MAP, Typical Set Decoding & Magnetization Enumerator in LDPC codes

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Outline

- We first introduce LDPC error correcting codes.

- Analysis of these codes is carried out using methods of statistical physics, exploring their theoretical and practical limits.

- Studying the entropy of the solutions we provide a link between MAP, MPM and typical set decoding results.

- Finally, we will briefly mention related work in this area.

The General Framework

- The message is a, possibly sparse, N dimensional Boolean (binary) vector $\boldsymbol{\xi}$.

- Encoded to an M dimensional Boolean (binary) vector and transmitted through a noisy channel. Here, we focus on binary symmetric channel (BSC).

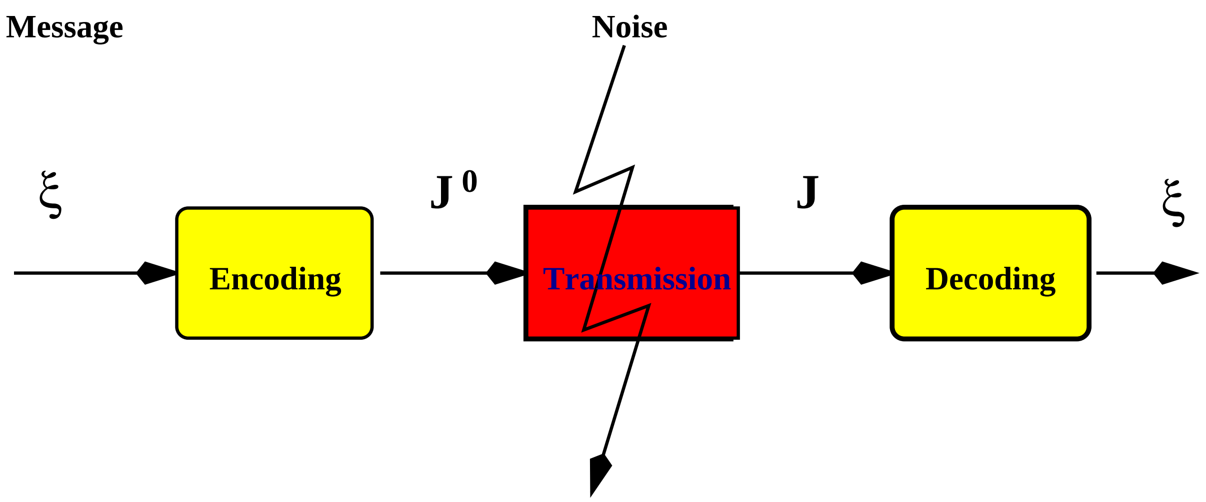
- The received massage $\boldsymbol{J} = \boldsymbol{J}^0 + \zeta$ (mod 2) is decoded to retrieve $\boldsymbol{\xi}$.

- The code rate is $R = N/M$.

- Error-free communication is theoretically possible below Shannon's bound

$$R_c = 1 + p \log_2 p + (1-p) \log_2 (1-p),$$

which has never been achieved.



Gallager/MN Codes

- Discovered by Gallager (1962) and MacKay-Neal (1996).

- Gallager's code - Construct two sparse matrices A and B of dimensionality $(M-N)\times N$ and $(M-N)\times(M-N)$ respectively.

- MN code -Construct two sparse matrices A and B of dimensionality $M\times N$ and $M\times M$ respectively.

- The matrix A has K non-zero (unit) elements per row and C per column, B has L per row/column.

$$A = \begin{array}{c} \\ M \left\{ \vphantom{\begin{pmatrix} 1\\0\\0\\0\\ \vdots \end{pmatrix}} \right. \\ \end{array} \begin{array}{c} N \\ \overbrace{\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & & \ddots \end{pmatrix}} \\ K \end{array} \begin{array}{c} \\ \downarrow C \\ \\ \end{array}, \quad B = \begin{array}{c} \\ M \left\{ \vphantom{\begin{pmatrix} 1\\0\\0\\0\\ \vdots \end{pmatrix}} \right. \\ \end{array} \begin{array}{c} M \\ \overbrace{\begin{pmatrix} 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & & & \ddots \end{pmatrix}} \\ L \end{array} \begin{array}{c} \\ \downarrow L \\ \\ \end{array}.$$

In Gallager's code we may refer to $\mathcal{A} = [\boldsymbol{A} \mid \boldsymbol{B}]$ and $K+L$ only.

Gallager's Code

Encoding

$$\underbrace{\boldsymbol{J}^{\boldsymbol{0}}}_{\in\{0,1\}^M} = \boldsymbol{G}^{\boldsymbol{T}} \underbrace{\boldsymbol{\xi}}_{\in\{0,1\}^N} \pmod 2.$$

The (dense) generator matrix is

$$\boldsymbol{G} = [\boldsymbol{I} \mid \boldsymbol{B}^{-1}\boldsymbol{A}] \pmod 2.$$

The received vector takes the form

$$\boldsymbol{J} = \boldsymbol{G}^{\boldsymbol{T}}\boldsymbol{\xi} + \boldsymbol{\zeta} \pmod 2.$$

Decoding

$$\boldsymbol{z} = \mathcal{A}\boldsymbol{J} = \underbrace{\mathcal{A}\boldsymbol{G}^{\boldsymbol{T}}\boldsymbol{\xi}}_{=0} + \mathcal{A}\boldsymbol{\zeta} \pmod 2,$$

$\mathcal{A} = [\boldsymbol{A} \mid \boldsymbol{B}]$ sparse $M \times M$ binary matrix.

The problem: $\boldsymbol{z} = \mathcal{A}\boldsymbol{\tau} \pmod 2$.

- MPM – $\widehat{\zeta}_j = \text{argmax}\ P(\tau_j \mid \boldsymbol{z})$
minimises $p_b = 1 - \frac{1}{M} \sum_{j=1}^{M} \delta[\widehat{\zeta}_j; \zeta_j]$.

- MAP – $\widehat{\boldsymbol{\zeta}} = \text{argmax}\ P(\boldsymbol{\tau} \mid \boldsymbol{z})$ minimises p_B.

Statistical Physics

- Mapping from Boolean to binary variables

$$\{0, 1, +\} \rightarrow \{1, -1, \times\} \, .$$

- We look for the conditional probability

Gallager

$$P(\boldsymbol{\tau} \mid \boldsymbol{z}) = \frac{1}{\mathcal{Z}} \, \delta\left(\boldsymbol{z}; \mathcal{A}\boldsymbol{\tau}\right) \underbrace{P(\boldsymbol{\tau})}_{prior}$$

- Can be represented as

$$P(\tau \mid z) = \frac{1}{\mathcal{Z}} \exp\left[-\beta\mathcal{H}(\tau; z)\right]$$

and

Statistical Physics - Gallager

- The ground state of the Hamiltonian

$$\mathcal{H} = \lim_{\gamma\to\infty} \gamma \sum_{\sigma} \mathcal{D}_\sigma \, \delta\left[-1 \; ; \; \mathcal{J}_\sigma \prod_{j\in\sigma} \tau_j\right] - F_\tau \sum_{j=1}^{M} \tau_j ,$$

corresponds to the Bayes-optimal estimate of the noise (binary representation).

- The tensor product $\mathcal{D}_\sigma \mathcal{J}_\sigma$, where $\mathcal{J}_\sigma = \prod_{j\in\sigma} \zeta_j$ and $\mathcal{D}_\sigma = 0, 1$ is the binary equivalent of $[A|B]\ \zeta$.

- $F_\tau = (1/2) \ln[(1-p)/p]$ is the prior knowledges about the noise .

The Free Energy

• One can then calculate the free energy

$$\frac{1}{M}\langle \ln \mathcal{Z}\rangle_{\{\mathcal{D},\zeta\}}\,;\,\mathcal{Z}(\mathcal{D},\zeta) = \mathrm{Tr}_{\{\boldsymbol{\tau}\}}\exp\left[-\beta\mathcal{H}\right]$$

using methods from statistical mechanics of diluted systems (replica method).

• The choice of $1/T = \beta = 1$ is finite temperature decoding (Nishimori's condition) - MPM.

• Averages give rise to an infinite set of order parameters. Using the replica symmetric ansatz we obtain saddle-point equations.

• Solutions to these equations (numerical or analytical) indicate the phase of the system:

– Ferro solution ($m = 1$): Perfect decoding
– Para solution ($m = 0$): Catastrophic failure
– Sub-optimal ferro solution ($m < 1$): failure

where $m = \frac{1}{M}\sum_{i=1}^{M}\zeta_i\tau_i$

Alternative Method - Gallager

- The Hamiltonian can also be represented as

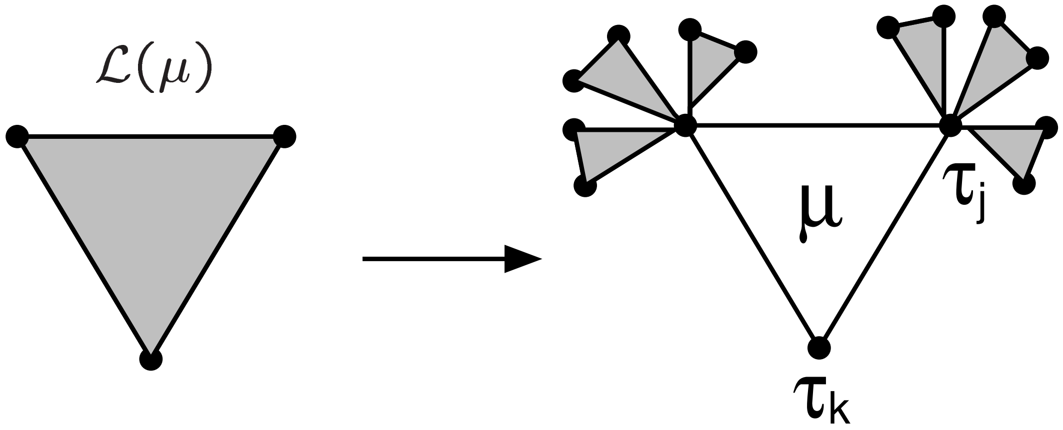
$$\mathcal{H} = -\lim_{\gamma\to\infty} \gamma \sum_{\mu=1}^{M-N} \left(z_\mu \prod_{j\in\mathcal{L}(\mu)} \tau_j - 1 \right) - \underbrace{F}_{prior} \sum_{j=1}^{M} \tau_j$$

- To estimate $\widehat{\zeta}_j$ we examine the local magnetization

$$\widehat{\zeta}_j = \mathrm{sign}(m_j) \qquad \underbrace{m_j}_{\text{local mag}} = \langle \tau_j \rangle$$

$$\overline{p_b} = \frac{1}{2} - \frac{1}{2} \underbrace{\left[\frac{1}{M} \sum_{j=1}^{M} \widehat{\zeta}_j \zeta_j \right]_{A,\zeta}}_{=m}$$

Husimi Cactus



- One constructs the recursion relations

$$P_{\mu k}(\tau_k) = \frac{1}{\mathcal{N}} P(\tau_k) \sum_{\{\tau_j\}} P(z_\mu \mid \{\tau_j\}, \tau_k) \prod_{\nu \in \mathcal{M}(j) \setminus \mu} \prod_{j \in \mathcal{L}(\mu) \setminus k} P_{\nu j}(\tau_j)$$

- A set of recursive equations is then derived

$$P_n(x) = \int \prod_{l=1}^{C-1} d\widehat{x}_l \, \widehat{P}_{n-1}(\widehat{x}_l) \left\langle \delta \left[x - \beta F \zeta - \sum_{l=1}^{C-1} \widehat{x}_l \right] \right\rangle_\zeta$$

$$\widehat{P}_{n-1}(\widehat{x}) = \int \prod_{j=1}^{K-1} dx_j \, P_{n-1}(x_j) \, \delta \left[\widehat{x} - \text{atanh} \left(\prod_{j=1}^{K-1} \tanh(x_j) \right) \right]$$

$P_n(x)$, $\widehat{P}_n(\widehat{x})$ are the n-th generation distribution of effective fields.

- The free energy is also calculated.

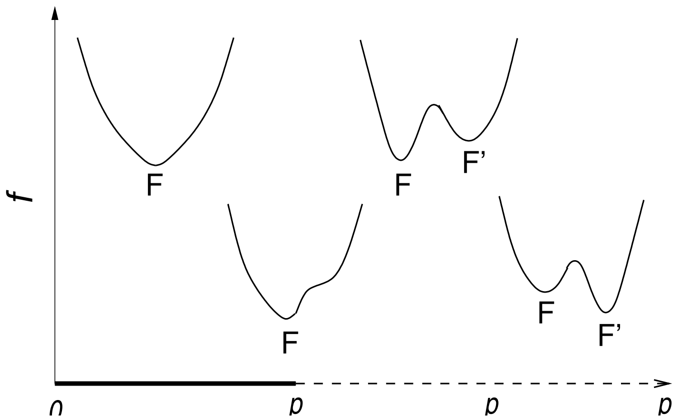
Solutions - Gallager

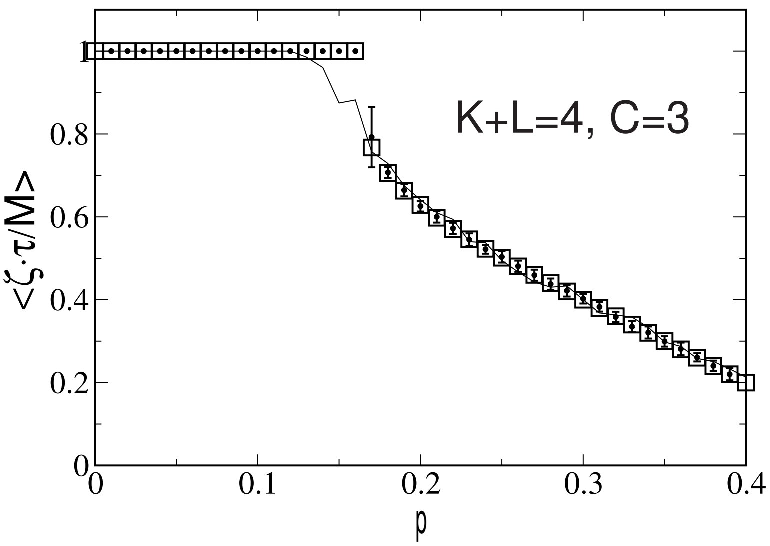
• Solutions to these equations (numerical or analytical) indicate the phase of the system:

– Ferro solution ($m = 1$): Perfect decoding

– Sub-optimal ferro solution ($m < 1$): failure

where $m = \frac{1}{M} \sum_{j=1}^{M} \widehat{\zeta}_j \zeta_j$

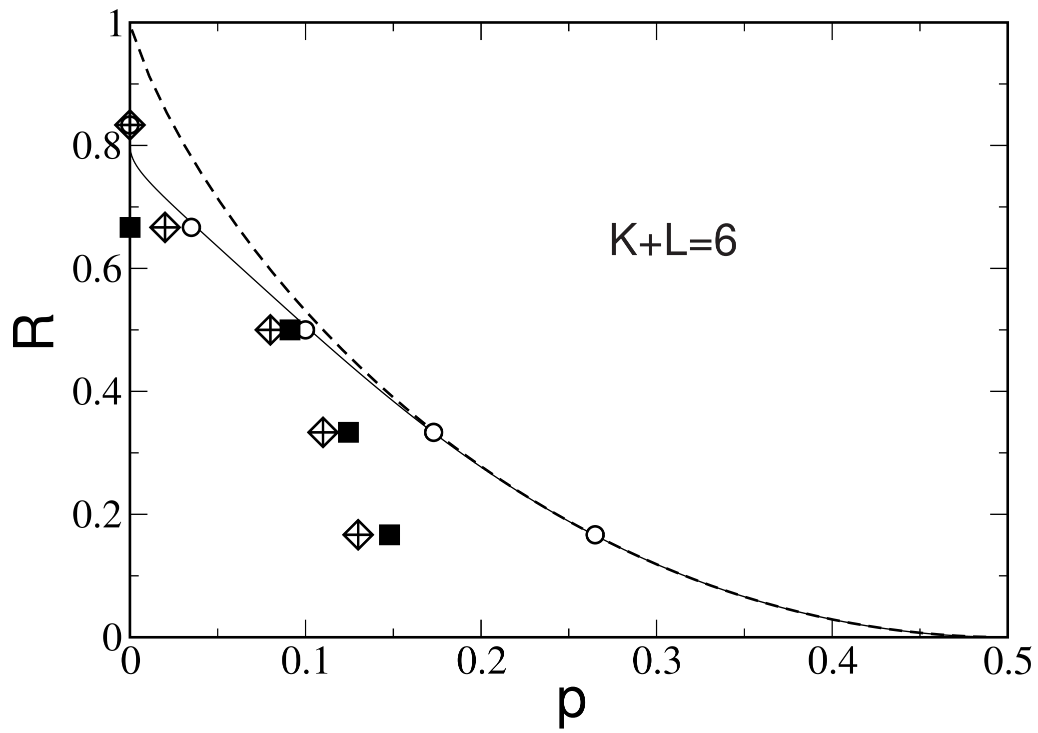




Theoretical and Practical Limits

Independently of the decoding method we identify:

- Theoretical limits - thermodynamical transition
- Practical limits - spinodal point



Decoding

• To examine different decoding methods within this framework we calculate the entropy.

• The magnetization enumerator $\mathcal{M}(m)$ can be calculated from the Hamiltonian

$$\mathcal{H} = - \lim_{\gamma\to\infty} \gamma \sum_{\sigma} \mathcal{D}_\sigma \, \delta\left[-1 \; ; \; \mathcal{J}_\sigma \prod_{j\in\sigma} \tau_j\right] ,$$

by forcing the constraint $\delta\left(\frac{1}{M}\sum_{j=1}^{M} \tau_j - m\right)$.

• To calculate the weight enumerator $\mathcal{W}(\omega)$ we use $\delta\left(\frac{1}{M}\sum_{j=1}^{M} \zeta_j \tau_j - \omega\right)$.

• The entropies can be calculated using the
replica method - $\frac{1}{M}\left\langle \ln \mathcal{Z} \right\rangle_{\{\mathcal{D},\zeta\}}$
or
annealed approximation - $\frac{1}{M} \ln \left\langle \mathcal{Z} \right\rangle_{\{\mathcal{D},\zeta\}}$.

Weight & Magnetization Enumerators

• Weight enumerator - independent of ζ after gauging, annealed approximation = replica result

$$\mathcal{W}(m) = \mathcal{M}_a(m)|_{p=0} = \mathcal{M}_q(m)|_{p=0} \ .$$

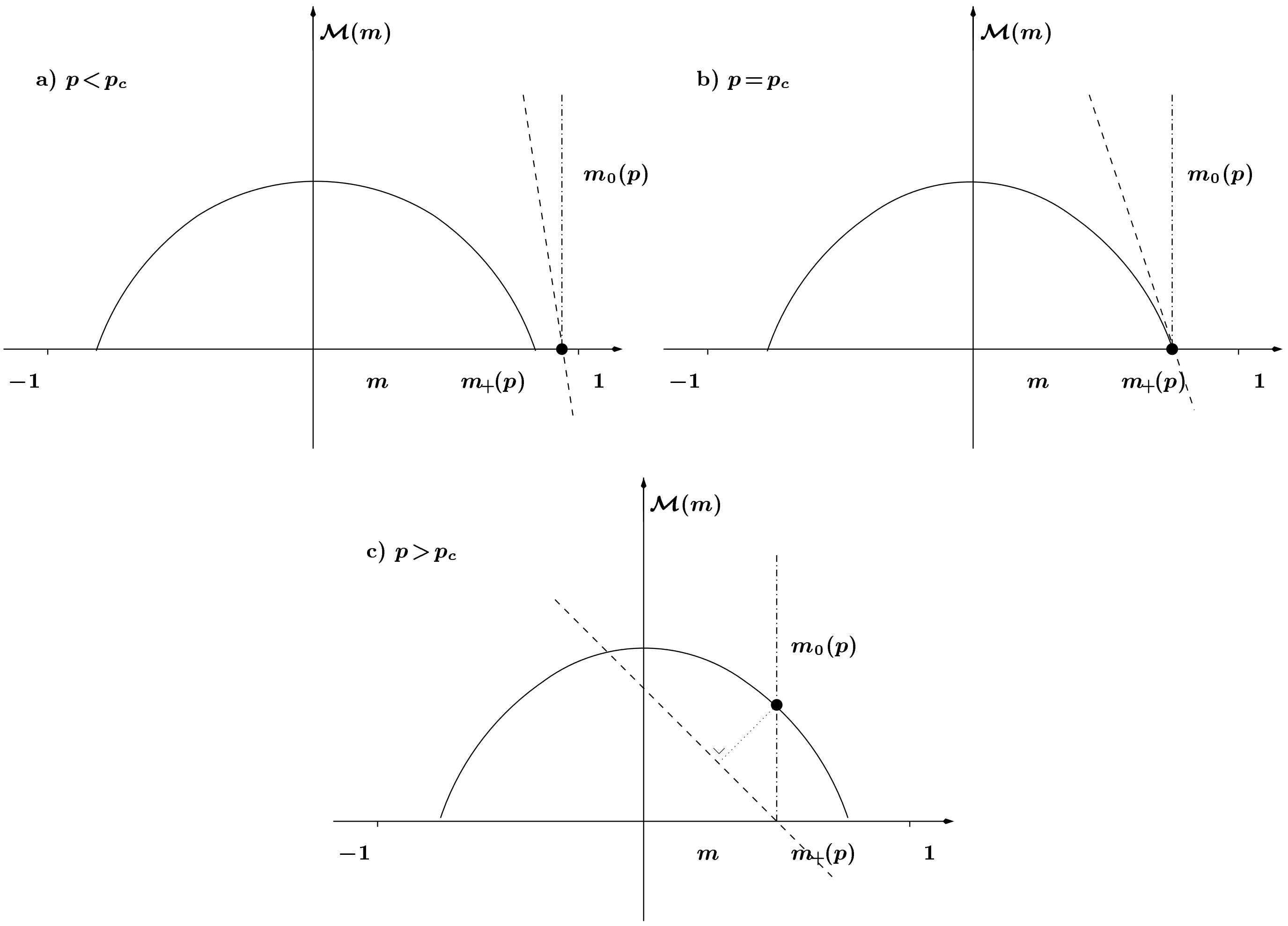
• Magnetization enumerator $\mathcal{M}(m)$ - different results for annealed approximation/replica method.

• Maximum likelihood (MAP) decoding - selects solution with highest magnetization.

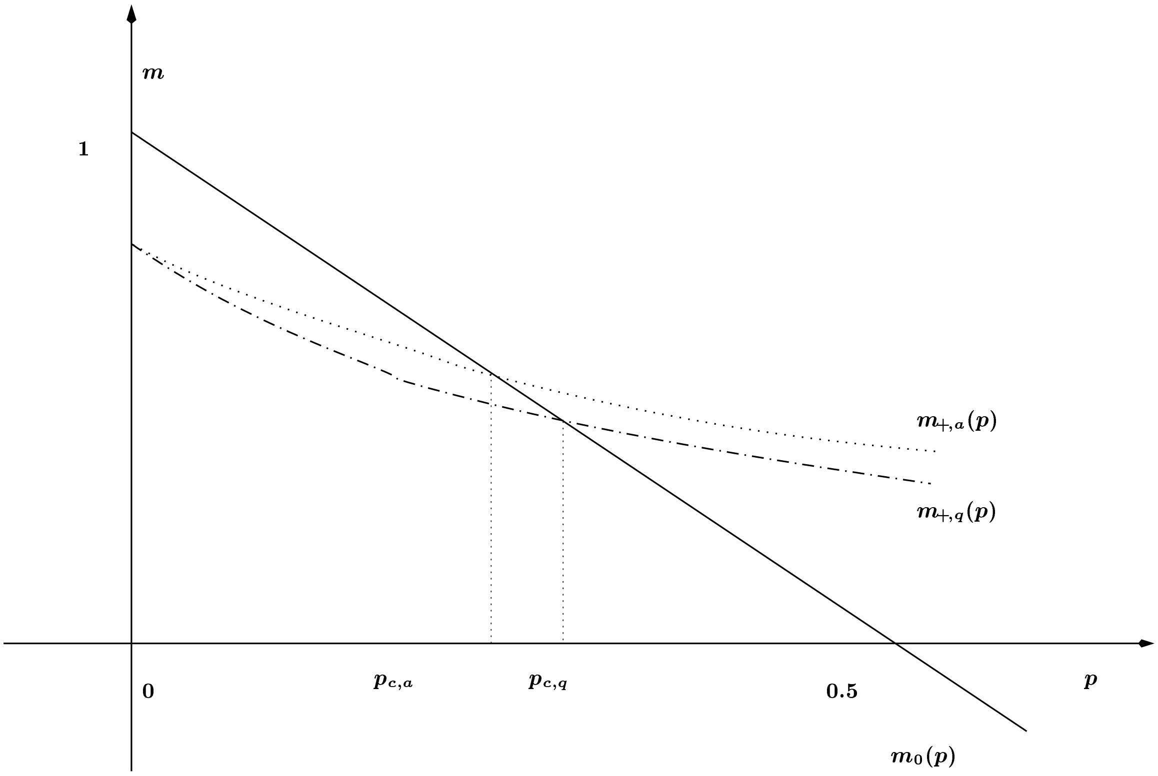
• Typical pairs decoding - selecting a (single) vector from typical set.

• Finite temperature (MPM) decoding - energy term $-Fm$ added, solution chosen by minimizing the free energy.

Entropy and decoding



Critical noise level



(K, C)	$(6, 3)$	$(5, 3)$	$(6, 4)$	$(4, 3)$
Code rate	$1/2$	$2/5$	$1/3$	$1/4$
IT ($\mathcal{W}_a$)	0.0915	0.129	0.170	0.205
SP	0.0990	0.136	0.173	0.209
$p_{c,a}$ ($\mathcal{M}_a$)	0.031	0.066	0.162	0.195
$p_{c,q}$ ($\mathcal{M}_q$)	0.0998	0.1365	0.1725	0.2095
Shannon	0.109	0.145	0.174	0.214

Summary

• Gallager codes saturate Shannon's bound only for $K+L \to \infty$, suboptimal perfect decoding is feasible for all configurations (MN theoretically saturate Shannon's bound for finite K, L, perfect decoding is feasible only in specific configurations).

• We have provided an intuitive explanation to the difference between MAP and MPM decoding based on the magnetization enumerator $\mathcal{M}(m)$.

• Critical noise levels for typical pair, MAP and MPM decoding must coincide.

• We have recently studied typical set decoding in LDPC and currently calculate the reliability exponent directly.

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