The Min-Max Voronoi Diagram of Polygons and Applications in VLSI Manufacturing*

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Abstract. We study the *min-max Voronoi diagram* of a set S of polygonal objects, a generalization of Voronoi diagrams based on the *maximum* distance between a point and a polygon and show that it is equivalent to the Voronoi diagram of S under the Hausdorff distance. We investigate the combinatorial properties of this diagram and give improved combinatorial bounds and algorithms. As a byproduct we introduce the *min-max hull* which relates to the min-max Voronoi diagram in the way a convex hull relates to the ordinary Voronoi diagram.

Problem statement. Given a set S of polygonal objects in the plane their *min-max* Voronoi diagram is a subdivision of the plane into regions such that the Voronoi region of a polygon $P \in S$ is the locus of points t whose maximum distance from (any point in) P is less than the maximum distance from any other object in S. The min-max Voronoi region of P is subdivided into finer regions by the farthest point Voronoi diagram of the vertex set of P. This structure generalizes both the ordinary Voronoi diagram of points and the farthest-point Voronoi diagram. The ordinary Voronoi diagram is derived if shapes degenerate to points and the farthest-point Voronoi diagram appears in the case where the set of points is the set of vertices of a single polygon ($|S| = 1$). The *min-max* Voronoi diagram can be defined equivalently on a collection S of sets of points instead of polygonal objects. It is equivalent to the Voronoi diagram of sets of points or polygonal objects under the Hausdorff metric.

Related work. The min-max Voronoi diagram problem was formulated in [8] where the *critical area* computation problem for *via-blocks* in VLSI designs was addressed via the L_∞ min-max Voronoi diagram of disjoint rectilinear shapes. This diagram had also been considered in [4] where it was termed the *Voronoi diagram of point clusters*, and in [1] where it was termed the *closest covered set diagram* for the case of disjoint convex shapes and arbitrary convex distance functions. In [4] general combinatorial bounds regarding this diagram were derived by means of envelopes in three dimensions. It was shown that the size of this diagram is $O(n^2\alpha(n))$ in general and $O(n)$ for clusters of points with disjoint convex hulls, where n is the number of points on the convex hulls of each cluster in S. The latter was also shown in [1] for disjoint convex shapes and arbitrary convex distances. Using a divide and conquer algorithm for computing envelopes in three dimensions, [4] concluded that the cluster Voronoi diagram can be constructed in $O(n^2\alpha(n))$ time. In [1] the problem for disjoint convex sets was reduced to abstract Voronoi diagrams and the randomized incremental construction of [5] was proposed for its computation. This approach results in an expected $O(kn \log n)$-time algorithm, where k is the time to construct the bisector of two convex polygons.

Summary of Results. In this paper we derive combinatorial properties of the min-max Voronoi diagram and provide tighter bounds and algorithms that improve in an output sensitive fashion the time complexity bounds given in [4] and [1]. Specifically, we show that for a set of arbitrary polygons or clusters of points the size of the min-max Voronoi diagram is $O(m)$, where m is the size of the *intersection graph* of S which reflects the number of intersections among the shapes of S in addition to the number of convex hull edges of each shape in S. In the worst case m is $O(n^2)$. Furthermore, we show that in case of *non-crossing*[1] polygons (not necessarily disjoint) the min-max Voronoi regions are connected and the size of the min-max Voronoi diagram is linear in the number of vertices on the convex hulls of the polygons in S. Thus, the connectivity and linearity of the diagram is maintained for a more general class of polygons than the ones shown in [4, 1]. We present a divide and conquer algorithm of time com-

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[1] Two polygons are called *non-crossing* if their convex hulls admit at most two supporting segments. Otherwise they are called *crossing*.

plexity $O((n + M + N + K)\log m)$ where n is the number of points on the convex hulls of shapes in S, M is the number of pairs of vertices of *crossing* shapes, K is the total number of vertices of *interacting*[2] pairs of shapes and N is $O(n\log n)$. The convex hulls of shapes in S are assumed precomputed in $O(|S|\log n)$-time. As a byproduct we introduce the *min-max hull* of a set of polygons which relates to the min-max Voronoi diagram in the way a convex hull relates to the ordinary Voronoi diagram. The min-max hull is computed in $O(n\log n)$ time making use of the notion of the Gaussian map [3]. The *merge curve* of the standard divide-and-conquer technique is shown to consist of unbounded chains and cycles. The unbounded portions are identified by the use of the min-max hull. The cycles are identified by heavily exploiting the special properties of the min-max Voronoi diagram and they dominate the time complexity of the merging process. The algorithm simplifies to $O((n+N+K)\log n)$ for *non-crossing* polygons which is the case of interest to our VLSI manufacturing application. In this case, N is the total number of vertices of shapes intersecting some dividing line in the divide-and-conquer scheme. In our application, VLSI contact shapes tend to be small in size and well spaced in which case N, K are both negligible compared to n. In the case of non-crossing polygons the conditions of abstract Voronoi diagrams are shown to be satisfied and thus the $O(kn\log n)$ randomized incremental construction of [5] could also be applied, where k is the time to construct the bisector of two convex polygons; k however can be high in general. A plane sweep approach of time complexity $O((n + K')\log n)$ for a non-crossing S, where K' is similar but slightly higher than K, is presented in a companion paper.

Motivation. Our motivation comes from an application in VLSI manufacturing, in particular *yield* prediction. The *critical area* of a VLSI layout is a measure reflecting the sensitivity of the design to spot defects due to dust and other contaminants during the manufacturing process. Its extraction poses the main computational bottleneck in the prediction of yield of a VLSI chip (see e.g.[6]) which reflects the percentage of functional chips over all chips manufactured. In [9, 8] the critical area computation problem for *shorts*, *opens*, and *via-blocks* was reduced to variations of L_∞ Voronoi diagrams of segments. In particular, the L_∞ min-max Voronoi diagram of disjoint rectilinear shapes was introduced as a solution to the critical area computation problem for via-blocks [8]. The L_∞ metric considerably simplified the construction of Voronoi diagrams [10] and reflected a natural model of square defects. However, the square defect model has been criticized for the case of via-blocks (see e.g. [7]) and thus the Euclidean version of the problem needs to be investigated. The construction of the Euclidean min-max Voronoi diagram remains realistic as robustness issues are similar to the construction of Voronoi diagrams of points and not to those of segments.

[2] A shape $Q \in S$ is *interacting* with $P \in S$ if Q is entirely enclosed in the minimum enclosing circle of P.

References

[1] M. Abellanas, G. Hernandez, R. Klein,V. Neumann-Lara, and J. Urrutia, *Discrete Computat. Geometry* 17, 1997, 307-318.

[2] "The Farthest Color Voronoi Diagram and Related Problems", M. Abellanas, F. Hurtado, C. Icking, R. Klein, E. Langetepe, L. Ma, B. Palop, V. Sacristán, *Abstracts 17th European Workshop Comput. Geom. CG 2001*, Freie Universität Berlin, Berlin 2001, 113–116.

[3] L.L. Chen, S.Y. Chou, and T.C. Woo, 1993, "Optimal parting directions for mold and die design," *Computer-Aided Design*, Vol. 26, No. 12, pp.762-768.

[4] H. Edelsbrunner, L.J. Guibas, and M. Sharir, "The upper envelope of piecewise linear functions: algorithms and applications", *Discrete Computat. Geometry* 4, 1989, 311-336.

[5] R. Klein,, K. Melhorn, S. Meiser, "Randomized Incremental Construction of Abstract Voronoi diagrams", *Computational geometry: Theory and Aplications* 3,1993, 157-184

[6] W. Maly, "Computer Aided Design for VLSI Circuit Manufacturability," *Proc. IEEE*, vol.78, no.2, 356-392, Feb. 90.

[7] C.H. Ouyang, W.A. Pleskacz, W. Maly, "Extraction of Critical Areas for Opens in Large VLSI Circuits", *IEEE Trans. on Computer-Aided Design*, vol. 18, no 2, 151-162, February 1999.

[8] E. Papadopoulou, "Critical Area Computation for Missing Material Defects in VLSI Circuits", *IEEE Transactions on Computer-Aided Design*, vol. 20, no.5, May 2001, 583-597.

[9] E. Papadopoulou and D.T. Lee, "Critical Area Computation via Voronoi Diagrams", *IEEE Trans. on Computer-Aided Design*, vol. 18, no.4, April 1999,463-474.

[10] E. Papadopoulou and D.T. Lee, "The L_∞ Voronoi Diagram of Segments and VLSI Applications", *International Journal of Computational Geometry and Applications*, Vol. 11, No. 5, 2001, 503-528.