Matching Planar Maps

Helmut Alt* Alon Efrat† Günter Rote* Carola Wenk†

1 Introduction

Patterns consisting of line segments occur in many applications of a geometric nature, like computer vision, geographic information systems, CAGD, etc. In many cases the problem occurs to determine whether some given pattern H is equal to or similar to some part of a larger pattern G. Here, we consider the cases that H is a polygonal curve and G is a geometric graph, or that both H and G are geometric graphs. We give both feasible distance measures as well as efficient algorithms for computing these distances. In this extended abstract we only sketch the results; more details can be found in [1]. These distance measures are generalizations of the well known Fréchet distance for curves, which has been investigated before in [2].

Definition 1.1. (Fréchet distance) *Let $f : I = [l_I, r_I] \to \mathbb{R}^2$, $g : J = [l_J, r_J] \to \mathbb{R}^2$ be two planar curves, and let $||.||$ denote the Euclidean norm. Then the* Fréchet distance *$\delta_F(f,g)$ is defined as*

$$\delta_F(f,g) := \inf_{\substack{\alpha:[0,1]\to I\\ \beta:[0,1]\to J}} \max_{t\in[0,1]} ||(f(\alpha(t)), g(\beta(t))||.$$

where α and β range over continuous and non-decreasing reparametrizations with $\alpha(0) = l_I$, $\alpha(1) = r_I$, $\beta(0) = l_J$, $\beta(1) = r_J$ only.

2 Matching a Curve in a Graph

As a first task we consider a given polygonal curve $\alpha : [0,p] \to \mathbb{R}^2$, and an undirected connected planar graph $G = (V,E)$ with a given straight-line embedding in $\mathbb{R}^2$, $|V| = q$. We wish to find a path π in G, which we identify with the corresponding polygonal curve, such that the Fréchet distance $\delta_F(\alpha,\pi)$ is minimized. [1] An application is to locate the path a person travelled, while recording a sequence of *GPS positions*, in a given roadmap. This is a prerequisite for incrementally constructing roadmaps from such GPS curves, which is especially interesting for roads such as hiking trails in a forest which are not visible on aerial pictures. We attack the minimization problem by first solving the decision problem for which we fix $\varepsilon > 0$ and wish to find a path (if it exists) in G such that the Fréchet distance is at most ε. Afterwards we apply parametric search, similar as in [2], to eventually solve the minimization problem. Our algorithm solves the decision problem in $O(pq\log q)$ time and $O(pq)$ space. Parametric search adds another factor of $\log(pq)$ to the runtime. Hence, our algorithm is only a logarithmic factor slower than the algorithm for computing the Fréchet distance between two curves, although the task seems to be much more challenging at first sight.

Definition 2.1. ([2]) *Let $f : I \to \mathbb{R}^2$, $g : J =\to \mathbb{R}^2$ be two curves; $I, J \subseteq \mathbb{R}$. The set $F_\varepsilon(f,g) := \{(s,t) \in I \times J \mid ||f(s) - g(t)|| \le \varepsilon\}$ denotes the* free space *of f and g. We call the partition of $I \times J$ into regions belonging or not belonging to $F_\varepsilon(f,g)$ the* free space diagram *$FD_\varepsilon(f,g)$.*

We call points in F_ε *white* or *feasible* and points in $FD_\varepsilon \backslash F_\varepsilon$ *black* or *infeasible*. In [2] it has been shown that $\delta_F(f,g) \le \varepsilon$ if and only if there exists a curve within $F_\varepsilon(f,g)$ from its lower left to its upper right endpoint, which is monotone in both coordinates. We call such a curve *feasible*.

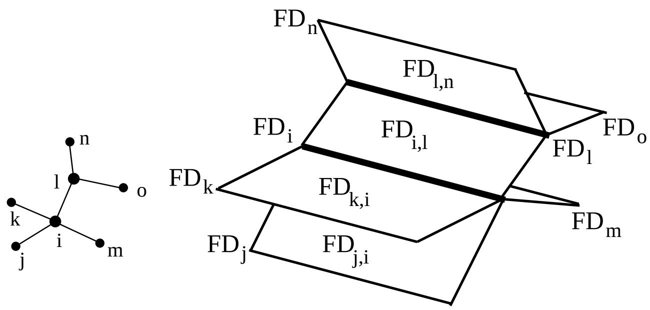
Let $s_{i,j} : [0,1] \to \mathbb{R}^2$ for all $(i,j) \in E$ be the line-segment embedding of (i,j). For every edge $(i,j) \in E$ consider the free space diagram $FD_{i,j}$ between α and $s_{i,j}$, which is a subdivision of $[0,p]\times[0,1]$. $FD_{i,j}$ consists of a row of p cells, each of which corresponds to a line segment of α. For a vertex $j \in V$ let FD_j be the one-dimensional free space diagram between α and v_j. Furthermore, let $\mathbf{L}_j$ be the left endpoint and $\mathbf{R}_j$ be the right endpoint of FD_j.

For each $i \in V$ the free space diagrams $FD_{i,j}$ and $FD_{j,i}$ for all $j \in \mathrm{Adj}(i)$ have the one-dimensional free space diagram FD_i in common - as the bottom of $FD_{i,j}$ and as the top of $FD_{j,i}$. Thus we can imagine to glue together the two-dimensional free space diagrams at the one-dimensional free space they have in common, according to the adjacency information of G. Like this we obtain a rather complex topological structure which we call the *free space surface* for G and α.

*Freie Universität Berlin, Insitut für Informatik, Takustr 9, 14195 Berlin, Germany. {alt, rote}@inf.fu-berlin.de

†University of Arizona, Computer Science Dept, 1040 E 4th Street, Tucson, AZ 85721. {alon, carolaw}@cs.arizona.edu

[1] Note that this definition allows a path π in G to travel the same edges in G multiple times.



We apply an approach related to [2] to our more general setting: We search for a feasible path in the free space surface. This path has to start at some white left corner $\mathbf{L}_k$ and has to end at some white right corner $\mathbf{R}_j$, for two vertices $j, k \in V$. Any path π in G selects a sequence of free space diagrams in the free space surface, whose concatenation yields $FD_\varepsilon(\alpha, \pi)$.

For every vertex $j \in V$ let $\mathcal{R}(j)$ be the set of all points $u \in F_j$ for which there exists a $k \in V$ and a path π from k to j in G such that there is a monotone feasible path from $\mathbf{L}_k$ to u in $F_\varepsilon(\alpha, \pi)$. We call points in $\mathcal{R}(j)$ *reachable*. We thus know that there is a path π in G with $\delta_F(\alpha, \pi) \leq \varepsilon$ iff there is a vertex $j \in V$ such that $\mathbf{R}_j \in \mathcal{R}(j)$.

In a preprocessing stage we first compute all one-dimensional free space diagrams FD_i for all $i \in V$. Further, for all $(i, j) \in E$ we compute for each white interval I of FD_i the leftmost point and the rightmost point on FD_j which can be reached from some point in I by a monotone feasible path in $FD_{i,j}$. [2] In the dynamic programming stage we decide whether there exists a feasible monotone path in the free space surface by computing parts of $\mathcal{R}(j)$, for all $j \in V$. Conceptually we sweep all $FD_{i,j}$ at once with a vertical sweep line from left to right. Let $0 \leq x \leq p$ denote the position of the sweep line. For each $i \in V$ we store a set $C_i \subseteq \mathcal{R}(i) \subseteq F_i$ of white points, which we compute in a dynamic programming manner. Throughout the algorithm we maintain the invariant that C_i consists of all reachable points $u \in \mathcal{R}(i) \subseteq FD_i$, such that $u \geq x$, and for which the last segment of their associated feasible monotone path crosses or ends at the sweep line. We maintain a priority queue Q of the first white intervals of each C_i – these are white intervals of FD_i which are known to be reachable. The priority of an interval is its left endpoint. We process all intervals in Q and update all C_j, until we either find a $j \in V$ such that $\mathbf{R}_j \in C_j$, or until Q is empty. In the latter case there is no path π in G with $\delta_F(\alpha, \pi) \leq \varepsilon$. In the first case we know that there exists such a path, and we reconstruct it using a backtracking variant.

[2] Similar reachability pointers have been used in [2] for attacking the case of closed curves.

Hence, our algorithm is a mixture of a sweep (since we are sweeping with a sweep line), dynamic programming (on the C_i we incrementally build up), and Dijkstra's algorithm for shortest paths (since we are computing paths using a priority queue to augment the path in a similar fashion to Dijkstra). In the more detailed version of this paper [1] we consider some variants of the original problem, one of which is a time-space tradeoff which also applies to the computation of the Fréchet distance for two polygonal curves.

3 Graph-to-Graph Distance

In this section we generalize the Fréchet distance to two embedded, connected graphs $H = (V_H, E_H)$ and $G = (V_G, E_G)$ with straight edges. Identifying H and G with the points lying on their edges we call a mapping $f : [0, 1] \to H$ which is continuous and surjective, a *traversal* of H. If f is not surjective we call it a *partial traversal*. The *traversal distance* from H to G is defined as

$$\delta_T(H, G) = \inf_{f,g} \max_{t \in [0,1]} \|f(t) - g(t)\|$$

where f ranges over all traversals of H and g over all partial traversals of G. If H and G are polygonal chains this definition corresponds to the weak Fréchet-distance, see [2]. Observe that the traversal distance is not a generalization of the Fréchet distance between a curve and a graph as defined in Section 2.

We show that for given $\varepsilon > 0$ we can decide whether $\delta_T(H, G) \leq \varepsilon$ in time $O(pq \log pq)$, where $p = |E_G|$ and $q = |E_H|$. Using parametric search the traversal distance can be computed in time $O(pq \log^2 pq)$.

For given $\varepsilon \geq 0$ we consider the free space surface of H and G. For edges $e \in E_H$ and $f \in E_G$, if a feasible path π traverses some cell $C_{e,f}$ of the surface, then let $I_{\pi,e,f}$ be the set of those points on e that are traversed by the corresponding motion on the graphs. e is called *satisfied* by π if the union $\bigcup_{f \in E_G} I_{\pi,e,f} = e$. It means that all points of e are eventually traversed by the motion on the graphs corresponding to π. Therefore we have to search for a feasible path π that satisfies all edges $f \in E_H$. We can construct such a path π by traversing the free space surface from edge to edge, and making sure at each edge to visit the leftmost and the rightmost point of the freespace; see [1] for details.

References

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[2] H. Alt and M. Godau. Computing the Fréchet distance between two polygonal curves. *Internat. J. Comput. Geom. Appl.*, 5:75–91, 1995.