Convex Boolean Functions and Clustering: Extended Abstract

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1 Introduction

2 Basic Concepts

We assume that the reader is familiar with the basic concepts of Boolean algebra, and we only introduce here the notions that we explicitly use in this paper.

A *Boolean function* f of n variables $x_1, \ldots, x_n$ is a mapping $\{0,1\}^n \rightarrow \{0,1\} = B$, where $B = \{0,1\}$, and where B^n is commonly referred to as the *Boolean hypercube*. The variables $x_1, \ldots, x_n$ and their complements $\bar{x}_1, \ldots \bar{x}_n$ are called *literals*. We shall sometimes denote x by x^1, and $\bar{x}$ by x^0. For two Boolean functions f and g we write $f \leq g$ iff for every 0-1 vector $\boldsymbol{x}$, $f(x_1, \ldots, x_n) = 1$ implies $g(x_1, \ldots, x_n) = 1$.

A Boolean function $f(x_1, \ldots, x_n) depends effectively$ on variable x_i iff there exist values $x_j^*(j = 1, \ldots, i-1, i+1, \ldots, n)$ for which

$$f(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*) \neq f(x_1^*, \ldots, x_{i-1}^*, \bar{x}_i, x_{i+1}^*, \ldots, x_n^*)$$

The *dual* of a Boolean function $f(\boldsymbol{x})$ is defined as

$$f^d(\boldsymbol{x}) = \bar{f}(\bar{\boldsymbol{x}}),$$

where $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is the *complement* of $\boldsymbol{x}$, and where $\bar{f}(\boldsymbol{y}) = 1$ if and only if $f(\boldsymbol{y}) = 0$ for each $0-1$ point of y.

The *true (false) set* of a function f, denoted by $\mathcal{T}_f$ $(\mathcal{F}_f)$, is the collection of the true (false) points of f, i.e.

$$\mathcal{T}_f = \{\boldsymbol{x} \in \{0,1\}^n : f(\boldsymbol{x}) = 1\} \quad \text{and} \quad \mathcal{F}_f = \{\boldsymbol{x} \in \{0,1\}^n : f(\boldsymbol{x}) = 0\}.$$

A *term*, or an *elementary conjunction*, is a conjunction of literals

$$\prod_{i \in P} x_i \prod_{i \in N} \bar{x}_i,$$

where P and N are disjoint subsets of $\{1, \ldots, n\}$. The *degree* of a term is the number of literals in it. We shall say that a term T *absorbs* another term T', iff $T \vee T' = T$, i.e. iff $T \geq T'$ (e.g. the term $x\bar{y}$ absorbs the term $x\bar{y}z$). A term T *covers* a 0-1 point $\boldsymbol{x}^*$ iff $T(\boldsymbol{x}^*) = 1$. A term T is called an *implicant* of a function f iff $T \leq f$. An implicant T of a function is called *prime* iff there is no distinct implicant T' absorbing T.

A *disjunctive normal form* (DNF) is a disjunction of terms. It is well known that every Boolean function can be represented by a DNF, and that this representation is not unique. A DNF representing a function f is called *prime* iff each term of the DNF is a prime implicant of the function. On the other hand, a DNF representing a function is called *irredundant* iff eliminating any one of its terms results in a DNF which does not represent the same function. The *length* of a DNF is the number of literals in it.

Two terms are said to *conflict* in x_i if x_i is a literal in one of them and $\bar{x}_i$ is a literal in the other. If two terms conflict in exactly one variable, i.e., they have the form x_iP and $\bar{x}_iQ$ and the elementary conjunctions P and Q have no conflict, then their *consensus* of P and Q is defined to be the term PQ. The *consensus method* applied to an arbitrary DNF Φ performs the following operations as many times as possible:

1. Consensus: If there exist two terms of Φ having a consensus T which is not absorbed by any term of Φ then replace the DNF Φ by the DNF $\Phi \vee T$.

2. Absorption: If a term T of Φ absorbs a term T' of Φ, delete T'.

It is easy to notice that all the DNFs produced at every step of the consensus method represent the same function as the original DNF. The following result plays a central role in the theory and applications of Boolean functions [1, 11]:

Proposition 2.1 *(***Blake, Quine***) The consensus method applied to an arbitrary DNF of a Boolean function f results in the DNF which is the disjunction of all the prime implicants of f.*

A classical hard problem concerning Boolean formulae is the *satisfiability problem (SAT)*. When working with DNFs, this problem can be formulated as follows: given as input a DNF Φ, is there an assignment *satisfying* Φ, i.e. is there a point $\boldsymbol{x}^* \in \{0,1\}^n$ such that $\Phi(\boldsymbol{x}^*) = 0$?

Throughout the text, the following notation will be used to represent terms:

Definition 2.1 *If $S = \{i_1, \ldots, i_{|S|}\} \subseteq \{1, \ldots, n\}$, and $\boldsymbol{\alpha}_S = (\alpha_{i_1}, \ldots, \alpha_{i_{|S|}}) \in \{0,1\}^{|S|}$ is an "assignment" of* 0-1 *values to the variables x_i $(i \in S)$, then the term $X^{\boldsymbol{\alpha}_S}$* associated to *$\boldsymbol{\alpha}_S$ is the conjunction $\Pi_{i \in S} x_i^{\alpha_i}$; if $S = \emptyset$, we define $X^{\emptyset} = 1$. Similarly, $X_S^{\bar{\boldsymbol{\alpha}}}$ is the term associated to the assignment $\bar{\boldsymbol{\alpha}}_S$.*

For example, if $S = \{1,3,5\}$ and $\alpha_1 = 1, \alpha_3 = 1, \alpha_5 = 0$, then the term corresponding to the above assignment is $x_1 x_3 \bar{x}_5$.

We shall frequently use in this paper the concept of *projection* of a DNF; the definition are given below:

Definition 2.2 *The restriction of $\Phi(x_1, \ldots, x_n)$ to $S \subseteq \{1, \ldots, n\}$ is the DNF Φ^S obtained by elimination from Φ all the variables c_i with $i \notin S$. If all the variables of a term of Φ are not in S, then the restriction is the constant 1. For completeness, the restriction to the empty set is defined to be 0 and any restriction of a constant is the constant itself.*

Definition 2.3 *Let $S = \{i_1, \ldots, i_{|S|}\} \subseteq \{1, \ldots, n\}$ and let $\boldsymbol{\alpha} = (\alpha_{i_1}, \ldots, \alpha_{i_{|S|}}) \in \{0,1\}^{|S|}$. The* projection *of a DNF $\Phi(x_1, \ldots, x_n)$ on $(S, \boldsymbol{\alpha})$ is the DNF $\Phi_{(S,\boldsymbol{\alpha})}$ obtained from Φ by the substitutions $x_i = \alpha_i$ for all $i \in S$.*

For example, the projection of $x_1x_2 \vee \bar{x}_2x_3$ on $(S, \boldsymbol{\alpha})$ where $S = \{2\}$ and $\boldsymbol{\alpha} = (0)$ is the DNF x_3.The relationship between the previous two DNFs can be generalized. Let $\bar{S} = \{1, \ldots, n\}$§. Then, for any DNF Φ,

$$\Phi_{(\boldsymbol{S},\boldsymbol{\alpha})} \leq \Phi^{\bar{\boldsymbol{S}}}$$

for any $\boldsymbol{\alpha} \in \{0, 1\}^{|\boldsymbol{S}|}$.

We shall make use later in the paper of two classes of DNFs for which a family of SAT problems can be solved polynomially.

Definition 2.4 *A class of DNFs has the* **extended SAT tractability** *property if there exists a polynomial time STA solving algorithm which can decide whether $\Phi' = 0$ has a solution for any projection Φ' of a DNF in this class.*

Definition 2.5 *A class of DNFs has the* **strong STA tractability** *property if there exists a polynomial time algorithm which can decide whether $\Phi' = 0$ has a solution for any DNF Φ' obtained from a DNF in this class by repeated applications of projections and restrictions.*

Some well known types of DNFs having both of the above properties are the quadratic, Horn, renamable Horn and q-Horn DNFs (see [2] and [3] for definitions).

Throughout this paper, we shall use the *Hammingmetric* which defines the distance between two Boolean vectors x and y as the number of components in which they differ:

$$d(x, y) = |\{i : x_i \neq y_i i \in [1 \ldots n]\}|$$

Two vectors x and y are called *neighbors* iff $d(x, y) = 1$. The point y is between x and z iff $d(x, y) + s(y, z) = d(x, z)$. A sequence of points $x_1, \ldots, x_k$ is called a *path* from x_1 to x_k iff any two consecutive points in this sequence are neighbors. A *shortest path* between x and y is a path of length $d(x, y)$. A *true(false) path* is a path consisting only of true (false) points of a Boolean function.

We say that two true (false) points are *convexly connected* iff all the shortest paths connecting them are true (false).

Definition 2.6 **A partially defined Boolean function(pdBf)** *is a pair of sets (T, F) of Boolean vectors where T denotes a set of true points (or positive examples) and F denotes a set of false points (or negative examples).*

Definition 2.7 *An* **extension** *of pdBf (T, F) is a Boolean function f such that f agrees with (T, F); in other words, if x is one of the data points from (T, F), then $f(x) = 1$ if and only if x is classified as true in (T, F).*

A **theory** *is an extension for which there exists a prime DNF representation ϕ with the property that every prime implicant in ϕ covers at least one true point in the given set (T, F).*

3 Convex Functions

The extremely powerful requirement of convexity puts a severe limitation on the number of functions with this property. in order to provide more flexibility, we introduced the following relaxation if the definition in [].

Definition 3.1 *For any integer $k \in \{2, \ldots n\}$, a Boolean function f is called*

- *k-convex iff any pair of true points at distance at most k is convexly connected*
- *co-k-convex iff any pair of false points at distance at most k is convexly connected i.e. $\bar{f}$ is k-convex*
- *strongly k-convex iff it is both k-convex and co-k-convex*

Example 3.1 The function $\bar{x}_1\bar{x}_2\bar{x}_3 \vee x_1x_2x_3$ is 2-convex, and its negation $x_1\bar{x}_2 \vee x_2\bar{x}_3 \vee \bar{x}_1x_3$ is co-2-convex.

The following results were all obtained in []. We present them here without proofs.

Theorem 3.1 *For any $k \geq 2$, a Boolean function f is k-convex iff any two prime implicants of f conflict in at least $k+1$ literals.*

It follows from Proposition 2.1 that,

Remark 3.1 A k-convex Boolean function has a unique irredundant prime DNF representation.

Remark 3.2 A k-convex function having an implicant of degree at most k is an elementary conjunction, and does not depend effectively on any variable not included in this conjunction.

With increasing values of k, the statement of Theorem3.1 gets stronger. In particular, an n-convex function consists of a single prime implicant. An $(n-1)$-convex function is either an elementary conjunction, or has the form of $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n} \vee x_1^{\bar{\alpha}_1}x_2^{\bar{\alpha}_2}\ldots x_n^{\bar{\alpha}_n}$. It follows by negation that co-n-convex functions are linear, while co-$(n-1)$-convex functions have the form $\bigvee_{i=1}^{n-1}(x_i^{\alpha_i}x_{i+1}^{\bar{\alpha}_{i+1}} \vee x_n^{\alpha_n}x_1\bar{\alpha}_1)$.

It is well known that given an arbitrary DNF of a positive Boolean function, we can obtain a positive DNF of it by simply "erasing" all the complemented variables from the given DNF. Obviously, the polynomial of this algorithm is a consequence of the a-priori knowledge of the positivity of the function.

In order to obtain a similarly efficient method for finding the prime implicants of a Boolean function which is assumed to be known to be k-convex, we shall use the following.

Definition 3.2 *If* $S = X^{\boldsymbol{\alpha}_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_C}$ *and* $T = X^{\overline{\boldsymbol{\alpha}}_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_D}$ *are elementary conjunctions with* $|a| \leq k$, *, then the* k**-convex hull** *of* S *and* T *is the elementary conjunction*

$$[S, T] = X^{\boldsymbol{\alpha}_B}$$

Note that when $B = \emptyset$, the convex hull is simply the constant 1. Obviously, $S \vee T \leq [S, T]_k$ for any k.

We shall describe now the $k-intersectionmethod$ for finding the k-convex hull of a DNF. Given any DNF $\Phi = \bigvee k = 1^m T_k$, if $T_i = X^{balpha_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_C}$ and $T_j = X^{\overline{\boldsymbol{\alpha}}_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_D}$ are two implicants of Φ such that $d(T_i, T_j) \leq k$, replace Φ such that $d(T_i, T_j] \leq k$,replace Φ by

$$\Phi' = [T_i, T_j] \vee (\bigvee_{k=1, k \neq i,j}^{m} T_k),$$

perform all possible absorptions, and repeat as many times as possible. Let $[\Phi]_k$ be the last DNF obtained after applying the $l - intersectionmethod.$ We call $[\Phi]_k$, the k-convex hull of Φ.

We shall show now that the k-convex hull of a DNF is the unique smallest k-convex majorant of the function represented by the given DNF.

Theorem 3.2 *Let a Boolean function* f *be represented by a DNF* Φ. *Let* $[\Phi]_k$ *be th* k*-convex hull of* Φ. *Then, the following are all true.*

(i) $[\Phi]_k$ *is a* k*-convex majorant of* Φ

(ii)$[\Phi]_k$ *is irredundant and prime,*

(iii)if Ψ *is a* k*-convex majorant of* Φ, *then* $[\Phi]_k \leq \Psi$

(iv)$[\Phi]_k$ *is unique,*

(v) If f *is* k*-convex for some* $k \geq 2$, *then* $[\Phi]_k \leq \Psi$,

We shall concentrate now on functions which are strongly k-convex, i.e. which are both k-convex and co-k-convex. The following two theorems indicate that this property is very strong, and consequently, the class of functions satisfying it is very small.

It is obvious from the definition that a strongly k-convex function is also strongly $(k-1)$-convex. Surprisingly, the next statement shows that the converse is also true:

Theorem 3.3 *A boolean function* f *is strongly* n*-convex if and only if it is stronly 2-convex.*

Corollary 3.1 *A Boolean function is strongly 2-convex if and only if it is either a constant or a variable* x *or* $\bar{x}$.

Corollary 3.2 *The number of strongly n-convex function is $an+2$.*

Theorem 3.4 *Given a DNF and number $k \in \{2, \ldots, n\}$, it is CoNP-hard tp decide whether the Boolean function represented by thid DNF is:*

(i) k-convex

(ii)co-k-convex

(iii)stronly k-convex

Theorem 3.5 *Let Φ be a DNF having the extended SAT tractability property and representing a function f. We can decide in polynomial time whether there exists a $k \geq 2$ for which f is k-convex. Moreover, we can determine a k^* with the property that f is k-convex for any $k \leq k^*$, and is not k-convex for any $k \geq k^*+1$.*

Lemma 3.1 *Let $f(x_1, \ldots, x_n)$ be a Boolean funtion which for some $k \geq 2$ is co-k-convex but not co-$k+1$-convex. Then, for any two false points of f at distance $k+1$, all the points are true.*

Theorem 3.6 *Let $\Phi = \bigvee_{i=1}^{m} T_i$ be a DNF having extended SAT tractability property and representing a function $f(x_1, \ldots, x_n)$. Then, for any fixed K, we can decide in polynomial time whether is co-K-convex.*

Theorem 3.7 *Let Φ be a DNF having extended SAT tractability property and representing a function f. For any $k = 2, \ldots, n$, we can decide in polynomial time whether is co-k-convex.*

It has been shown that the set of the true points if k-convex functions consists of entire subcubes at large Hamming distances form each other. Perhaps not suprisinfly, the class of strongly k-convex functions were shown to contain very few functions; in fact, the only stronly k-convex functions are those that depend effectively on at most one varible. Finally, it has been shown that there are only two kinds of connected functions: some which are k-convex for any k, and the others which are not k-convex for any k.

4 Convex theories of PDBFs

Theorem 4.1 *Given k, we can decide in polynomial time if a pdBf (T,F) has a k-convex extension and if so find a DNF representation for a minimum theory of it.*

Proof. Let

$$\Phi = \bigvee_{S \in T} \wedge_{i=1}^{n} x_i^{s_i}$$

in other word, Φ is the minterm expansion of the set T. Obviously, Φ represents a minimum theory of (T,F). Let us apply the k-intersection method to Φ to obtain the k-convex hull of it , $[\Phi]_k$. By construction, $[\Phi]_k$ represents the minimum k-convex theory of (T,F) provided that no term in $[\Phi]_k$ covers a false point from (T,F). □

Given a pdBf (T,F) and a point x not belonging to $T \cup F$, we can find out if there exists a k-convex extension which classifies this point as true. We simply add this point to the true set T and apply the procedure of the previous proof.

Theorem 4.2 *Given a pfBf (T,F) and a minimum k-convex theory of this pdBf, say f_m, we can check in polynomial time if this theory is the unique k-convex theory for the given pdBf.*

Proof. Without loss of generality, let us assume that $f_m = \bigvee_{i=1}^{s} P_i$. If f_m is not the unique k-convex theory of the pfBf (T,F), then there exists another theory, say f_u which is a majorant of f_m simply due to the construction of f_m. In other words, $f_u = \bigvee_{i=1}^{s} P_i' \vee \bigvee_{j=1}^{t} Q_j$ and suppose that Q_1 is a prime implicant in a representation of f_u. Since f_u is a theory, Q_1 must cover a true point from the set T, say x_1. Since $x_1 \in T$, it must be covered by one of the prime implicants in the given representation of f_m, say P_l. Since $P_l' \geq P_l$, x_1 must also be covered by P_l'. However, this is not possible since P_l' and Q_1 must conflict in at least $k+1$ literals. We have thus established that each term in a representation of f_u must majorise a term of the DNF $\bigvee_{i=1}^{s} P_i$. For each term P_i, and for each literal in this term, first drop this literal and run k-intersection method on the resulting DNF to obtain a minimum k-convex majorant of this DNF. If the output of the k-intersection method is a DNF which covers some points from F i.e. false points, we conclude that this literal cannot be dropped and we try another one and so on. If there exists at least one literal, after dropping of which we have another k-convex theory, we cnclude that f_m is not unique. Otherwise it is. □

Theorem 4.3 *Given a pdBf (T,F) and an unknown point x i.e. $x \notin T \cup F$, we can check in polynomial time if all the k-convex theories of (T,F) agree on x.*

Proof. Without loss of generality assume that the theory f_m constructed by the application of algorithm in the proof of ?? is not unique.

If f_m classifies x as a true point, then ao should any k-convex theory since f_m is the minimum among them. Hence, we may assume that f_m classifies this point as false. We then check if x could be a true point in some k-convex theory of (T,F). To do this, we add x to T and try to find the minimum k-convex theory of (T',F), say f_m', where $T' = T \cup x$ by applying the procedure in ??. If we fail to do so, we conclude that x is classified as false in all the k-convex theories of (T,F). If f_m' is also a theory of (T,F), we know that f_m' and f_m disagree on x. So we assume that f_m' is not a theory of (T,F). In other words, there exists a prime implicant of f_m', say P, which covers only the true point x from T'. All the other prime implicants of f_m' cover at least one point from T. We will try to merge P with one of the other prime implicants of f'_m where by merging two prime implicants we mean replacing their disjunction with the term which is the conjunction over the common literals in both terms. We merge P with the other prime implicants one by one. After each merge, we apply the k-intersection method and check if the resulting DNF covers any false points from F. If not, then the current DNF represents a k-convex theory of (T,F) which disagrees with f_m. Otherwise, we try the merge process with another prime implicant. If none of the merges resluts in a k-convex theory of (T,F), we conclude that all k-convex theories of (T,F) must classify x as false. □

We now look for more structure in our extensions.

Given a pdBf (T,F), we would like to obtain a k_P-convex function f_P which covers all true points and a co-k_P-convex function f_N which covers all negative points such that f_P and f_N do not intersect i.e. share the same true point.

Given a pdBf (T,F), let h_k be the k-convex theory of (T,F) obtained through the algorithm in ?? and g_k be the co-k-convex theory of (T,F) obtained when we apply the parallel version of this algorithm to negative points.

For fixed k_P starting from 2, we first obtain h_{k_P}. then for $k_N = 2, 3, \ldots$ and so on, we get g_{k_N} and check that h_{k_P} and g_{k_N} do not intersect i.e. a prime implicant of h_{k_P} conflicts with all the prime implicant of g_{k_N}. For each fixed k_P, we increment k_N as much as we can. Then we repeat the same steps for $k_P + 1$ and so on. In this way, we can output in polynomial time all possible non-dominated pairs (k_P, k_N) for which there exists a k_P-convex function covering all points in T and a co-k_N-convex function covering all points in N and such that these two functions do not intersect. By our construction, if (k_P, k_N) is a possible pair, then so is $(k_P - 1, k_N)$ and $(k_P, k_N - 1)$.

Note that given f_P and f_N, we can check easily if they together constitute an extension of (T,N), in other words, if any unknown point can be classified using these functions. To this extent, we simply add the number of true points of the functions and compare with 2^n. This is true simply because the functions do not intersect and have orthagonal DNF representations.

5 Probablistic Properties

Let $\mathcal{F}(n, k)$ denote the class of k-convex function for fixed n.

Definition 5.1 *The* **Vapnik-Chervonenkis dimension (VC-dimension)** *of a set of functions is the lasrgest integer d such that the number of different classifications these functions induce on an arbitrary set of d points in B^n is 2^d.*

When the set of functions under consideration is $\mathcal{F}(n, k)$, we have

$$d \leq \log_2 |\mathcal{F}(n, k)|$$

References

[1] A. Blake, *canonical expressions in Boolean algebra*, Ph.D. Thesis, University of Chicago (Aug. 1937).

[2] E. Boros, Y. Crama and P.L. Hammer, "Polynomial-time inference of all valid implications for Horm and related formulae", *Annals of Mathematics and Artificial Inteligence"*, **1** (1990) 21-32.

[3] E. Boros, P.L. Hammer and X. Sun, "Recognition of q-Horn formulae in linear time", *Discrete Applied Mathematics*, **55** (1994) 1–13.

[4] R.K. Brayton, G.D. Hachtel, C.T. McMullen and A.L. Sangiovanni-Vincentelli, *Logic Minimization Algorithms for VLSI Synthesis* (Kluwer, Dordrecht, 1984).

[5] M.R. Garey and D.D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, San Francisco, 1979.

[6] P.L Hammer and A. Kogan, "Horn functions and their DNFs", *Information Processing Letters*, **44** (1992), 23–29.

[7] P.L Hammer and A. Kogan, "Quasi-acyclic propsitional Horn knowledge bases: Optimal compression", *IEEE Transactions on Knowledge and Data Engineering*, **7,3** (1995).

[8] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research* (Springer-Verlag, Berlin, Heidelberg, New York, 1968).

[9] T. Hegedüs and N. Megiddo, On the geometric separability of Bollean functions, *Discrete Applied Mathematics*, **66** (1996), 205–218.

[10] S. Muroga, *Threshold Logic and its Applications* (Wiley-Interscience, New York, 1971).

[11] W. Quine. A way to simplify truth functions, *American Mathematical Monthly*, **62** (1955), 627–631.