Sufficiently Fat Polyhedra are Not 2-Castable*

David Bremner†and Alexander Golynski‡

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In this note, we consider the problem of manufacturing a convex polyhedral object via casting. Casting is a common manufacturing process where some molten substance is poured or injected into a cavity (called a *mold*), and then allowed to solidify. In many applications it is desirable to remove the cast object from the mold without destroying the mold. In general this requires the mold be partitioned into several parts, which are then translated away from the cast object.

In the simplest case, a mold for polyhedron P is partitioned into two parts using a plane. If a successful partition (i.e. both parts can be removed by translations without collisions), we say that P is 2-*sand castable*. Here we consider a generalization of the sand casting process where the object can be manufactured by gluing together two 1-castable objects. We allowed to glue object via identical faces only. Polyhedron P is called 1-*castable* if it can be made using a single mold. That is, there is a half-space H containing P, such that P can be separated from the mold $H \setminus P$ by a translation. The boundary plane of H intersects P by one of its facets (F), let us call F a *casting facet* and call P castable through F. The polyhedra which can be obtained by gluing two 1-castable is called 2-*castable*.

Guided by intuition about smooth objects, one might suspect that all convex polyhedra are 2-sand castable. This turns out not to be the case. Bose, Bremner and van Kreveld give an example of a 12 vertex convex polyhedron that is not 2-sand castable (the proof is computer based). Majhi, Gupta and Janardan give a simpler example with only 6 vertices.

Here, we provide a general class of convex polyhedra that are not 2-castable. In particular, we show that polyhedra is not 2-castable if
(i) it is can be approximated by a sphere within some factor and
(ii) it is in general position.
Polyhedra is approximated by a sphere if it can be enclosed between two concentric spheres with the ratio of their radii less than a fixed number (here, 1.07). In this context, polyhedra is said to be in general position if
(i) there are at most 2 faces parallel to a line and
(ii) if three of its edges lie in a plane then they belong to the same facet.

Here is an outline of the proof. Assume that P is 2-castable and it satisfies the above conditions for approximation factor α. Let P_1 and P_2 denote the corresponding 1-castable

*Research funded in part by NSERC Canada
†University of New Brunswick. `bremner@unb.ca`
‡University of Waterloo. `agolynsk@math.uwaterloo.ca`

polyhedra. Let C denote their common facet. It is assumed to be horizontal, and P_1 is above C, P_2 is below C.
(i) If α is close enough to 1, then P_1 can only be castable through C.
(ii) If P_1 is castable through C, then the volume of P_2 an not be too small (bounded in terms of α).
(iii) Therefore, P_2 can only be castable through C, provided that α is close enough to 1 (the requirement on α here is stronger than in (i))

It remains to consider the case where both P_1 and P_2 are castable through C. We reduce problem of castability (i.e. searching for a proper translation vector) of P_1 and P_2 to linear programs LP_1 and LP_2. Since they are castable through the same facet the linear programs are somewhat related. For almost every constraint of the form

$$\mu_x d_x + \mu_y d_y - \mu_z \leq 0$$

in LP_1 there is a counterpart in LP_2

$$\mu_x d_x + \mu_y d_y + \mu_z \leq 0$$

Where μ is the outward normal to a facet of P, and d is vector of translation (its z coordinate d_z is assumed to have absolute value 1). If LP_1 has an inner feasible point p, then the point $-p$ does not satisfy most of the constraints in LP_2. Roughly speaking, if LP_2 has an inner feasible point too (say q), then it follows that LP_2 is unbounded along the ray from $(-p)$ to q. By symmetry, LP_1 is also "roughly unbounded", which makes a contradiction (more details will be provided in the full paper).

References

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