Linear programming bounds for spherical codes and designs

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ABSTRACT

(Outline of the talk)

The linear programming techniques for estimating the size of codes and designs on the n-dimensional Euclidean sphere $\mathbf{S}^{n-1}$ were introduced after 1975. The result was quite impressive: the new bounds appeared to be, in a sense, universal, and easy for calculation.

The background for the linear programming approach was developed by Delsarte, Goethals and Seidel [11] and Kabatiansky and Levenshtein [12]. They proved that real polynomials which possess certain properties (to be explained in the talk) can be used for obtaining upper bounds on the size of spherical codes (for fixed dimension and minimum distance) and lower bounds on spherical designs (for fixed dimension and strength). Moreover, suitable polynomials were proposed and universal bounds have been obtained.

In section 2 we discuss the logic of the linear programming bounds and the logic of the universal bounds, i.e. the Levenshtein bound for codes [13, 14] and the Delsarte-Goethals-Seidel bound for designs [11]. Althought extremal in some sense and the best possible in (infinitely) many cases, these bounds appear not to give the full strength of the linear programming [16, 4, 6, 10]. We show how the distance distributions of codes and designs attaining the universal bounds can be computed. This can be used for proving nonexistence results [2, 3, 5, 7, 8].

In section 3 we describe necessary and sufficient conditions for existence of better than the universal bounds. Such conditions are investigated and it can be shown that they are satisfied in (infinitely) many cases. However, the properties of these functions are not completely studied and we give some conjectures about their behaviour. When improvements are possible, we describe a method for calculating corresponding new bounds. A computer program for such computations will be presented.

Section 4 is devoted to a search of analytic form of the new bounds (in this way they would become universal in the same sense that the Levenshtein bound and the Delsarte-Goethals-Seidel bound are universal). This problem is solved for designs [15] and we explain our agruments toward analogous solutions for codes.

In section 5 we go beyond the pure linear programming. Indeed, the linear programming techniques can be combined with other (geometric, for example) ideas for investigations on the structure of putative good codes and designs. Again the problem for designs is easier and we describe improvements (either asymptotic and in concrete cases, such as the first open ones) of the Delsarte-Gothals-Seidel bound [9]. In other application, linear programming techniques are used to estimate other param-

eters of codes such as their distance distribution. This resulted [1] in obtaining new bound on the exponent of error probability of decoding for the best possible codes in the Gaussian channel.

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