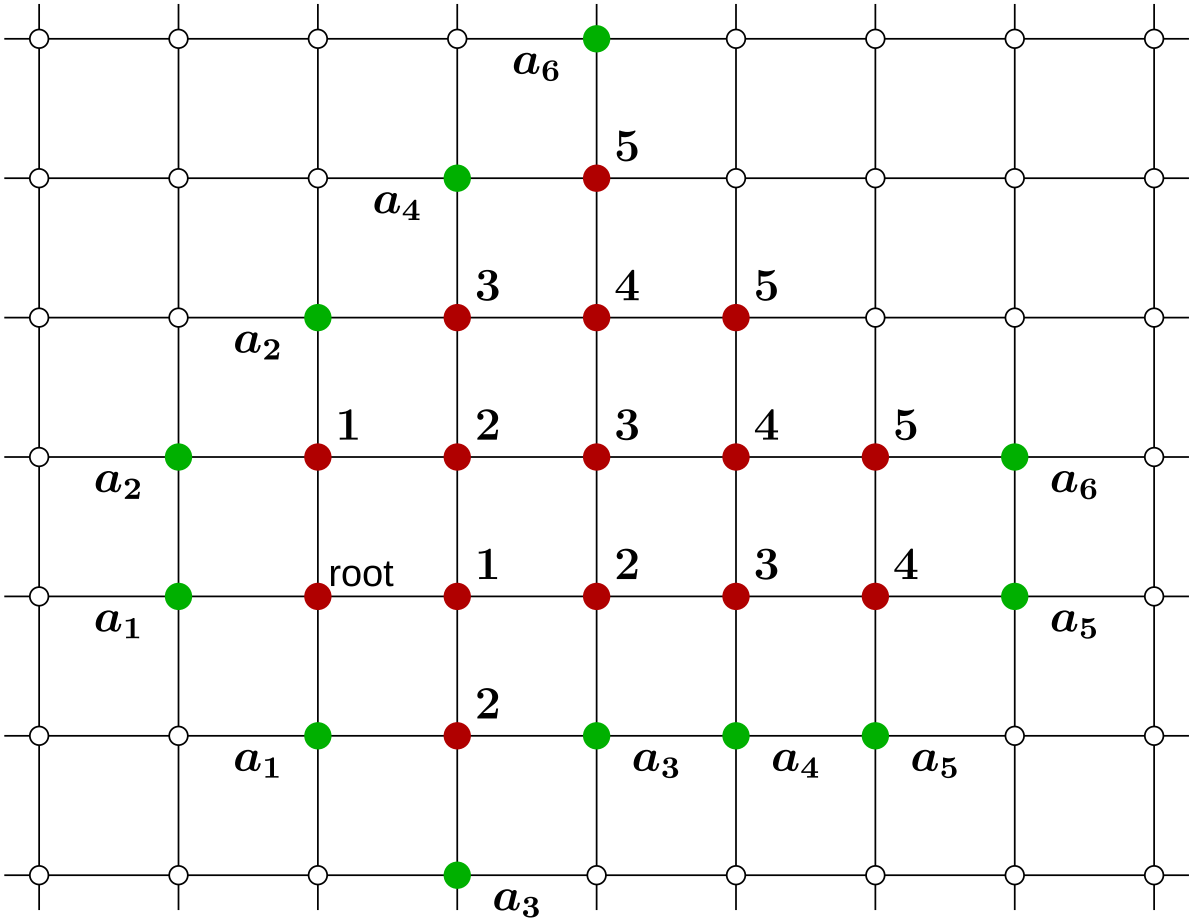

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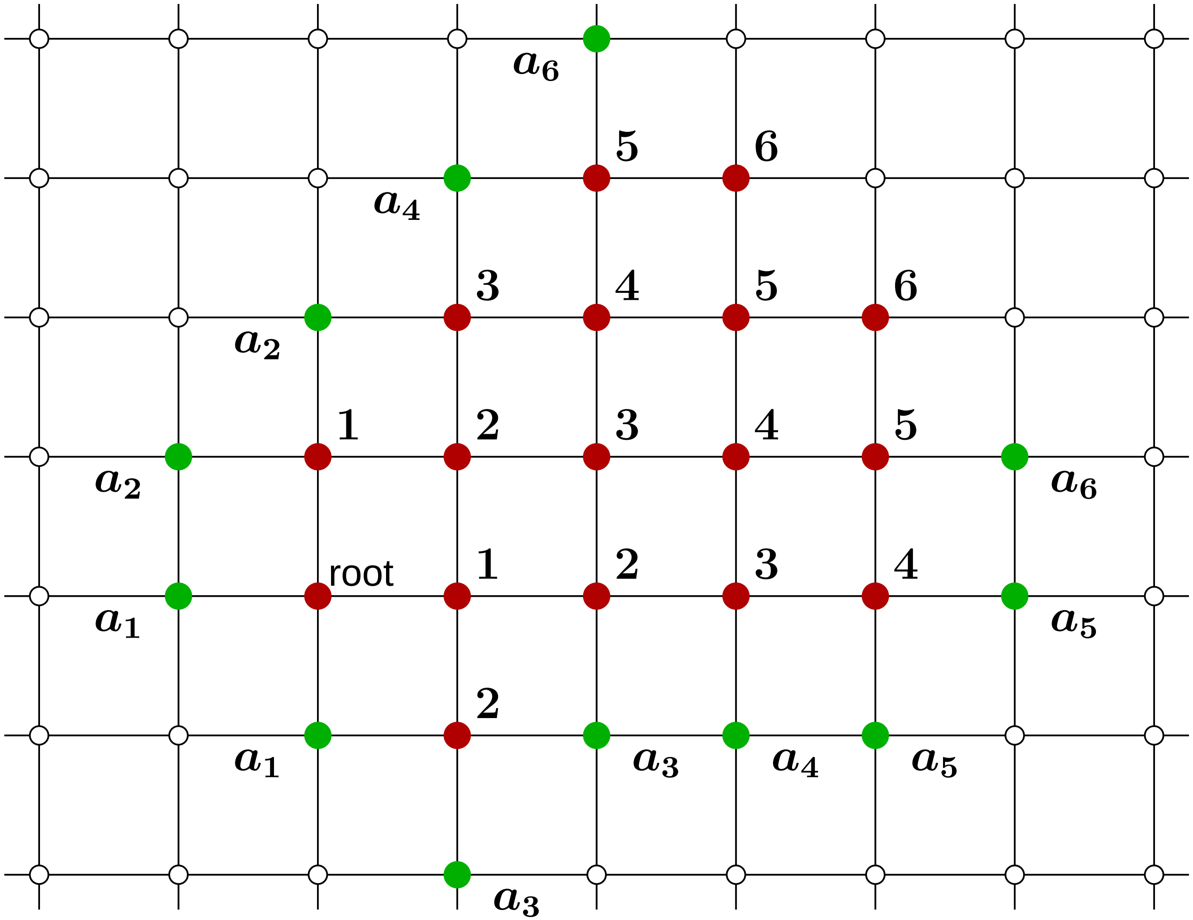

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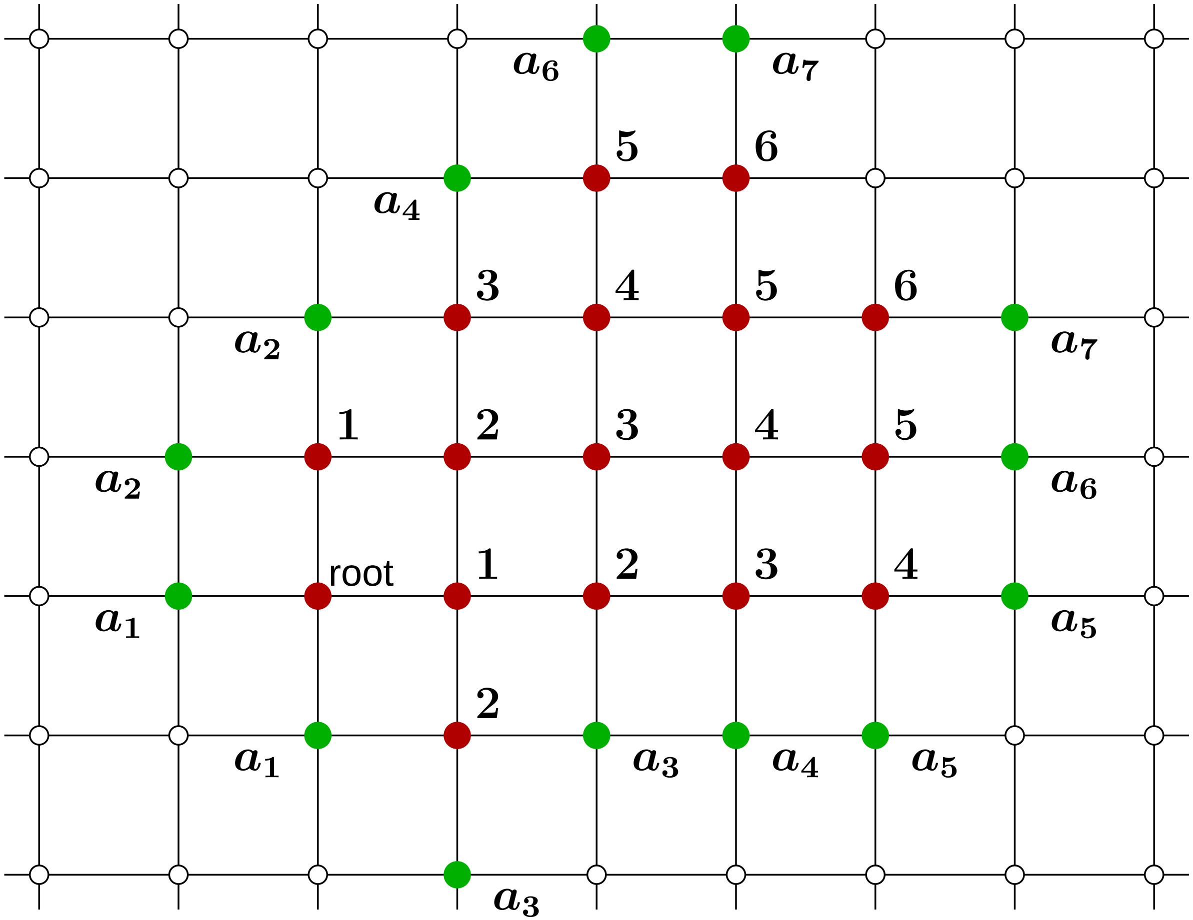

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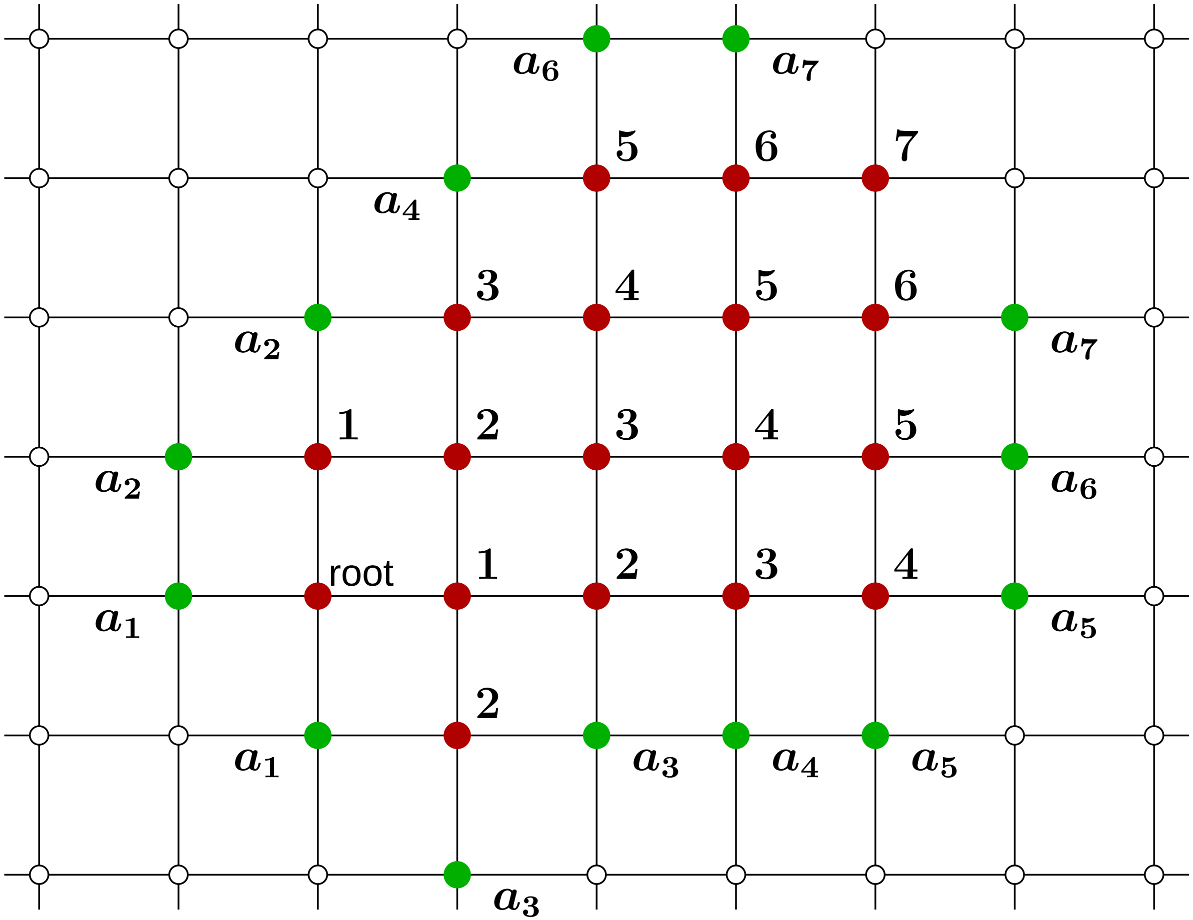

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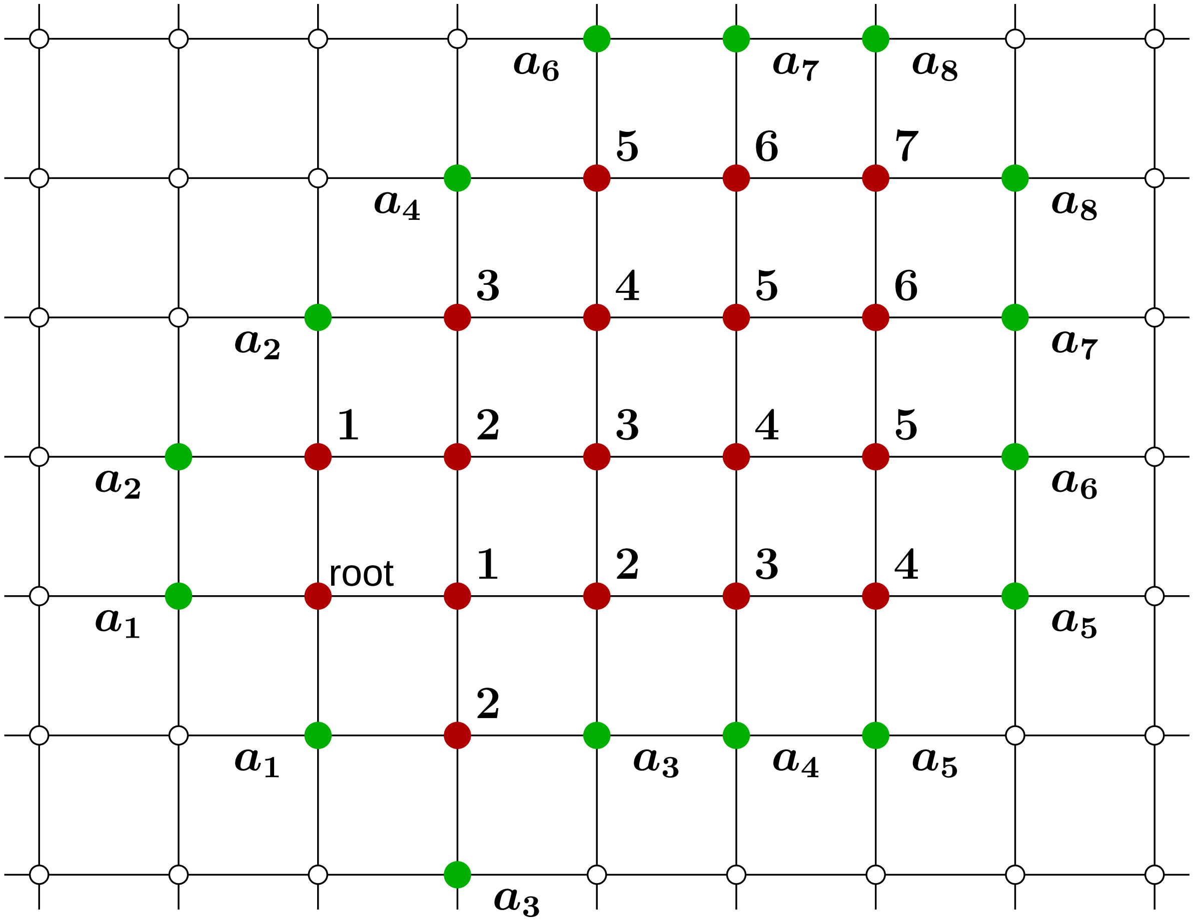

Hartnell, Finbow, and Schmeisser proved that $f = 2$ suffices.

Two Dimensional Grid

What is the minimum number of infected vertices in L^2 with 2 vaccinations per time step?

What is the minimum number of time steps until containment?

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Wang and Moeller (2002) used to case analysis to show 8 time steps were necessary and sufficient. Their solution allowed 18 infected vertices.

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What is the minimum number of infected vertices in L^2 with 2 vaccinations per time step?

What is the minimum number of time steps until containment?

Wang and Moeller (2002) used to case analysis to show 8 time steps were necessary and sufficient. Their solution allowed 18 infected vertices.

Using an integer program to simulate the disease spread, we have

Thm. (H) 18 vertices must be infected.

Integer Program

minimize $\sum_{x \in L} b_{x,T}$

subject to: $b_{x,t} + d_{x,t} - b_{y,t-1} \geq 0,$ for all $x \in L$, $y \in N(x)$, $1 \leq t \leq T$,

$b_{x,t} + d_{x,t} \leq 1,$ for all $x \in L$ and $1 \leq t \leq T$,

$b_{x,t} - b_{x,t-1} \geq 0,$ for all $x \in L$ and $1 \leq t \leq T$,

$d_{x,t} - d_{x,t-1} \geq 0,$ for all $x \in L$ and $1 \leq t \leq T$,

$\sum_{x \in L} (d_{x,t} - d_{x,t-1}) \leq 2,$ for $1 \leq t \leq T$,

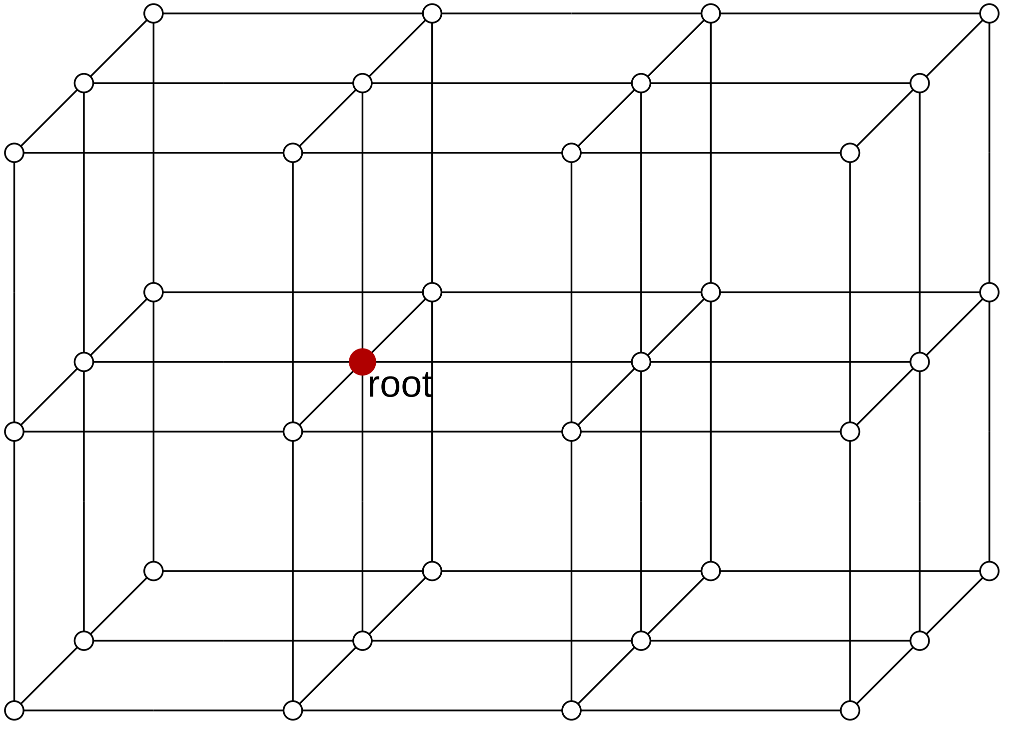
$b_{x,0} = \begin{cases} 1 & \text{if } x \text{ is the origin,} \\ 0 & \text{otherwise,} \end{cases}$ for all $x \in L$,

$d_{x,0} = 0,$ for all $x \in L$,

$b_{x,t}, d_{x,t} \in \{0, 1\},$ for all $x \in L$ and $0 \leq t \leq T$.

Three Dimensional Grids

Wang and Moeller (2002) showed $\mathbf{2d-1}$ vaccinations suffice for $\mathbb{L}^{\boldsymbol{d}}$.



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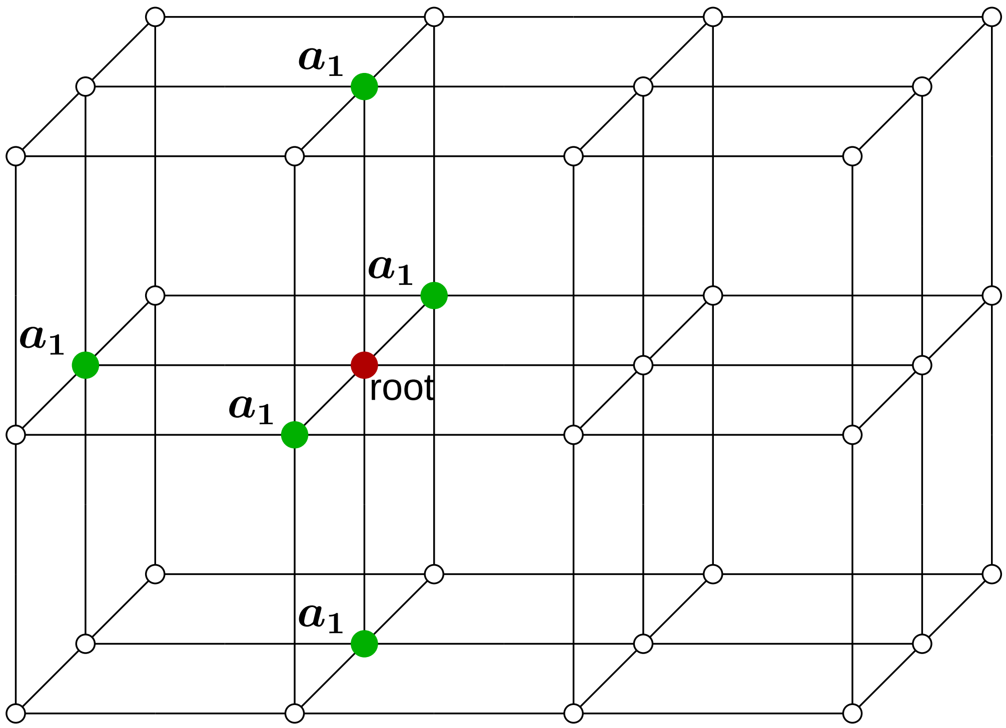

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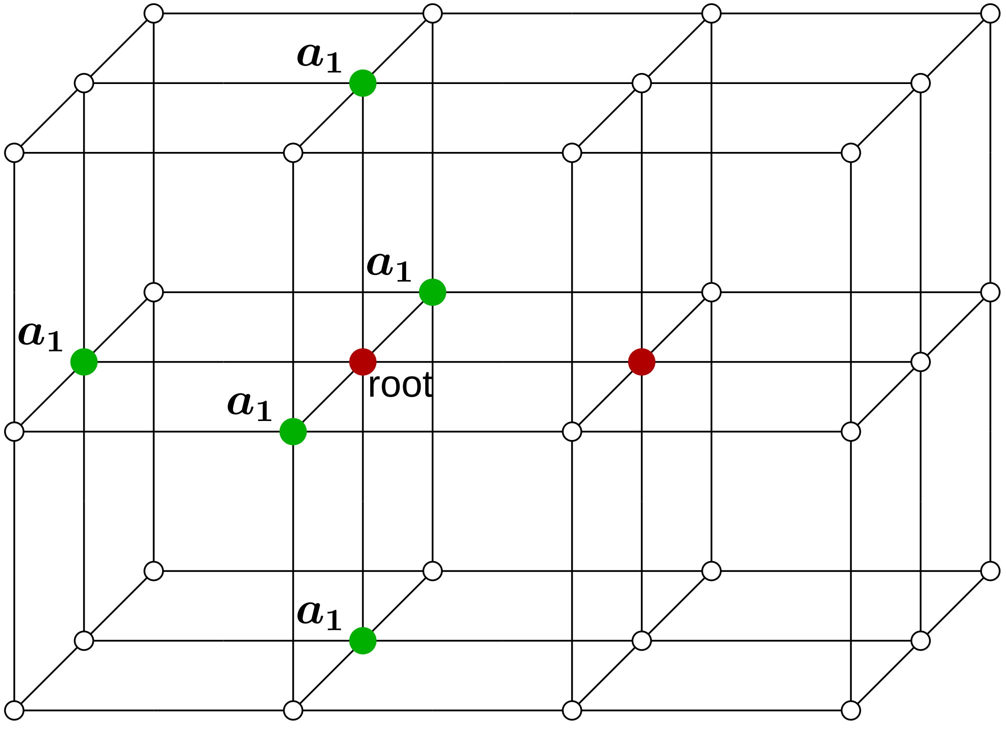

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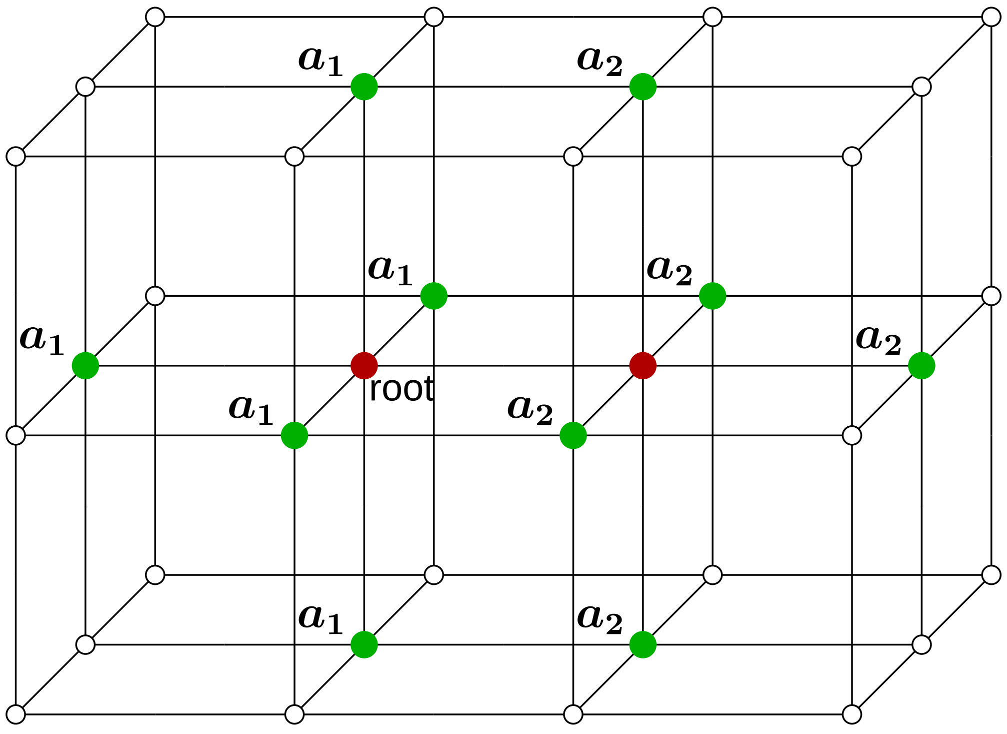

Higher Dimensional Grids

Wang and Moeller conjectured the following:

Thm. (Develin and H) $\mathbf{2}\boldsymbol{d}-\mathbf{1}$ vaccinations are *necessary* to contain an outbreak starting at a single vertex in $\boldsymbol{L}^{\boldsymbol{d}}$, for $\boldsymbol{d} \geq \mathbf{3}$.

Method

We strengthen a “Hall-type” condition of Fogarty (2003):

Def. Let D_k denote the vertices at distance k from the root.

Lem. If every $A \subseteq D_k$ satisfies

$$|N(A) \cap D_{k+1}| \geq |A| + f,$$

then an outbreak starting at the root *cannot* be contained with f vaccinations per timestep.

The disease grows faster than any containment effort using only f vaccination per timestep.

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We strengthen the lemma by making the expansion non-uniform.

We consider initial growth of the vaccination that is faster than f so that the disease reaches a "critical mass" and can sustain growth by f from then on.

Larger Outbreaks

What about larger outbreaks?

Fogarty showed that $\mathbf{2}$ vaccinations per time step suffice to contain *any* finite outbreak in L^2.

Larger Outbreaks

What about larger outbreaks?

Fogarty showed that $\mathbf{2}$ vaccinations per time step suffice to contain *any* finite outbreak in $\boldsymbol{L^2}$.

However, we prove

Thm. (Develin and H) For every $\boldsymbol{f > 0}$, there is a finite outbreak in $\boldsymbol{L^d}$, $\boldsymbol{d \geq 3}$, such that $\boldsymbol{f}$ vaccinations per time step cannot contain the outbreak.

Lem. Let $A \subseteq D_k \subseteq L^3$. If

$$|A| > \frac{1}{2}(f-1)(f-2),$$

then $\left|N(A) \cap D_{k+1}^{+}\right| \geq |A| + f$.

In other words, the "front" of the disease grows by at least $\Omega(\sqrt{x})$.

Further Directions for Research:

- Consider the problem on other classes of graphs besides trees and grids.
  - Of interest: small-world and scale-free graphs, random geometric models.
- Consider a probabilistic model of disease spread.
- Characterize the “most vulnerable” vertices in a graph for an outbreak to start at (without explicitly calculating the response).

Disease Containment by Progressive Vaccination on Trees and Grids

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