Stochastic Modeling and Chaotic Epidemic Outbreaks

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Outline

- SEIR model - a model for epidemics in childhood diseases (Yorke and London (1973); May and Anderson (1979); Schwartz (1983); Grenfell et al. (2000); Hethcote (2000))
- Add stochastic perturbations to represent noise in population size
- Generic setting for bifurcation to stochastic chaos
- Possible vaccination strategies to control and prevent future outbreaks

Modeling Epidemics: Assumptions

The population:

- Assume the population is large and well mixed.
- Variables and parameters:

S:	Susceptibles	α^{-1}:	mean latent exposed period
E:	Exposed	γ^{-1}:	mean infectious period
I:	Infectives	μ:	birth and death rate
R:	Recovered	β:	contact rate (for S & I)

- Normalize the population: **S** + **E** + **I** + **R** = 1

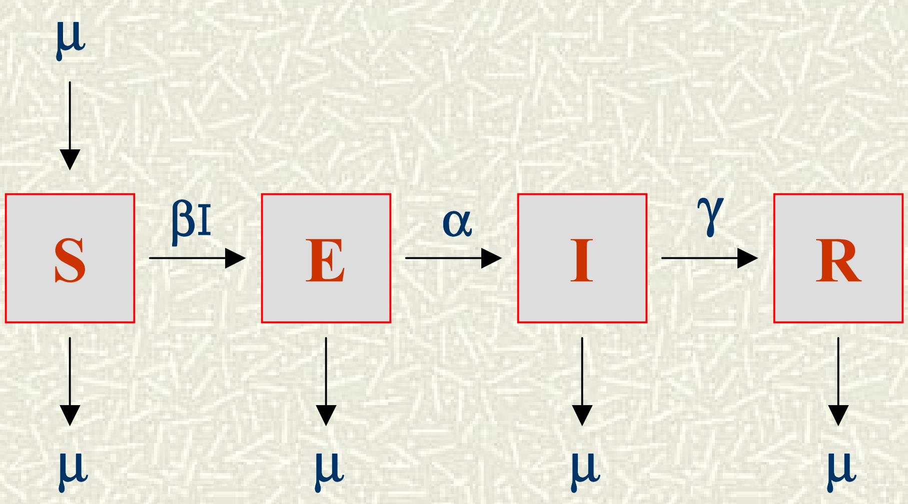
The SEIR model

$$\frac{dS}{dt} = \mu - \beta(t)IS - \mu S$$

$$\frac{dE}{dt} = \beta(t)IS - \alpha E - \mu E$$

$$\frac{dI}{dt} = \alpha E - \gamma I - \mu I$$

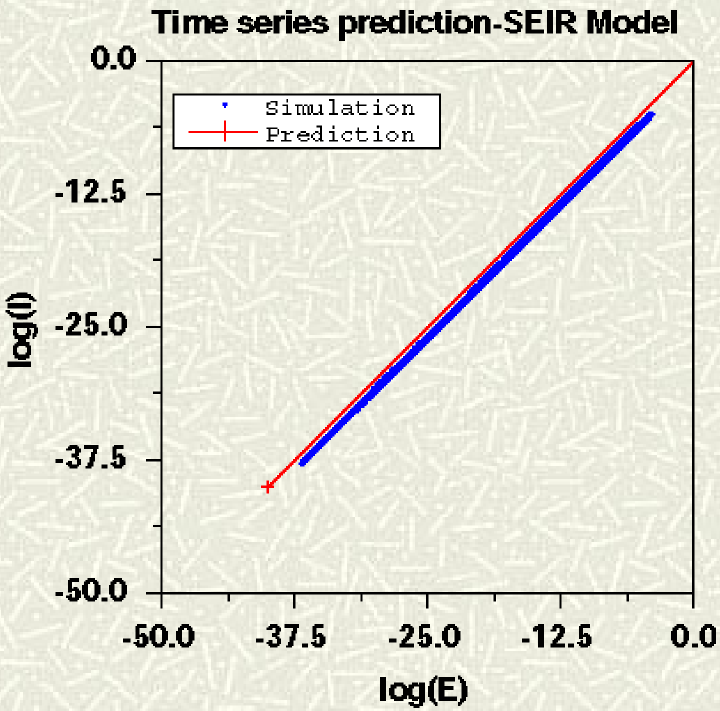
$$\frac{dR}{dt} = \gamma I - \mu R$$



The SEIR model

- The contact rate: $\beta(t) = \beta(t+1) = \beta_0(1 + \delta \cos 2\pi t)$
- The infectives are roughly proportional to the exposed
  [Schwartz, J. Math. Biol. 1985]

$$I(t) \approx \left(\frac{\alpha}{\mu + \gamma} \right) E(t)$$



The SEIR model

- The final model (MSI)

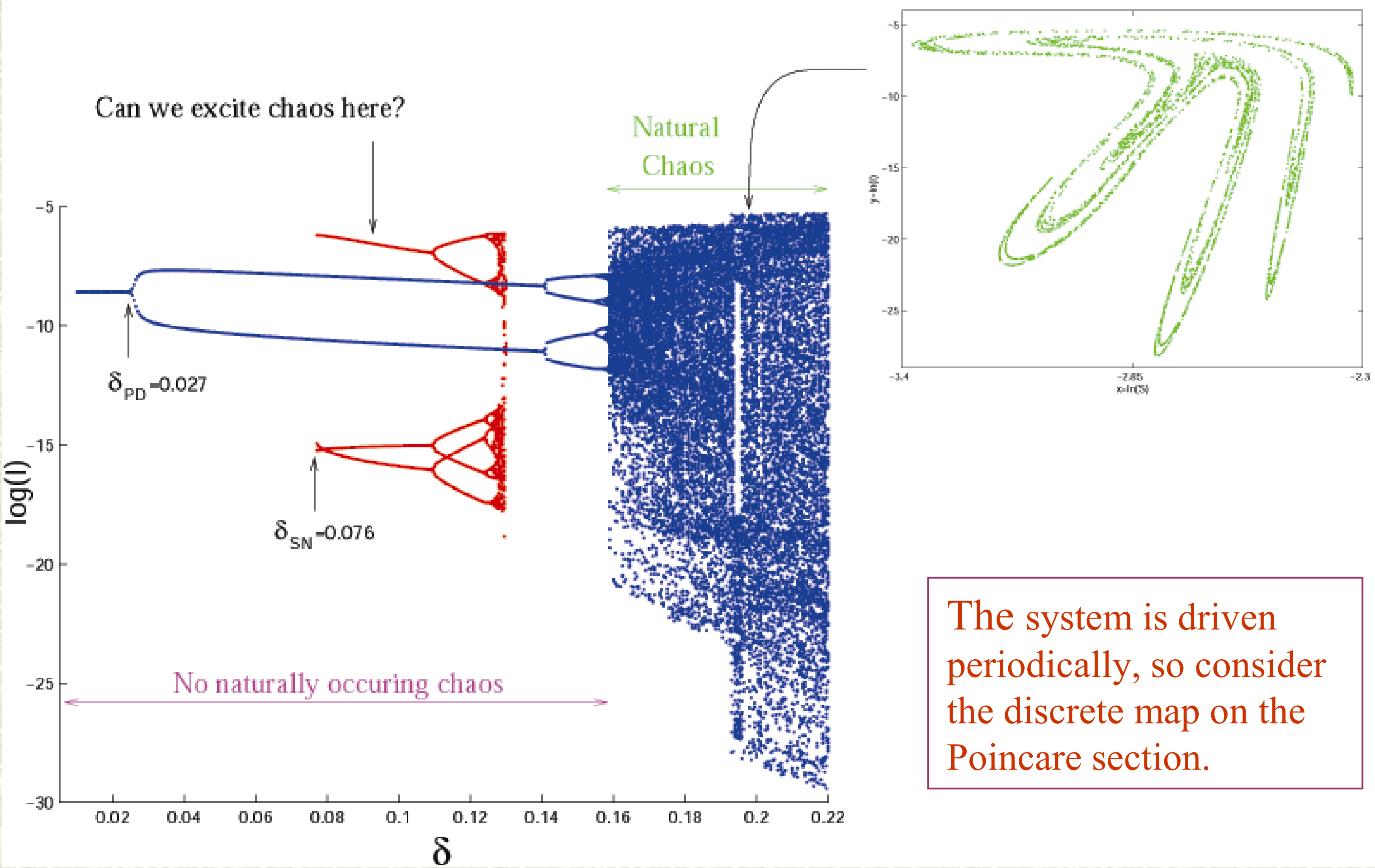
$$\frac{dS}{dt} = \mu - \beta(t)IS - \mu S$$

$$\frac{dI}{dt} = \left(\frac{\alpha}{\mu + \gamma}\right)\beta(t)IS - (\mu + \alpha)I$$

$$\beta(t) = \beta_0(1 + \delta\cos(2\pi t))$$

The paremeter we vary

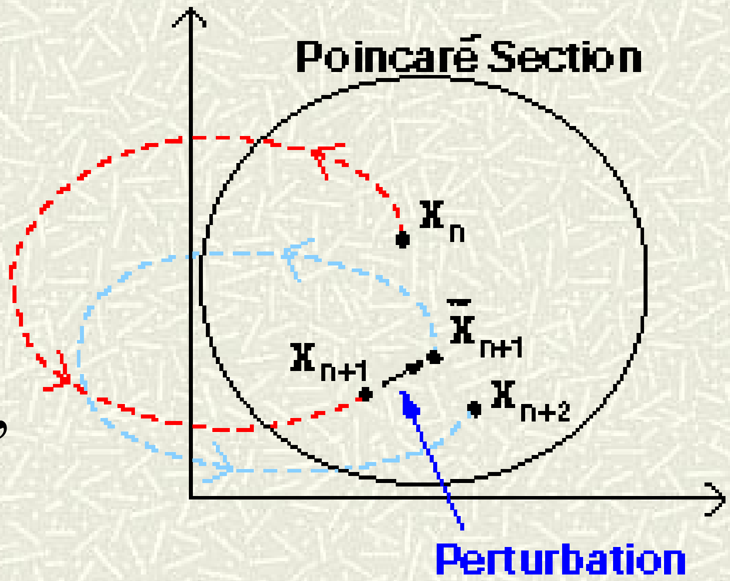
Bifurcation diagram



The system is driven periodically, so consider the discrete map on the Poincare section.

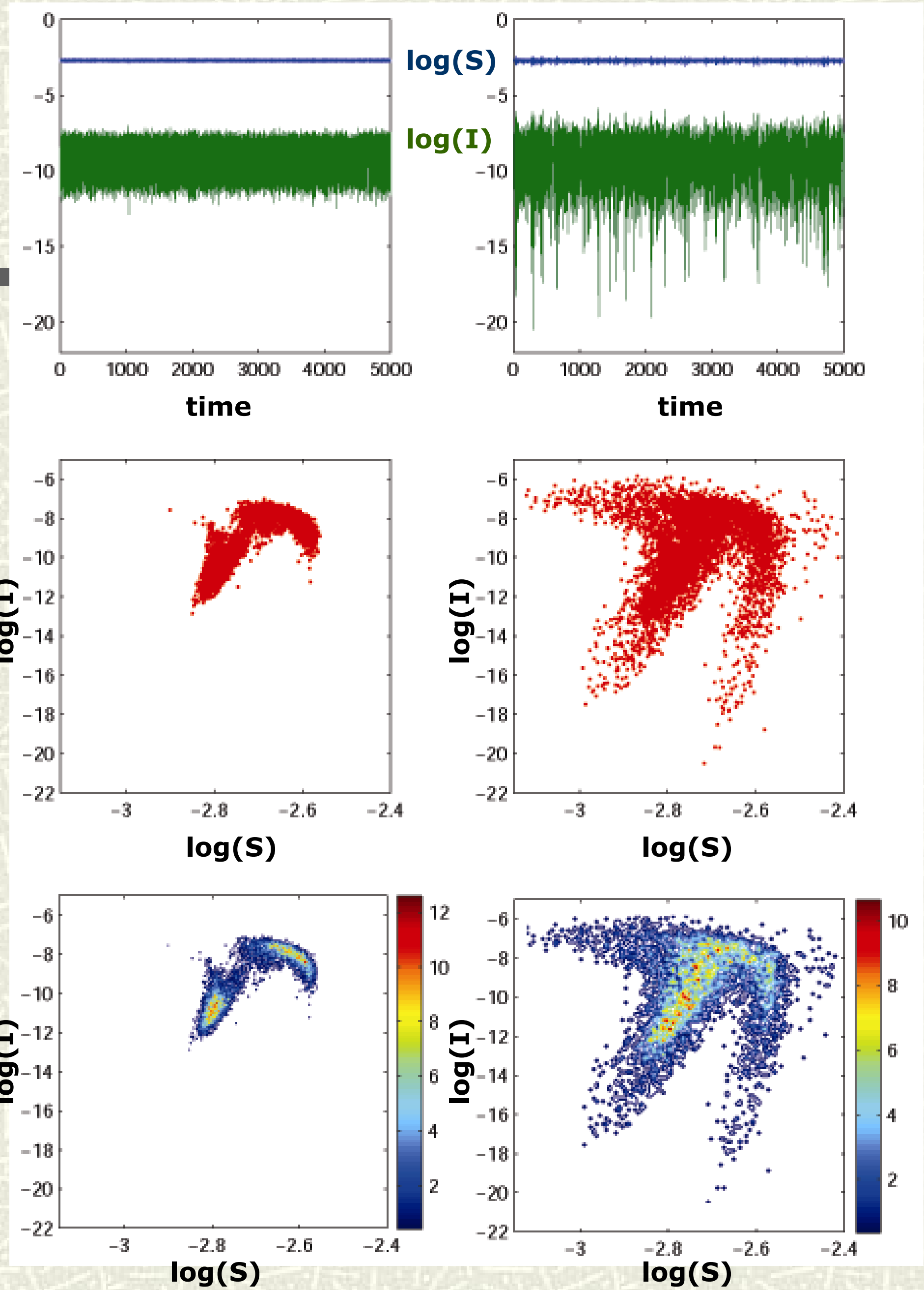
Adding noise

- The system is driven periodically, so add noise as if it is a map. (Additive noise)
- Noise: normal distribution, mean=0, vary the standard deviation (σ)

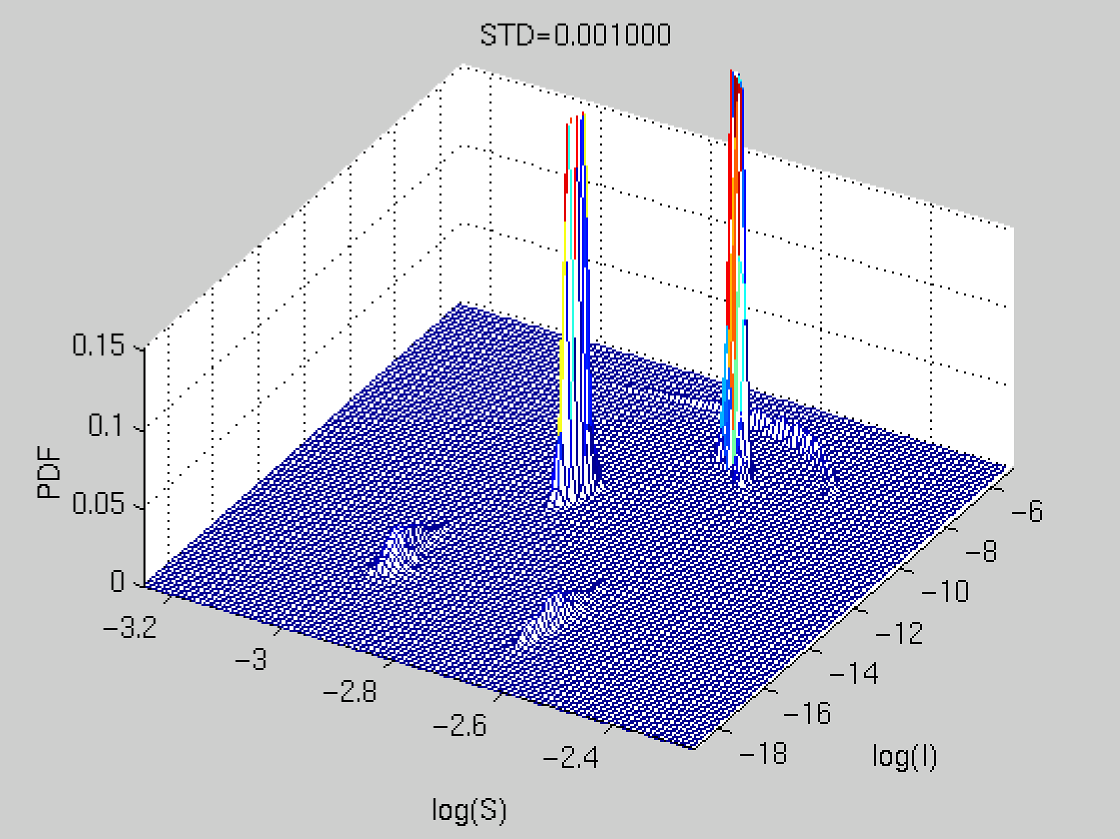


Noisy dynamics

- Time series
- Phase space diagram
- Probability density function



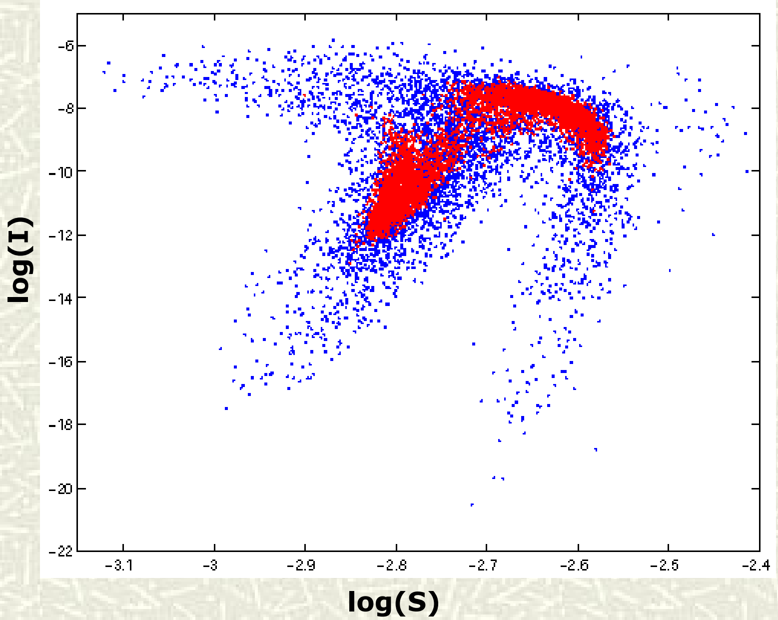
Probability Density Function



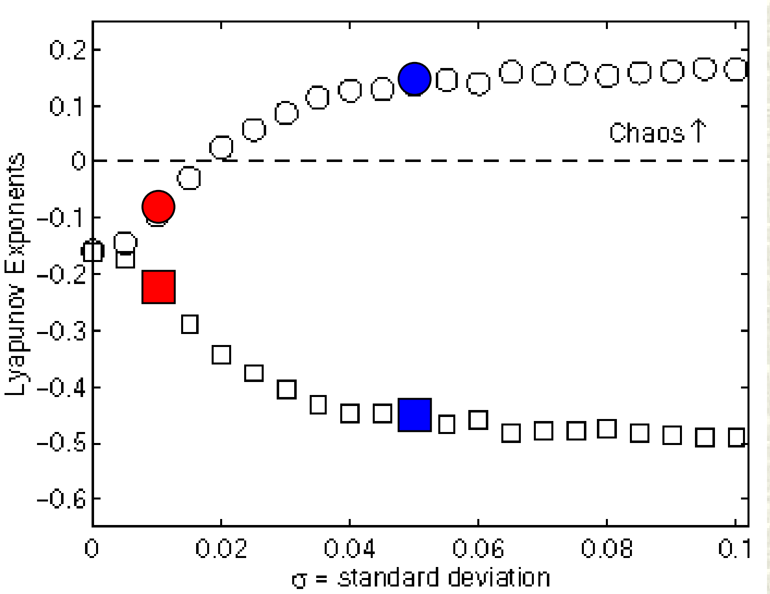
Stochastic Chaos?

- Deterministic definition (numerical)
  - Compact set
  - Positive Lyapunov exponent
  - Not asymptotically periodic
- Stochastic version?
  - Compact set
  - Positive Lyapunov exponent
  - Homoclinic/heteroclinic topology (makes chaotic orbits possible)

Lyapunov exponents



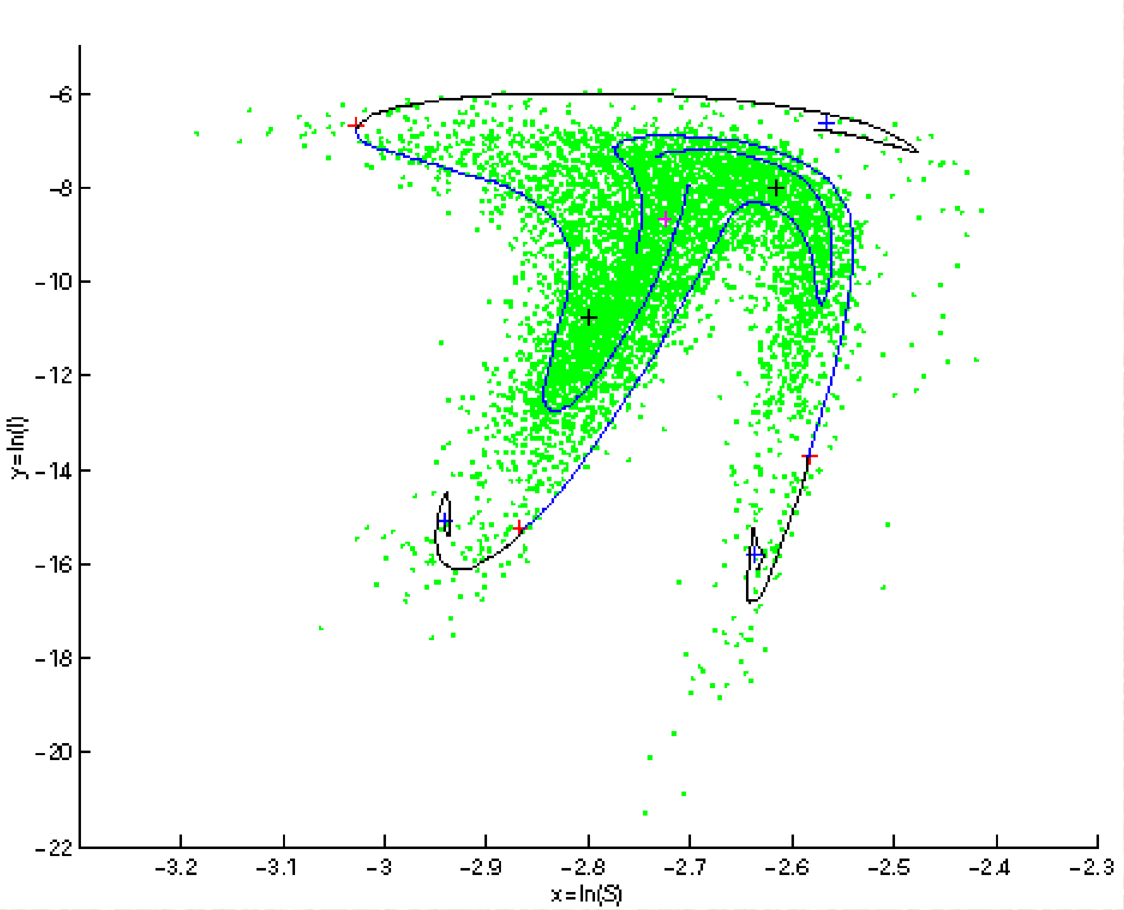
- Red: $\sigma = 0.01$ (noisy)
- Blue: $\sigma = 0.05$ (chaotic)



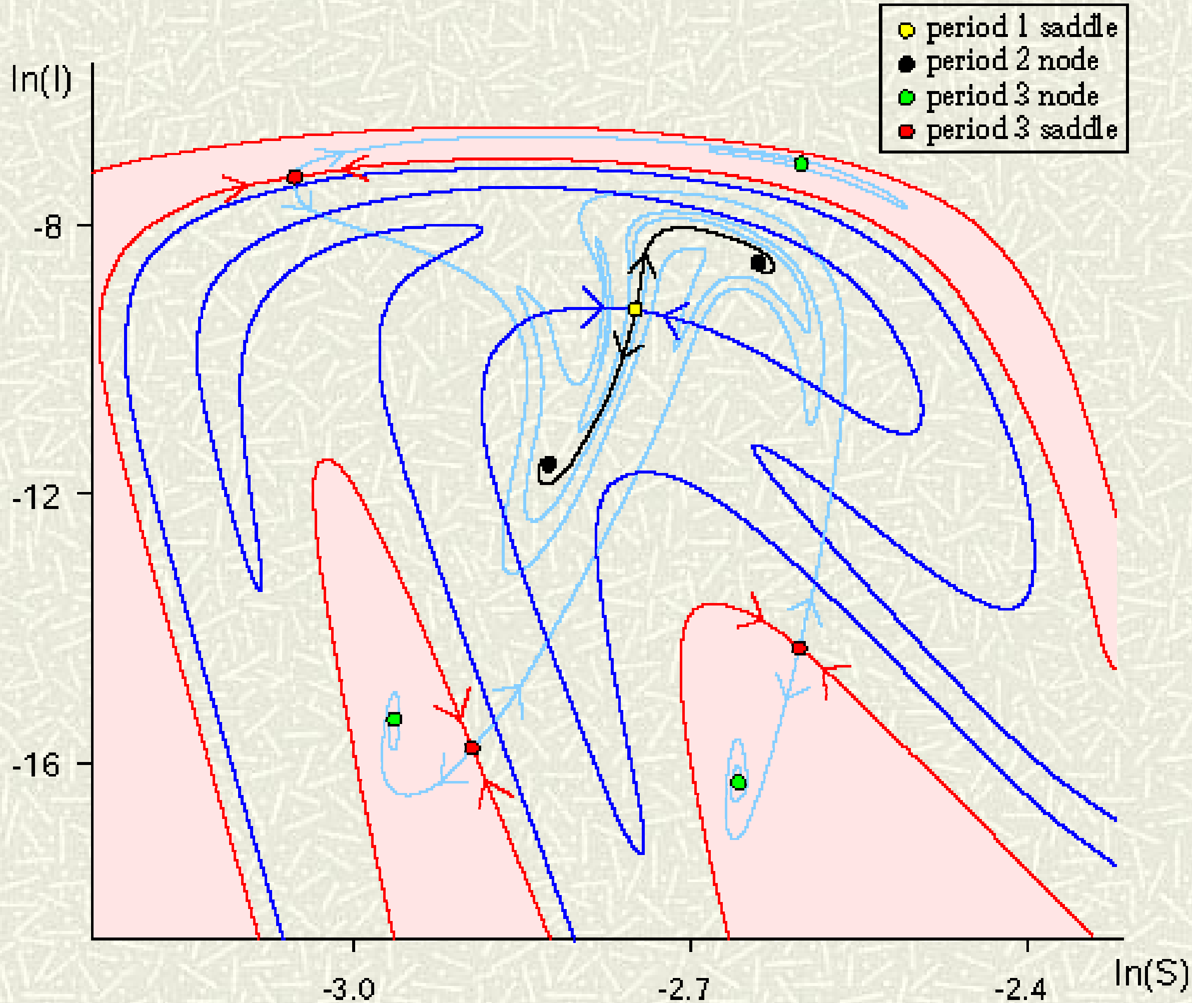
But Lyapunov exponents can yield false results

Unstable manifolds

- Random trajectories follow the unstable manifolds of the period three saddle
- What is the role of the manifolds?

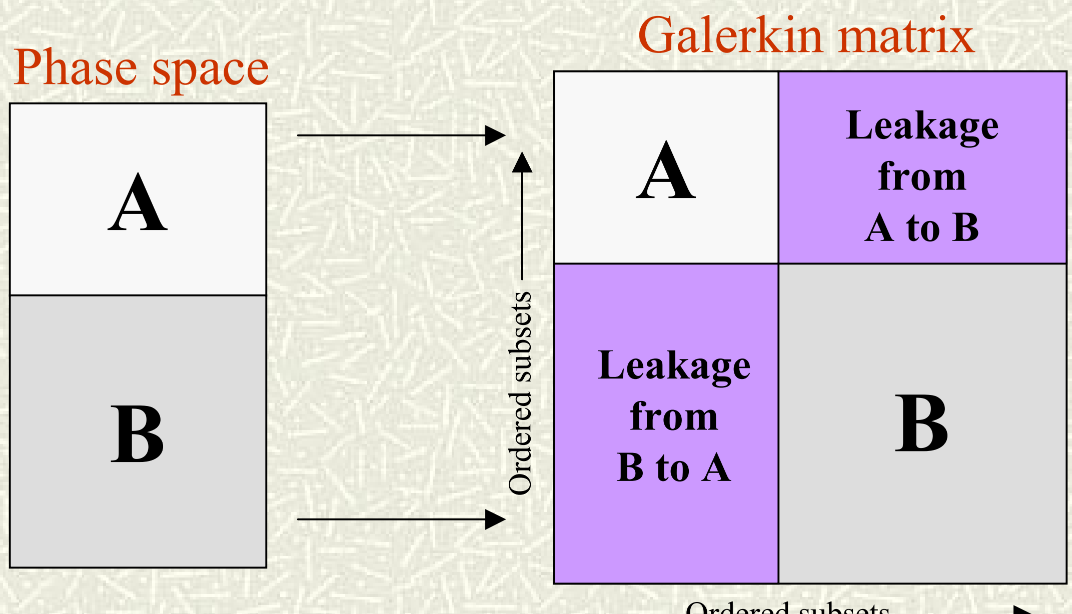


Stochastic Chaotic Saddle



New tool to detect transport

- Use the Galerkin approximation Stochastic Frobenius-Perron Operator to detect the flux across a basin boundaries and predict the most probable regions of transport created by noise.



Theory for Galerkin Matrix

- Add noise using a random variable η, standard deviation σ

$$F: M \to M,\ \ x \to F(x) + \eta$$

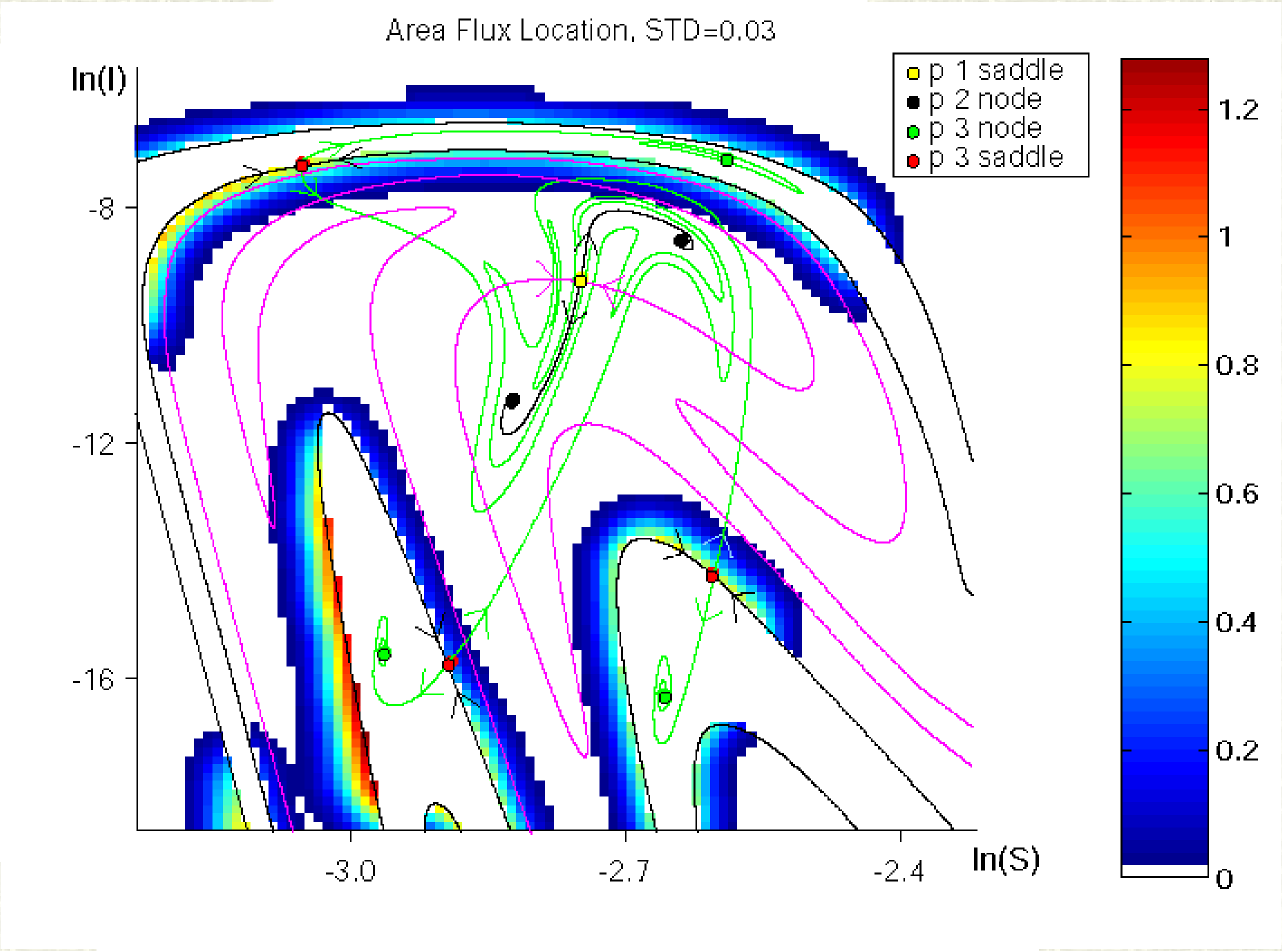
- Stochastic Frobenius-Peron operator

$$P_{F_\sigma}[\rho(x)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_M e^{-\frac{\|(x-F(y))\|^2}{2\sigma^2}} \rho(y)\,dy$$

- Galerkin approximation

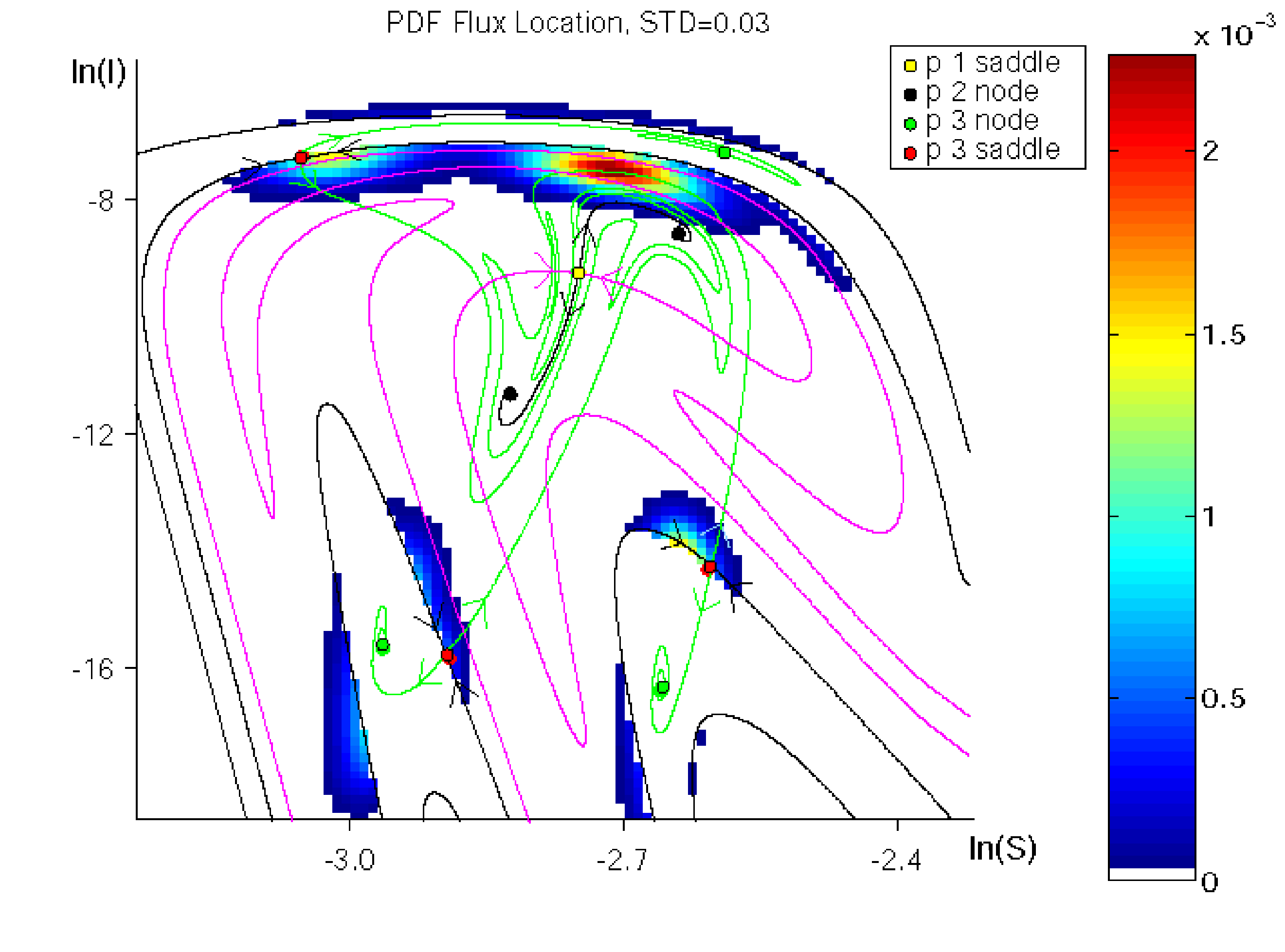
$$A_{i,j} = (P_{F_\sigma}[\varphi_j], \varphi_i) = \int_M P_{F_\sigma}[\varphi_j(x)], \varphi_i(x)\,dx$$

Area Flux



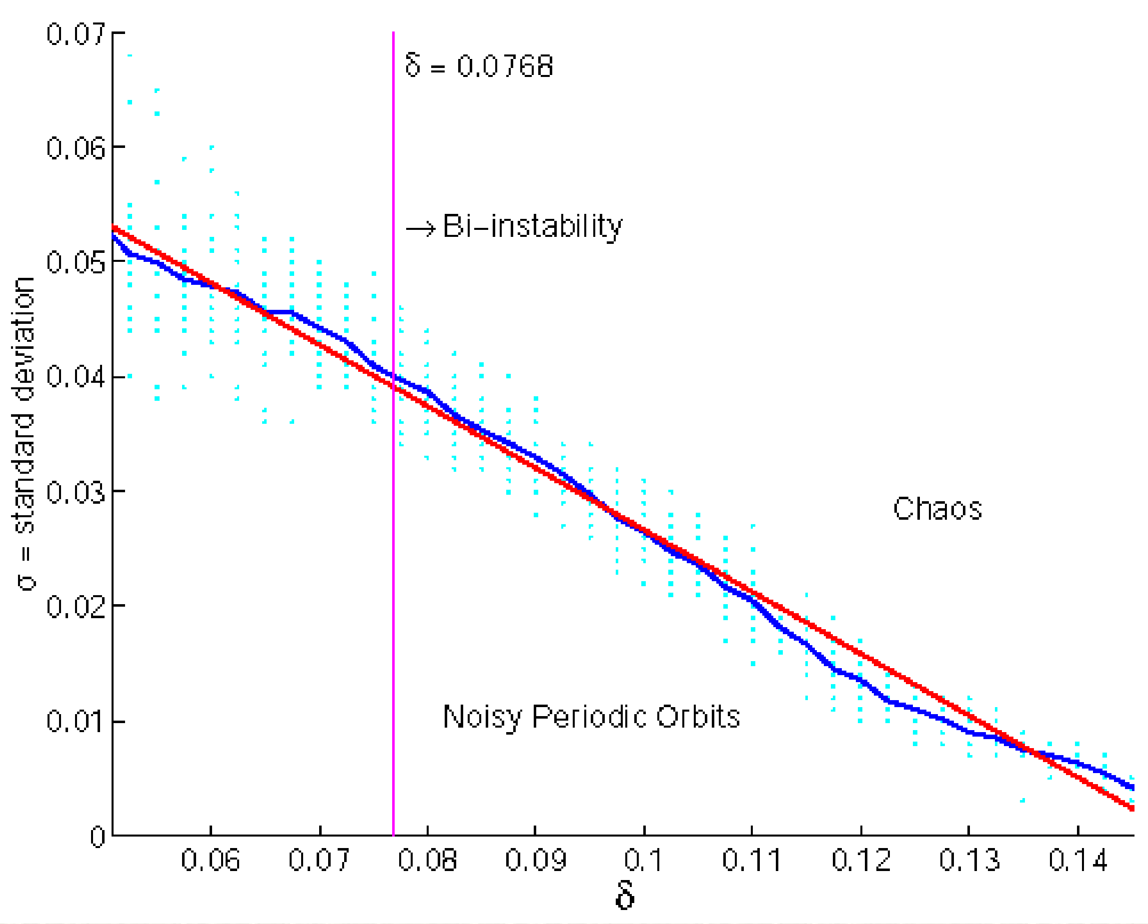
Area Flux Location, STD=0.03
ln(I)
-8
-12
-16
-3.0
-2.7
-2.4
ln(S)
p 1 saddle
p 2 node
p 3 node
p 3 saddle
1.2
1
0.8
0.6
0.4
0.2
0

PDF Flux



PDF Flux Location, STD=0.03
ln(I)
-8
-12
-16
-3.0
-2.7
-2.4
ln(S)
p 1 saddle
p 2 node
p 3 node
p 3 saddle
x 10^{-3}
2
1.5
1
0.5
0

Predicting chaos



Conclusions

- Stochastic perturbations can induce new, emergent dynamics in models
- Chaos-like behavior is induced in both models by additive noise
- We have shown a new stochastic bifurcation to chaos (completion of heteroclinic orbit)
- We have presented a be tool that detects transport (Galerkin matrix)