Variants on Alternating Segment Paths

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Abstract
It was shown recently that there is an alternating path of length $\Theta(\log n)$ in the segment endpoint visibility graph $\mathrm{Vis}(S)$ of any set S of n disjoint line segments in the plane, and this bound best possible apart from a constant factor. This talk focuses on two related problems: (1) How long is the longest alternating path that $\mathrm{Vis}(S)$ contains for any set S of n disjoint axis-parallel line segments? (2) What is the largest subset $S' \subseteq S$ such that $\mathrm{Vis}(S')$ contains a Hamiltonian alternating path? We give partial answers to both questions.

1 Introduction

Given a set S of disjoint line segments in the plane, an *alternating paths* is a simple polygonal path $p = (v_1 v_2, \ldots, v_k)$ such that $v_{2i-1}v_{2i} \in S$, $i = 1, \ldots, \lfloor k/2 \rfloor$ and $v_{2i}v_{2i+1}$ does not cross any segment of S for $i = 1, \ldots, \lfloor (k-1)/2 \rfloor$.

It is known that there are sets of disjoint segments which do not admit an alternating Hamiltonian path. Answering a question of Bose [1, 6], it was shown recently [3], that for any set S of n disjoint line segments in the plane, there is an alternating path running through at least $\lceil \log_2(n+2) \rceil - 1$ segments of S, and this bound is best possible apart from a constant factor.

The $O(\log n)$ upper bound construction [6, 3] is a set S of line segments arranged so that every segment $s \in S$ has two endpoints on the convex hull $\mathrm{conv}(\bigcup S)$, and therefore any alternating path containing segments from both sides of s should go through s as well. In this construction n segments have $\Omega(n)$ different orientations. If all segments are axis-parallel, there is a polynomial size alternating path:

Theorem 1 *For any set S of n disjoint axis-parallel segments in the plane, there is an alternating path running through $\sqrt[3]{n/2}$ segments of S.*

The best upper bound, known to me, is $O(n^{\log_5 4}) = O(n^{.861})$, which follows from a simple recursive construction.

The condition that no second (visibility) segment $v_{2i}v_{2i+1}, i = 1, \ldots, \lfloor (k-1)/2 \rfloor$, crosses any segment of S can be relaxed: The visibility segments still should not cross any segment of S in the path p, since the alternating path p must be a simple polygonal path. Deleting all segments which are not in a path, we arrive to a question: What is the largest subset of S that admits a Hamiltonian alternating path?

Theorem 2 *For any set S of n disjoint segments in the plane, there is a subset $S' \subseteq S$ of at least $\sqrt[3]{n}$ segments admitting a Hamiltonian alternating path.*

The best known upper bound, $2\sqrt{n}$, follows from a well known construction: n segments arranged in groups of $\sqrt{n}$ along the sides of a convex $\sqrt{n}$-gon.

2 Axis-parallel segments

We may assume that there are at least $n/2$ horizontal segments in S, let H, $H \subseteq S$ denote their set. For two segments $s \in H$ and $t \in H$, we say that s is above t if the y-coordinate of s is bigger than that of t and there is a vertical line segment connecting s and t which does not cross any segment of H [5]. We say that $s \succ t$ (or $t \prec s$) iff there is a sequence $(s = s_0, s_1, s_2, \ldots, s_r = t)$ in H such that s_i is above s_{i+1}, $i = 0, 1, 2, \ldots r-1$ (Fig. 1). The relation $\succ$ is a partial order in H.

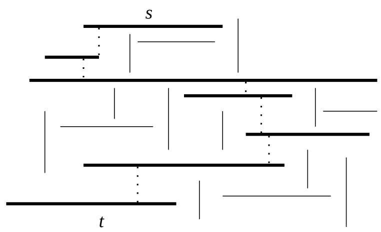

Figure 1: $s \succ t$.

By Dilworth theorem [2], there is either (i) a chain of size $(n/2)^{1/3}$, or (ii) an anti-chain of size $(n/2)^{2/3}$ with respect to $\succ$. In both cases, we show that all segments in the chain or at least $(n/2)^{1/3}$ segments from the anti-chain can be linked together in a common alternating path.

In case (i), let $s_1, s_2, \ldots, s_r$ be a sequence of $r \geq \sqrt[3]{n/2}$ segments of H such that s_i is above s_{i+1}. Denote the left and right endpoint of s_i by a_i and b_i. Let $v_i w_i$, be the vertical segment connecting s_i and s_{i+1} with $v_i \in s_i$ and $w_i \in s_{i+1}$ (Fig. 2). For every i, $i = 1, 2, \ldots, t-1$, we connect a_i and b_{i+1} by a path $\pi(i)$: Place a rubber band pinned down at a_i, v_i, w_i, b_{i+1} and release the pin

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points at v_i and w_i such that the band cannot cross any segment of S (Fig. 3). The rubber band forms a polygonal path $\pi(i)$ through segment endpoints lying between s_i and s_{i+1}. Notice that the portion of $\pi(i)$ left (resp., right) to $v_i w_i$ is concave (resp., convex) function in both x- and y-coordinates.

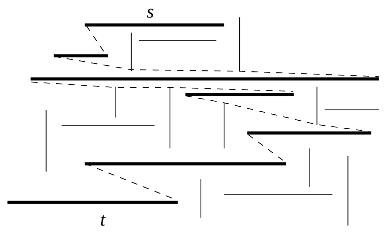

Figure 2: The initial paths $\pi(i)$.

We expend recursively every $\pi(i)$ into an alternating path starting with s_i such that the r resulting alternating paths are pairwise disjoint (see Fig. 3 and 4).

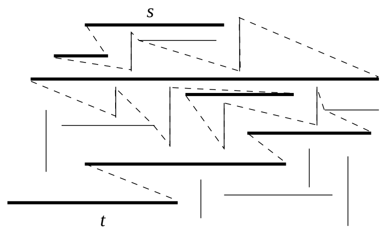

Figure 3: Paths $\pi(i)$ after one iteration.

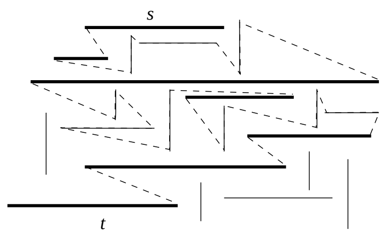

Figure 4: The resulting alternating path.

In case (ii), note that any two segments in an anti-chain are separated by a vertical line, therefore there is a linear order in our anti-chain of size $(n/2)^{2/3}$.

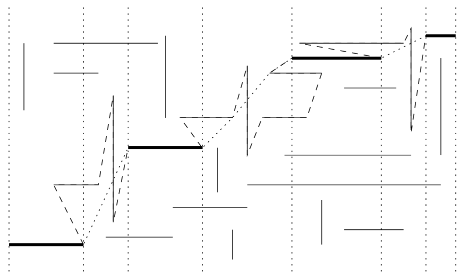

Figure 5: Linear oder in an anti-chain.

Let $A = \{s_1, s_2, \dots, s_r\} \subset H$ be an anti-chain of $r \geq \sqrt[3]{n/2}$ segments such that both the x- and the y-coordinates of points of s_i are monotone in i (Fig. 5). Denote the left and right endpoint of s_i by a_i and b_i. For every i, $i = 1, 2, \dots t-1$, we connect b_i and a_{i+1} by a rubber band $\varrho(i)$ that can cross vertical segments but must avoid segments of H. Observe that $\varrho(i)$ can be places such that it forms monotone curve, because $s_i \not\prec s_{i+1}$. Recursively, at every intersection point of $\varrho(i)$ with a vertical segment vw, we force $\varrho(i)$ to pass trough v and w. In this way, the every section of ϱi between segments is still monotone, and $\varrho(i)$ forms a polygonal path from b_i to a_{i+1} not crossing any segment of S.

Finally, we expend recursively every $\varrho(i)$ into an alternating path starting with s_i such that the r resulting alternating paths are pairwise disjoint (Fig. 5).

3 Subset with Hamiltonian alternating path

Pach and Rivera-Campo [4] have shown that every set of n disjoint line segments in the plane contains at least $\sqrt[3]{n}$ elements $s_1, s_2, s_3, \dots, s_r$ such that either (i) for every i, $i = 1, 2, \dots, r-1$, a vertical line separates s_i and s_{i+1}; or (ii) a common vertical line crosses $s_1, s_2, s_3, \dots, s_r$ in this order and their slopes are monotone increasing or decreasing. We show that in either case, that there is a Hamiltonian alternating path through $s_1, s_2, s_3, \dots, s_r$.

In case (i), $s_1, s_2, \dots, s_r$ can clearly be linked by a common alternating path from left to right.

In case (ii), induction on i can show that an alternating path can link $s_1, s_2, \dots, s_i$ in this order for every $i = 1, 2, \dots, r$. More precisely: For every i, $i = 1, 2, \dots r$, there are two alternating paths p_i^1 and p_i^2 through $s_1, s_2, \dots, s_{i-1}$ such that the last vertex of p_i^1 and p_i^2, respectively, are the two endpoints of the segment s_i.

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