Three Observations on Geometric Permutations*

Boris Aronov† Shakhar Smorodinsky‡

October 11, 2002

1 Summary

A *body* is a closed convex set in $\mathbb{R}^d$. Let S be a family of bodies in $\mathbb{R}^d$. A line ℓ is said to be a *transversal* for S if it intersects every member of S. If S consists of pairwise disjoint convex bodies, then a line transversal for S induces two linear orderings on S – the two orders in which the members of S are met by ℓ, corresponding to the two orientations of ℓ. Katchalski *et al.* [3] were the first to study such orderings and called them *geometric permutations*. (When counting geometric permutations, we do not distinguish the two permutations corresponding to opposite orientations of a transversal.)

In this note we settle a simple version of a long-standing open problem in transversal theory, namely that of determining the largest number $g_d(n)$ of geometric permutations that a family of n pairwise disjoint bodies in $\mathbb{R}^d$ can admit. We solve a much easier variant, namely we find the exact expression for the maximum number $\hat{g}_d(n)$ of those geometric permutations to a family of n pairwise disjoint bodies that are induced by *lines passing through the origin*.

Further, we show that the same expression gives the maximum number of connected components of the space of those line transversals to a family of n possibly overlapping bodies that pass through the origin. We provide only proof outlines due to space limitations.

Finally, we disprove a conjecture of Sharir and Smorodinsky [7] on the number of neighbors in geometric permutations.

2 Geometric permutations

The following asymptotic bounds on $g_d(n)$ are known: $g_2(n) = 2n-2$ [1], and in higher dimensions $g_d(n)$ is $\Omega(n^{d-1})$ [4] and $O(n^{2d-2})$ [9]. It has been conjectured that $g_d(n)$ is $O(n^{d-1})$ (or at least that the current upper bound is far from the truth). Better bounds are known for some restricted families of convex bodies; a recent survey of results on transversals and geometric permutations can be found in [10].

In [8] it was shown that the lower bound of $\Omega(n^{d-1})$ on $g_d(n)$ can be attained by families of balls. Furthermore, the construction described there is such, that all geometric permutations are induced by lines passing through the origin. In this paper we examine this restricted version of the problem. Namely, we determine the exact expression for $\hat{g}_d(n)$. Specifically, letting $K(n,d) = \Theta(n^d)$ denote the maximum number of d-cells in an arrangement of n great hyperplanes in the real projective

*Work on this paper has been partially supported by a grant from the U.S.-Israeli Binational Science Foundation.

†Computer and Information Science Department, Polytechnic University, Brooklyn, NY 11201, USA; *aronov@poly.edu*. First author's work on this paper has also been supported by NSF Grant CCR-9972568 and NSF ITR Grant CCR-0081964.

‡School of Computer Science, Tel-Aviv University, Tel-Aviv, 69978, Israel; *smoro@tau.ac.il*. Work on this paper was done while the second author was a Ph.D. student under the supervision of Prof. Micha Sharir.

d-space $\mathbb{RP}^d$, which is also the exact number of d-cells for simple arrangements [2], we prove the following:

Theorem 1. *The maximum number $\hat{g}_d(n)$ of geometric permutations of a set of n pairwise disjoint bodies in $\mathbb{R}^d$ induced by lines passing through the origin is $K(n, d-1) = \Theta(n^{d-1})$.*

The lower bound is realized by a variant of the construction from [8], while the upper bound comes from considering the arrangement of a certain set of n great hyperspheres on the sphere $\mathbb{S}^{d-1}$ of line directions and noticing that each cell of the arrangement can produce at most one geometric permutation and that antipodal cells correspond to opposite orientations of the same permutation.

3 Path-connected components of the space of transversals through the origin

We now count the number of path-connected components of the transversal space, for transversals through the origin. We contrast Theorem 2 with the simple observation that for the space of *unconstrained* transversals, the number of path-connected components is unbounded, even for a family of *four* bodies in $\mathbb{R}^3$.

Theorem 2. *The space of all transversals through the origin to a family S of n possibly overlapping, bounded bodies has at most $K(n, d-1)$ connected components. The bound is tight in the worst case.*

The lower bound follows from Theorem 1, while the upper bound can be obtained, for example, as follows: As only transversals through the origin are considered, scaling the bodies by independent homotheties with center at the origin does not affect the transversal space. Hence we can assume the bodies are pairwise disjoint. At this point we observe that (1) (trivially, by continuity) directed transversals in the same path-connected component of the transversal space must yield the same geometric permutation, but in fact conversely (2) directed transversal yielding the same (signed) geometric permutation lies in the same path-connected component of the space of directed transversals (in fact, the component is convex).

4 Neighbors in geometric permutations

Given a family S of n pairwise-disjoint bodies, two bodies of S are *neighbors* if there is a geometric permutation in which they are consecutive; the number $p = p(S)$ of neighbor pairs is trivially at most $\binom{n}{2}$. Sharir and Smorodinsky [7], proposed a plan of attack for improving the upper bound on $g_d(n)$ by showing that the number of geometric permutations of S is bounded by $p(S)^{d-1}$ and conjecture that $p = o(n^2)$ and perhaps even linear, always (indeed they prove that in the plane p is either $\Theta(n)$ or 0). We disprove the conjecture by constructing already in $\mathbb{R}^3$ a family of pairwise-disjoint balls with at least $n^2/4$ neighbor pairs.

Theorem 3. *There exists a collection S of n pairwise disjoint unit balls in $\mathbb{R}^3$ in which the number $p(S)$ of pairs in geometric permutations is at least $\lfloor n/2 \rfloor \times \lceil n/2 \rceil$. In fact, that many pairs already appear in geometric permutations of S realized by lines through the origin.*

The construction again is based on that from [8]—two planar families of disks with many geometric permutations through the origin are placed in orthogonal planes through the origin and then each is "lifted" to three dimensions.

References

[1] H. Edelsbrunner and M. Sharir. The maximum number of ways to stab n convex non-intersecting sets in the plane is $2n-2$. *Discrete Comput. Geom.*, 5(1):35–42, 1990.

[2] B. Grünbaum, *Convex Polytopes.* John Wiley & Sons, New York, NY, 1967.

[3] M. Katchalski, T. Lewis, and A. Liu. Geometric permutations and common transversals. *Discrete Comput. Geom.*, 1:371–377, 1986.

[4] M. Katchalski, T. Lewis, and A. Liu. The different ways of stabbing disjoint convex sets. *Discrete Comput. Geom.*, 7:197–206, 1992.

[5] M. Katchalski, T. Lewis, and J. Zaks. Geometric permutations for convex sets. *Discrete Math.*, 54:271–284, 1985.

[6] S. Smorodinsky. Geometric permutations and common transversals. Master's thesis, Tel-Aviv University, School of Mathematical Sciences, Tel-Aviv, Israel, July 1998.

[7] M. Sharir. and S. Smorodinsky On neighbors in geometric permutations, In *Proc. 8th Scandinavian Workshop on Algorithms Theory*, July 3-5, 2002.

[8] S. Smorodinsky, J.S.B Mitchell, and M. Sharir. Sharp bounds on geometric permutations of pairwise disjoint balls in $\mathbb{R}^d$. *Discrete Comput. Geom.*, 23(2):247–259, 2000.

[9] R. Wenger. Upper bounds on geometric permutations for convex sets. *Discrete Comput. Geom.*, 5:27–33, 1990.

[10] R. Wenger, Helly-type theorems and geometric transversals. In *Handbook of Discrete and Computational Geometry*, J. E. Goodman and J. O'Rourke, Eds. CRC Press LLC, Boca Raton, FL, 1997, ch. 4, pp. 63–82.