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COMPLEX NETWORKS: TOPOLOGY, DYNAMICS AND SYNCHRONIZATION

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Dramatic advances in the field of complex networks have been witnessed in the past few years. This paper reviews some important results in this direction of rapidly evolving research, with emphasis on the relationship between the dynamics and the topology of complex networks. Basic quantities and typical examples of various complex networks are described; and main network models are introduced, including regular, random, small-world and scale-free models. The robustness of connectivity and the epidemic dynamics in complex networks are also evaluated. To that end, synchronization in various dynamical networks are discussed according to their regular, small-world and scale-free connections.

Keywords: Complex network; topology; dynamics; synchronization.

1. Introduction

The current study of complex networks is pervading all kinds of sciences today, ranging from physical to biological, even to social sciences.

Generally speaking, a network can be simply defined as a set of interconnected nodes, where a node is a basic element or a fundamental unit with detailed contents depending on the nature of the specific network under consideration.

Nodes in a network can be anything depending on the context of discussion. They can be routers (or subnetworks) in the Internet, which are connected by some physical links such as optical fibers [Faloutsos *et al.*, 1999]; they can be document files (e.g. webpages) in the World Wide Web (WWW), joined by the so-called hyper-links [Albert *et al.*, 1999; Huberman & Adamic, 1999; Barabási *et al.*, 2000a, 2000b; Broder *et al.*, 2000; Tadic, 2001]; they can also be substrates and enzymes in some metabolic networks connected through chemical interactions [Jeong *et al.*, 2000]; they even can be scientific papers in the scientific citation network, where links correspond to citations among different papers [Redner, 1998]. To continue, they can be scientists in the scientific collaboration network, where two nodes are connected if two scientists have joint works such as coauthoring articles [Newman, 2001a, 2001b, 2001c]; they can be individuals, organizations or countries in the social network connected by social interactions [Wasserman & Faust, 1994]. Simply, there are too many examples to list. Among them, there are some typical and representative examples that have recently motivated the scientific community to investigate. Current investigation has focused on some generic features that characterize the formation and topology of various complex networks.

Traditionally, complex networks have been studied in a branch of mathematics, known as the graph theory. A network with complex topology is described by a random graph, and the most widely investigated random graph model was perhaps the one introduced by Erdös and Rényi (ER) [Erdös

& Rényi, 1960]. Although our intuition clearly indicates that many real complex networks are neither completely regular nor completely random, the ER random graph model has dominated scientists' thinking about complex networks for nearly 40 years, due in large part to the absence of detailed topological information about large-scale real networks [Bollobas, 1985].

In the past few years, the computerization of data acquisition and the availability of high computing power have led to the emergence of large databases on complex topology of various real networks, including the above examples. The availability of these huge amount of real data has in turn stimulated great interest in trying to uncover the generic properties of complex networks [Strogatz, 2001; Albert & Barabási, 2001; Dorogovtsev & Mendes, 2001]. In this endeavor, two significant recent discoveries are the small-world effect and the scale-free feature of most complex networks.

In 1998, in order to describe the transition from a regular lattice to a random graph, Watts and Strogatz (WS) introduced the concept of small-world network [Watts & Strogatz, 1998]. It is notable that the small-world phenomenon is very common. An interesting experience is that, oftentimes, soon after meeting a stranger, one is surprised to find that they have a common friend in between. So they both cheer: "What a small world!" Indeed, we all are connected through a short chain of acquaintances. The most popular manifestation of such "small-world effect" is the so-called "six degrees of separation" concept, uncovered by the social psychologist Milgram in the late 1960s [Milgram, 1967]. He concluded, through a simple experiment, that six was the average number of acquaintances separating most pairs of people in the United States. He thereafter conjectured that a similar separation might characterize the relationship of any two people in the human world. In the past few years, the small-world pattern has been shown to be ubiquitous in many real networks, including the above examples and many others, such as the food web [Montoya & Solé, 2000], electronic circuits [Cancho *et al.*, 2001], and even the human language [Cancho & Solé, 2001].

A prominent common feature of the ER random graph and the WS small-world model is that the connectivity distribution of the network peaks at an average value and decays exponentially. Such networks are called "exponential networks." An exponential network is homogeneous in nature: each node has roughly the same number of connections. Another significant recent discovery in the field of complex networks is the observation that a number of large-scale complex networks, including the Internet, WWW, and metabolic networks are scale-free, that is, their connectivity distributions are in the power-law form [Barabási & Albert, 1999; Barabási *et al.*, 1999]. A scale-free network is inhomogeneous in nature: most nodes have very few connections and only a few nodes have many connections.

It is very important to characterize the structure of complex networks. Why is it so? Because structure always affects the function and the behavior of a dynamical system, especially a complex one, which can only be understood by looking at its intrinsic interactions among the massive individual parts. The apparent ubiquity of complex networks leads to a fascinating set of common problems concerning how the network structure facilitates and constraints the network behaviors. For example, it has been argued that the inhomogeneous feature makes the connectivity of a scale-free network error-tolerant but vulnerable to attacks [Albert *et al.*, 2000; Callaway *et al.*, 2000; Tu, 2000; Cohen *et al.*, 2000, 2001].

As the next step towards understanding the dynamical behaviors of complex networks, one may introduce dynamical elements into the network nodes. In recent years, networks of coupled dynamical systems have received a great deal of attention in nonlinear dynamics since they can exhibit many interesting phenomena, such as the Turing patterns, autowaves, spiral waves, and spatio-temporal chaos, and they are important in modeling many large-scale real systems [Chua, 1995]. In particular, over the past decade, the increasing interest in synchronization of chaotic dynamical systems has led many scientists to consider the synchronization phenomenon in large-scale networks of coupled chaotic oscillators (see, e.g. [Heagy *et al.*, 1994; Wu & Chua, 1995; Hu *et al.*, 1998; Pecora *et al.*, 2000]). These networks are usually described by systems of coupled ordinary differential equations or maps, with completely regular topological structures such as chains, grids, lattices, and globally coupled graphs. Two typical cases are the discrete-time coupled map lattice (CML) [Kaneko, 1992] and the continuous-time cellular neural (or nonlinear) networks (CNN) [Chua, 1998]. The main benefit of these simple architectures is that it allows us to focus on the complexity caused by the nonlinear dynamics of the nodes, without considering additional complexity in the network structure itself. Another appealing feature is the possibility of realizing such regularly coupled networks by integrated circuits, with the obvious advantage of compactness.

The topology of a network, on the other hand, often plays a crucial role in determining its dynamical features. For example, despite the evidence that a strong enough diffusive coupling will result in synchronization within an array of identical nodes [Wu & Chua, 1995], this cannot explain why many real-world complex networks exhibit strong tendency toward synchronization even with a relatively weak coupling. As an instance, it was observed that the apparently independent routing messages from different routers in the Internet can easily become synchronized while the tendency for routers towards synchronization may depend heavily on the topology of the Internet [Floyd & Jacobson, 1994]. One way to break up synchronization is for each router to add a (sufficiently large) random component to the period between routing messages. However, the tendency to synchronization in the Internet is so strong that by changing one deterministic protocol to correct the synchronization is likely to generate another synchrony. This suggests that a more efficient solution requires a better understanding of the nature of the synchronization behavior in complex networks. Recently, synchronization in different small-world dynamical network models have been carefully studied, including the small-world network of phase oscillators [Watts, 1999], coupled map lattice with small-world interaction [Gade & Hu, 2000], small-world neural networks [Lago-Fernandez *et al.*, 2000], and small-world network of chaotic oscillators [Wang & Chen, 2002a]. More recently, the synchronizability of a class of continuous-time scale-free dynamical networks has been investigated in detail [Wang & Chen, 2002b]. These studies may shed some new light on synchronization phenomena in real complex networks.

The objective of this article is to review some recent advances in the field of complex networks, with emphasis on the relationship between the dynamics and the topology of a complex network. Section 2 describes three basic quantities of complex networks — the average path length, clustering coefficient, and degree distribution. Three typical examples are discussed — the Internet, WWW and scientific collaboration network. Section 3 then introduces four major kinds of network models — the regular, random, small-world and scale-free models. Robustness of connectivity of complex networks and the spreading property of complex networks are discussed in Secs. 4 and 5, respectively. Section 6 further introduces a class of continuous-time dynamical network models. Synchronization in dynamical networks according to regular, small-world and scale-free connections are studied in Sec. 7, and other related works are briefly summarized in Sec. 8. Conclusions are finally given in Sec. 9, with a sufficient number of references provided at the end of the article.

2. Some Basic Concepts about Complex Networks

Although many quantities and measures of complex networks have been proposed and investigated in the past few years, three spectacular quantities — the average path length, clustering coefficient, and degree distribution — play a key role in the recent development of complex networks theory and modeling. In fact, the original attempt of Watts and Strogatz in their work on small-world networks was to construct some networks with small average path length as random graphs and relatively large clustering coefficients as regular lattices [Watts & Strogatz, 1998]. On the other hand, the discovery of scale-free networks was based on the observation that the degree distributions of many real networks have a power-law form [Barabási & Albert, 1999]. Therefore, these three quantities are of particular importance and so are first being reviewed.

2.1. *The average path length*

In a network, the distance d_{ij} between two nodes i and j is defined as the number of edges along the shortest path connecting them. The *diameter* D of a network, on the other hand, is the maximal distance between any pair of its nodes. The *average path length* L of the network, then, is defined as the distance between two nodes, averaged over all pairs of nodes (Fig. 1). Here, L determines the effective "size" of a network, the typical separation of pairs of nodes. In a friendship network, for instance, L is the average number of friendships in the shortest chain connecting two people. Clearly, $L/D \leq 1$, and $L/D = 1$ if and only if the network is globally coupled in the

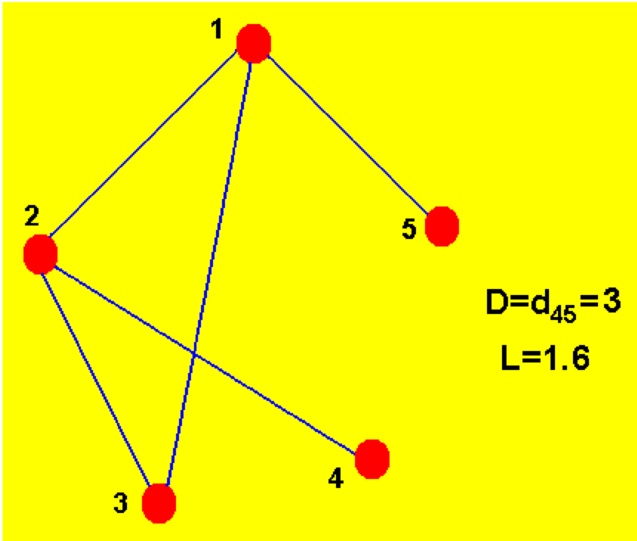


Fig. 1. The diameter and average path length of a simple network consisting of five nodes and five edges.

sense that any two different nodes in the network are connected directly. One of the most important discoveries in the field of complex networks is that the average path lengths of most large-scale real complex networks are relatively small, even though they have much fewer edges than a globally coupled network with equal number of nodes. This smallness is usually referred to as the *small-world effect*.

2.2. *The clustering coefficient*

In your friendship network, it is quite possible that your friend's friends are also your direct friends; or, to put it another way, two of your friends are also friends with one another. This property is called the *clustering* of networks. More precisely, one can define a *clustering coefficient* C as the average fraction of pairs of neighbors of a node that are also neighbors of each other. Suppose that a node i in the network has k_i edges, which connect it to k_i other nodes. These nodes are the neighbors of node i. Clearly, at most $k_i(k_i-1)/2$ edges can exist between them, and this occurs when every neighbor of node i is connected to every other neighbor of node i. The clustering coefficient C_i of node i is defined as the ratio between the number E_i of edges that actually exist between these k_i nodes and the total number $k_i(k_i-1)/2$, namely,

$$C_i = \frac{2E_i}{k_i(k_i-1)}\,. \tag{1}$$

The clustering coefficient C of the whole network is the average of C_i over all i. For friendship networks, C_i reflects the extent to which friends of i are also friends of each other. Thus, C measures the cliquishness of a typical friendship circle. Clearly, $C \le 1$, and $C = 1$ if and only if the network is globally coupled, which means that everyone in the network knows everyone else. In a completely random network, $C \sim N^{-1}$, which is very small as compared to most large networks. It has been found that most large-scale real networks have a tendency to clustering, in the sense that their clustering coefficients are much greater than $O(N^{-1})$, although they are significantly less than one.

2.3. *The degree distribution*

The simplest and most intensively studied one-node characteristic is the concept of *degree*. The degree k_i of a node i is the total number of its connections. The average of k_i over all i is called the *average degree* of the network, and is denoted as $\langle k\rangle$. The spread in node degree is characterized by a distribution function $P(k)$, which gives the probability that a randomly selected node has exactly k edges.

A regular lattice has a simple degree sequence. All the nodes have the same number of edges, and so a plot of the degree distribution contains a single sharp spike. Any randomness in the network will broaden this peak. In the limiting case of a completely random network, the degree sequence obeys the Poisson distribution, with

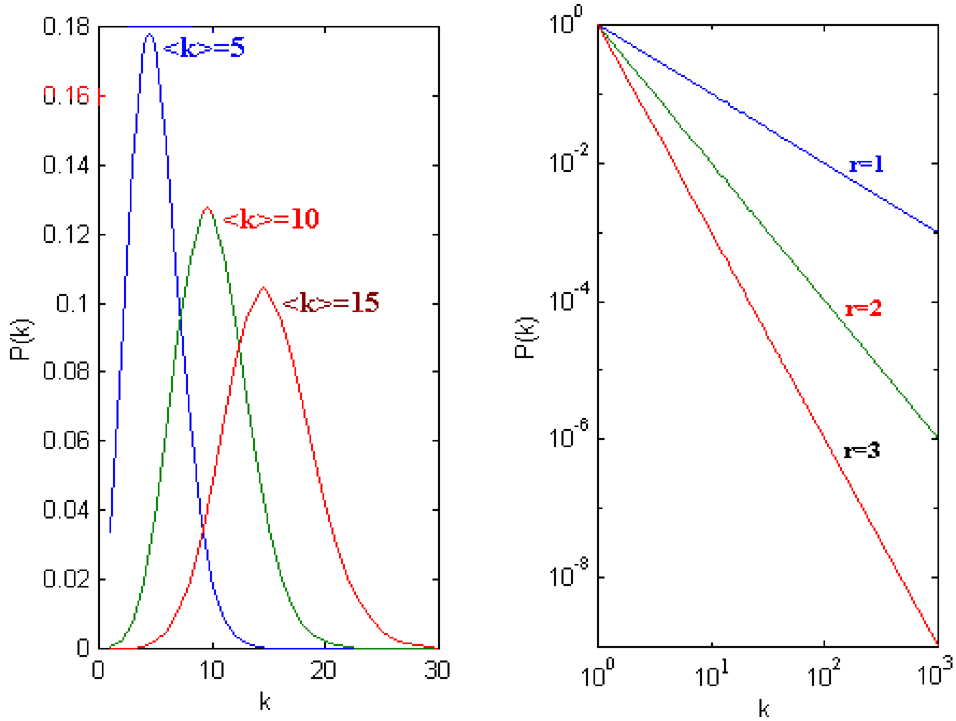
$$P(k) \simeq e^{-\langle k\rangle}\frac{\langle k\rangle^k}{k!}\,. \tag{2}$$

The shape of the Possion distribution falls off exponentially, away from the peak value $\langle k\rangle$. Because of this exponential decline, the probability of finding a node with k edges becomes negligibly small for $k \gg \langle k\rangle$ [Fig. 2(a)].

In the past few years, a lot of empirical results had showed that for most large-scale real networks the degree distribution significantly deviates from the Possion distribution. In particular, for a number of networks, including the WWW, Internet, and metabolic networks, the degree distribution is more accurately described by a power law of the form

$$P(k) \sim k^{-\gamma}\,. \tag{3}$$

This power-law distribution falls off more gradually than an exponential one, allowing for a few nodes of



(a) (b)

Fig. 2. (a) Guassian degree distribution with $\langle k \rangle = 5$, 10 and 15, respectively. (b) Power-law degree distribution with $r = 1$, 2 and 3, respectively.

very large degree [Fig. 2(b)]. For example, suppose that there is a network with $N = 100\,000$ nodes that obeys a power-law degree distribution with exponent 2. Then, one-ninth of the nodes, or about 11 111 nodes, will be triplely linked, and one ten-thousandth of the 100 000 nodes — a total of 10 — will have 100 links. Since power-laws are free of a characteristic scale, a network with a power-law degree distribution is called a "scale-free network."

2.4. *Three typical examples*

Three representative examples are discussed in this section — the familiar Internet, WWW, and scientific collaboration networks.

2.4.1. *The Internet*

Without a doubt, the Internet and WWW are two of the greatest inventions of our time. The sustained explosive growth of the Internet and WWW over the past decade has made them a part of our life and the modern human society (Fig. 3). They have changed the way we do business, communication, entertainment, retrieving information, and even our education. Perhaps we can hardly imagine what life was like without such huge artificial networks in our real world today.

However, given the crucial role that the Internet and WWW play in our world, we know them much less than one may believe. Due to the Internet's structure complexity, simulating how the global Internet behaves has been an immensely challenging undertaking [Floyd & Paxson, 2001]. The Internet is designed to operate over different underlying communication technologies, including those yet to be introduced, and to support multiple and evolving applications and services. Two factors shield the users of the Internet from the complex reality of the network: the use of the Internet Protocol (IP) to communicate across different kinds of networks, and the efforts of the companies and organizations that operate the Internet's different networks to keep its elements interconnected. This has enormous advantages for users who need not worry about the complexities of the networks that they are using. However, the fact that IP masks the vast heterogeneity of the Internet from a user's perspective does not make them go away! IP gains uniform connectivity at the price of diversity, but not its uniformness. Indeed, the greater the IP is

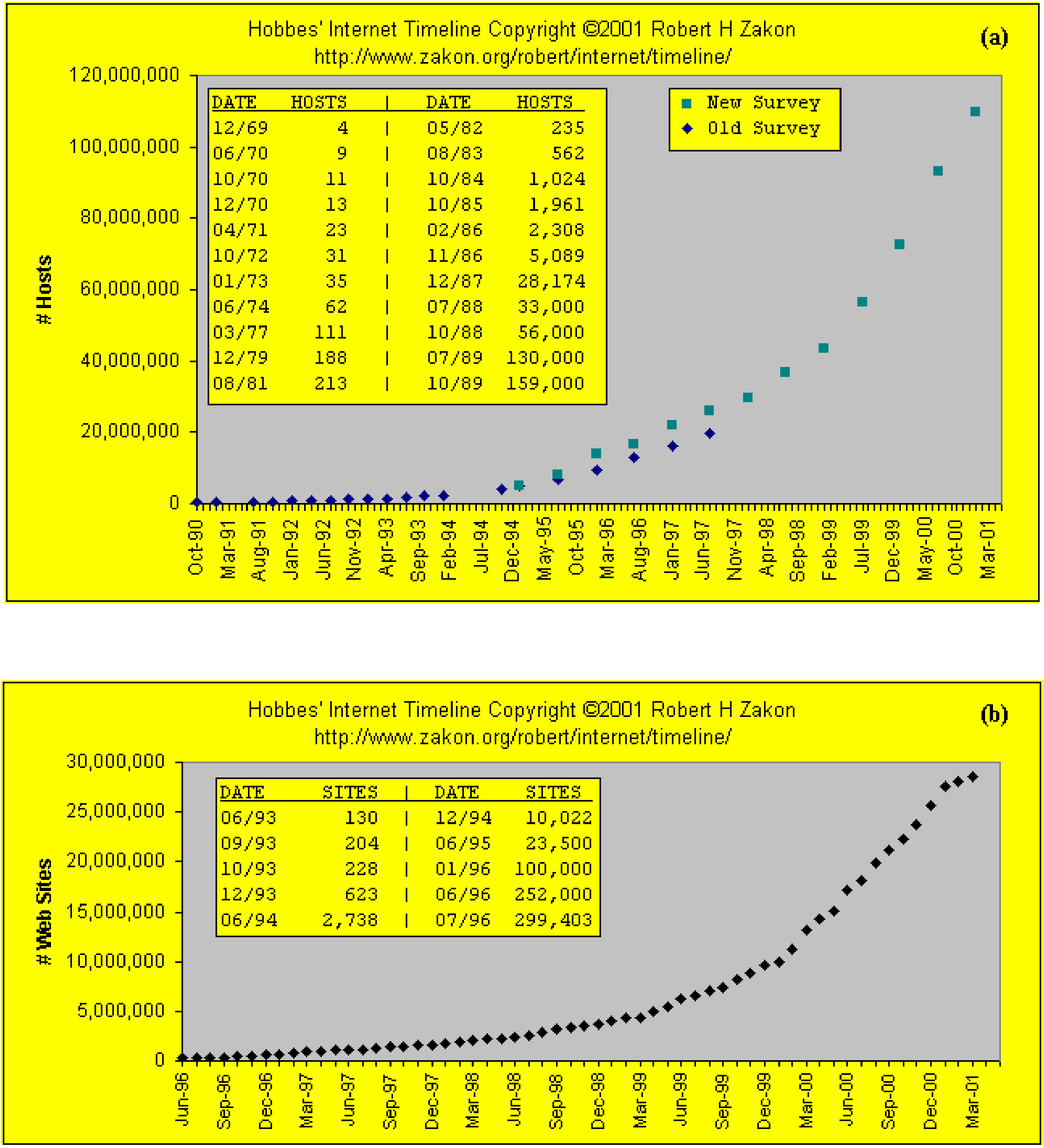


Fig. 3. Internet hosts and WWW websites growth.

successful in unifying diverse networks, the harder is the problem of understanding how a large IP network behaves.

The topology of the Internet has been studied at two different levels (Fig. 4). At the router level the nodes are the routers, and edges are the physical connections between them. At the interdomain (or autonomous system) level each domain, composed of hundreds of routers and computers, is represented by a single node, and an edge is drawn between two domains if there is at least one route that connects them. Although a few Internet

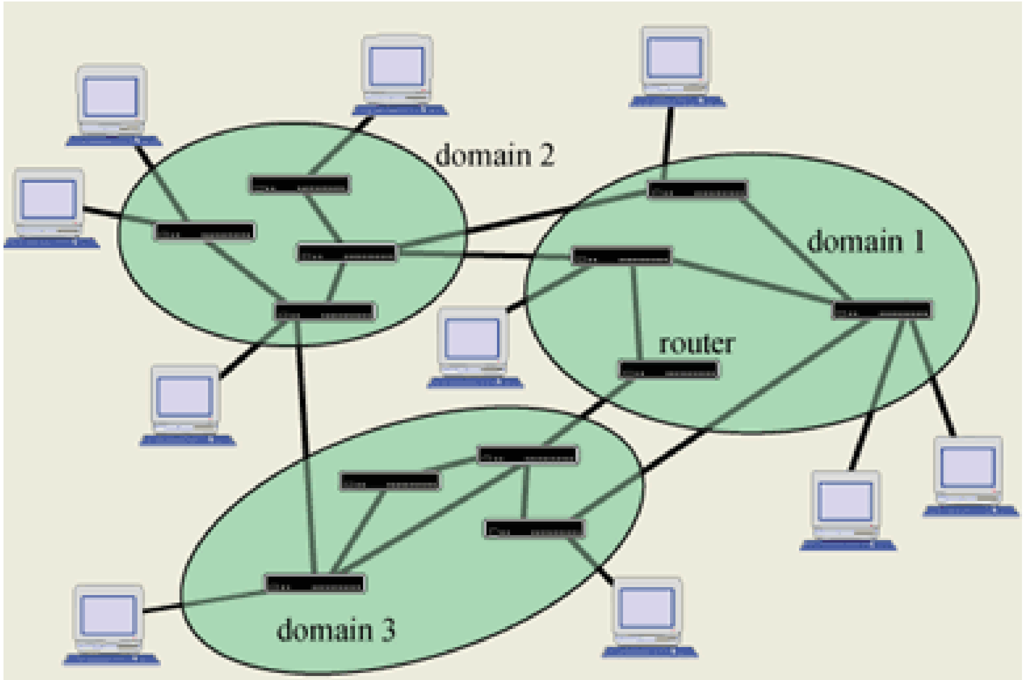


Fig. 4. The Internet is a network of routers that navigate packets of data from one computer to another. The routers are grouped into domains.

topology generators have been proposed in recent years [Calvert *et al.*, 1997], our ability to design good topology generators is limited by the lack of good understanding of the basic mechanisms that shape the Internet's large-scale topology. In 1999, Faloutsos *et al.* [1999] studied the degree distribution of Internet at both levels, concluding that in each case the degree distribution follows a power-law of the form $P(k) \sim k^{-\gamma}$. The interdomain topology of the Internet, captured on three different days between 1997 and the end of 1998, resulted in a degree exponent $\gamma_{I^d} \approx 2.2$. The 1995 survey of the Internet topology at the router level, containing 3888 nodes, found that $\gamma_{I^r} \approx 2.48$. The discovery of the scale-free feature of the Internet has motivated the development of a new brand of Internet topology generators [Medina *et al.*, 2000; Jin *et al.*, 2000; Yook *et al.*, 2001b].

Pastor-Satorras *et al.* [2001] studied the clustering and small-world effect of the Internet at the domain level. Between 1997 and 1999, the clustering coefficient of the Internet ranged between $C_I = 0.18$ and $C_I = 0.3$, to compare with $C_{\text{rand}} \approx 0.001$ for random networks of similar parameters. The average path length of the Internet at the domain level was found to be about 4, compared to $L_{\text{rand}} \approx 10$ for the corresponding random graph.

2.4.2. *The WWW*

The WWW represents a large-scale real network, the largest of its kind today, for which topological information is currently available. The structure of the WWW was researched experimentally by several groups worldwide. These studies have covered different subgraphs of the Web. The nodes of the network are the webpages and the edges are the hyperlinks that point from one page to another (Fig. 5). The small-world effect in the WWW was first reported by Albert *et al.* [1999]. They developed a software, "robot," that traversed the WWW and collected all the links from a webpage and then followed them to their destinations. The process was repeatedly involving 325729 webpages, with a picture of the interconnectivity of the WWW emerged. They found that the average path length for this network was 11.2, and they predicted, using a finite-size scaling, that for the full WWW of 800 million nodes in 1999 the average path length would be around 19.

Perhaps the WWW's biggest problem is a result of its success, quite ironically. There is so much information out there that it is often very hard to find what you want. The small-world property of the WWW may prove useful to search engine companies who would like to develop a new type of

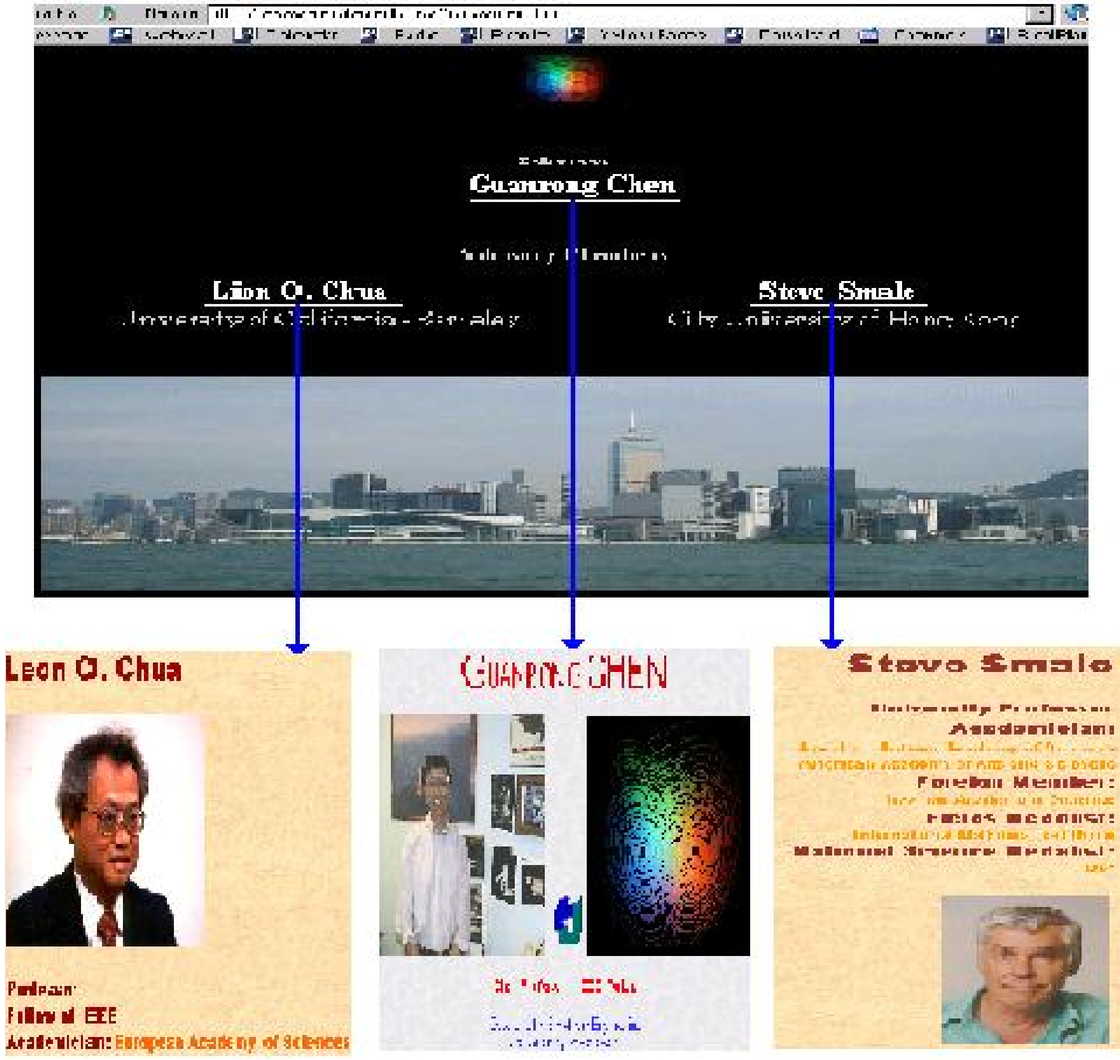


Fig. 5. The nodes of the WWW are webpages, each of which is identified by a unique uniform resource locator (URL). Most webpages contain URLs that link to other pages. These URLs represent outgoing links, three of which are shown for the webpage, http://www.ee.cityu.edu.hk/∼cccs/main.htm (blue arrows). The number of incoming links of a webpage characterizes the popularity of the page.

search engine that is much more powerful and efficient. The new intelligent search engine is expected to outperform all the conventional search engines, which index the text of many individual pages and use them to find specified terms.

The WWW is characterized by two degree distributions: the distribution of outgoing edges, $P_{\text{out}}(k)$, signifies the probability that a webpage has k outgoing hyperlinks, and the distribution of incoming edges, $P_{\text{in}}(k)$, is the probability that k hyperlinks point to a certain webpage. Both the in- and out-degree distributions are found to be in a power-law form:

$$P_{\text{in}}(k) \sim k^{-\gamma_{\text{in}}}; \quad P_{\text{out}}(k) \sim k^{-\gamma_{\text{out}}}. \tag{4}$$

As mentioned, Albert *et al.* [1999] studied a subset of the WWW containing 325729 nodes, while Kumar *et al.* [1999] investigated a 40-million document crawl prepared by the Alexa Inc. Broder *et al.* [2000] further studied 200-million document obtained by two 1999 Altavista crawls. These studies have showed that the in-degree distribution of the WWW is $\gamma_{\text{in}} = 2.1$, while its out-degree distribution is ranged somewhere in between $\gamma_{\text{out}} = 2.38$ and $\gamma_{\text{out}} = 2.72$.

The directed nature of the WWW does not allow us to measure the clustering coefficient using Eq. (1). One way to avoid this difficulty is to let the network be undirected, making each edge bidirectional. It was found that the clustering coefficient is much higher than that of a random graph of the same sizes and edges, although it is still significantly less than 1.

2.4.3. *The scientific collaboration network*

In a scientific collaboration network, the nodes are scientists and a link between two scientists is established by their coauthorship of one or more scientific papers.

The idea of constructing a network of coauthorship is certainly not new. Many mathematicians have the concept of the *Erdös number*, named after Paul Erdös (1913–1996), one of the most prolific mathematicians in history. Reportedly, Erdös thought and wrote mathematics for nineteen hours a day in average until the day he died [Hoffman, 1998]. He published over 1500 papers with over 500 co-authors. It has become a popular cocktail party pursuit for mathematicians to calculate how far away they were in terms of coauthored publications from Erdös.[1] The Erdös number of a scientist measures how many coauthorship steps the scientist has from Paul Erdös. Those who had published a paper with Erdös were given an Erdös number of 1. Those who had published a paper with one of Erdös' coauthors but not with Erdös himself were given an Erdös number of 2, and so on. If there is no chain of coauthorships connecting someone with Erdös, then that person's Erdös number is said to be infinite. The set of all mathematicians with a finite Erdös number is called the *Erdös component* in the whole mathematics research collaboration network, which has all mathematicians as its nodes. Given this construction, it has been conjectured that the Erdös component contains almost all publishing mathematicians alive today, except perhaps those who just published papers on their own. The average Erdös number of the Erdös component was calculated to be about 4.7, and the maximum finite Erdös number is known to be only 15. The present author, for example, has an Erdös number of 3, via Charles Kam-Tai Chui and Guanrong Chen [Borosh *et al.*, 1978; Chui & Chen, 1987; Wang & Chen, 1999].

Recently, Newman [2001a, 2001b, 2001c] constructed some collaboration networks for scientists in a variety of fields. He investigated four databases:

(i) *Los Alamos e-Print Archive*: a database of unrefereed preprints in physics, self-submitted by their authors, running from 1992 to the present. This database is subdivided into specialties within physics, such as astro-physics (*astro-ph*), condensed matter (*cond-mat*) and high-energy physics (*hep-th*).
(ii) *Medline*: a database of articles on biomedical research published in refereed journals, stretching from 1961 to the present.
(iii) *Stanford Public Information Retrieval System* (*SPIRES*): a database of preprints and published papers in high-energy physics running from 1974 to the present.
(iv) *Network Computer Science Technical Reference Library* (*NCSTRL*): a database of preprints in computer science, submitted by participating institutions and stretching back about ten years.

In each case, Newman examined the papers that appeared within a five year window from 1995 to 1999 inclusively. The size of the databases ranges from 2-million papers for MEDLINE to 13000 for NCSTRL. More recently, Barabási *et al.* [2001] investigated the collaboration network of mathematicians and neuroscientists. The databases contain article titles and authors of all relevant journals in the field of mathematics and neuro-science, published between 1991 and 1998. In mathematics, the database contains 70905 different authors and 70901 papers; in neuro-science, the number of different authors is 209293 and the number of papers is 210750. All these networks show small average path length, indicating the small-world effect of these scientific collaboration networks. These networks are also highly clustered, meaning that two scientists are much more likely to have collaborated — if they have a third common collaborator — than are two scientists chosen at random from the community. This may indicate that the process of scientists introducing their collaborators to one another is important to the development of scientific communities. The degree distribution $P(k)$ of a scientific collaboration network represents the probability that a scientist has k collaborators. The degree distribution for the SPIRES, the mathematics or neuroscience database fits a power law well, while the degree distribution corresponds to the Medline, the NCSTRL, or the Los Alamos Archive is well fitted by an exponentially truncated power

[1]For the Erdös Number Project, see http://www.oakland.edu/~grossman/Erdöshp.html

law of the form, $P(k) \sim k^{-\tau} e^{-k/k_c}$, where τ and k_c are constants.

3. Models of Some Complex Networks

3.1. *Regular coupled networks*

Measuring properties of a complex network such as the average path length L, the clustering coefficient C, and the degree distribution $P(k)$, is a first step toward understanding its structure. The next step, then, is to develop a mathematical model with a topology of similar statistical properties. It is known that a globally coupled network has the smallest average path length $L_{\rm gc} = 1$ and the largest clustering coefficient $C_{\rm gc} = 1$ [Fig. 6(a)]. Although globally coupled network models capture the small-world and clustering properties of many real networks, it is easy to notice its limitations: a globally coupled network with N nodes has $N(N-1)/2$ edges, while most large-scale real networks tend to be sparse, that is, the number of edges in a real network is generally of order N rather than N^2.

An widely studied, sparse, and regular network model is the nearest-neighbor coupled network (lattices), which is a regular graph in which every node is joined to a few neighbors. The term lattice may raise the impression about a two-dimensional square grid, but actually it can have other geometries. A minimal lattice is a one-dimensional structure, like a row of people holding hands. A nearest-neighbor lattice with a periodic boundary condition consists of N nodes arranged in a ring, where each node i is adjacent to its neighboring nodes $i \pm 1, i \pm 2, \ldots, i \pm K/2$, with K an even integer [Fig. 6(b)]. For a large K, such a network is highly clustered: the clustering coefficient of the nearest-neighbor coupled network is

$$C_{\rm nc} = \frac{3(K-2)}{4(K-1)} \sim \frac{3}{4}\,. \qquad (5)$$

However, the nearest-neighbor coupled network is not a small-world network. On the contrary, its average path length is

$$L_{\rm nc} \sim \frac{N}{2K} \to \infty, \quad \text{as} \quad N \to \infty\,. \qquad (6)$$

This may help explain why it is difficult to achieve any dynamical process (like synchronization) that requires global coordination in such a locally coupled network.

The degree distribution of a nearest-neighbor coupled network is a delta function centered at K, i.e.

$$P(k) = \begin{cases} 1, & k = K\,; \\ 0, & k \neq K\,. \end{cases}$$

Does there exist a regular network that is sparse and clustered, and has a small average path length? The answer is *Yes*. An simple example is a star coupled network [Fig. 6(c)], in which there is a center node and each of the $N-1$ other nodes only connect to this center. Its average path length is

$$L_{\rm star} = 2 - \frac{2(N-1)}{N(N-1)} \to 2, \quad \text{as} \quad N \to \infty\,, \qquad (7)$$

and its clustering coefficient is

$$C_{\rm star} = \frac{N-1}{N} \to 1, \quad \text{as} \quad N \to \infty\,. \qquad (8)$$

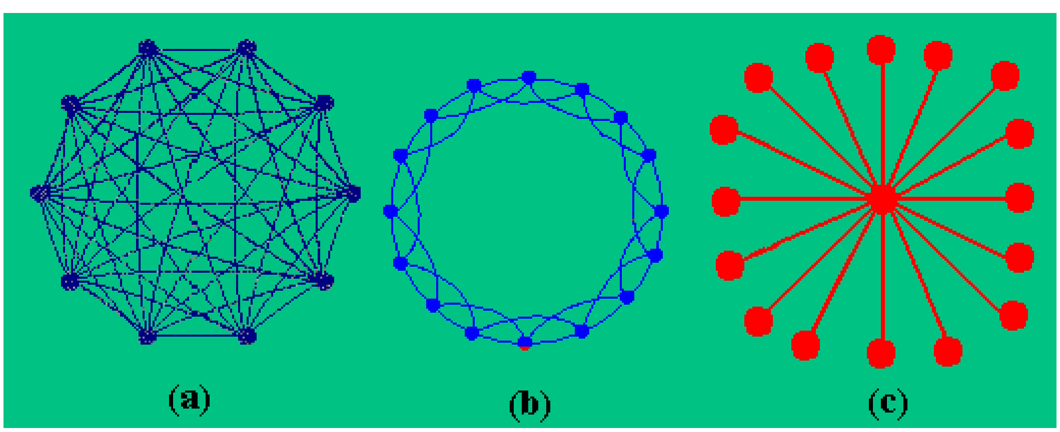


Fig. 6. (a) Globally coupled network; (b) nearest-neighbor coupled network; (c) star coupled network.

The star model captures the sparse, clustering, small-world, and some other properties of many real networks. In this sense, it is better than the regular lattice as a model of real networks.

3.2. *Random graphs*

At the opposite end of the spectrum from a completely regular network is a network with a completely random graph, which was first studied by Erdös and Rényi [1960] about 40 years ago. Try to imagine that you have a large number ($N \gg 1$) of buttons scattered on a floor. With the same probability p, you tie every pair of buttons with a thread. The result is a physical example of an ER random graph with N nodes and about $pN(N-1)/2$ edges (Fig. 7). The main goal of random graph theory is to determine at what connection probability p will a particular property of a graph most likely arise. The greatest discovery of Erdös and Rényi was that many important properties of random graphs appear quite suddenly. For example, if you lift up a button, how many other buttons will you pick up thereby? Erdös and Rényi showed that if the probability p is greater than a certain threshold, $p_c \sim \ln(N)/N$, then almost every random graph is connected, which means that you will pick up all the buttons on the floor by randomly lifting up just one button.

The average degree of the random graph is

$$\langle k \rangle = p(N-1) \cong pN\,. \tag{9}$$

Let L_{rand} be the average path length of a random network. Intuitively, about $\langle k \rangle^{L_{\mathrm{rand}}}$ nodes of the random network are at a distance L_{rand} or closer to it. Hence, $N \sim \langle k \rangle^{L_{\mathrm{rand}}}$, which means that

$$L_{\mathrm{rand}} \sim \frac{\ln(N)}{\ln(\langle k \rangle)}\,. \tag{10}$$

This logarithmic increase in the average path length with the size of the network is typical of the small-world effect. Since $\log N$ increases slowly with N, it allows the average path length to be quite small even in a very large system.

On the other hand, in a completely random network, the probability that two of one's friends will be friends of one another is no greater than the probability that two randomly chosen people will be so. Hence, the clustering coefficient of the ER model is

$$C = p = \frac{\langle k \rangle}{N} \ll 1\,. \tag{11}$$

This means that a large-scale random network does not show clustering.

For a large N, the ER algorithm generates a homogeneous network, whose connectivity approximately follows a Poisson distribution described by Eq. (2). However, the degree distribution of most real networks deviates significantly from the Possion distribution of the random graph. To capture the degree distribution of real networks, one can generalize the ER random graph by constructing random graphs with given degree distributions. Such models are called *generalized random graphs* [Molloy & Reed, 1995; Newman *et al.*, 2001]. In particular, a general approach to random graphs with given degree distributions has been developed, by Newman *et al.*, using a generating function formalism [Newman *et al.*, 2001].

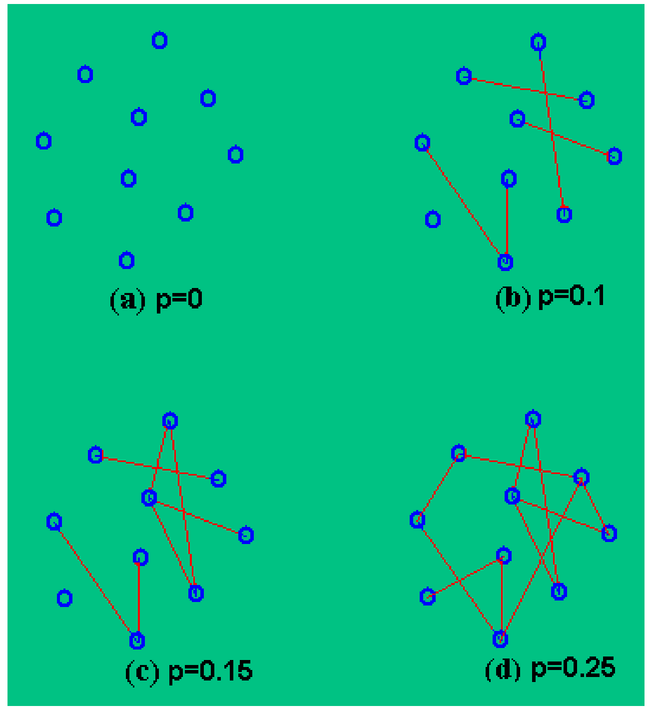


Fig. 7. Evolution of a random graph. Given $N = 10$ isolate nodes (a), one connects every pair of nodes with probability (b) $p = 0.1$, (c) $p = 0.15$ and (d) $p = 0.25$, respectively.

3.3. *Small-world networks*

The regular lattice is clustered, but does not exhibit the small-world effect. On the other hand, the random graph shows the small-world effect, but does

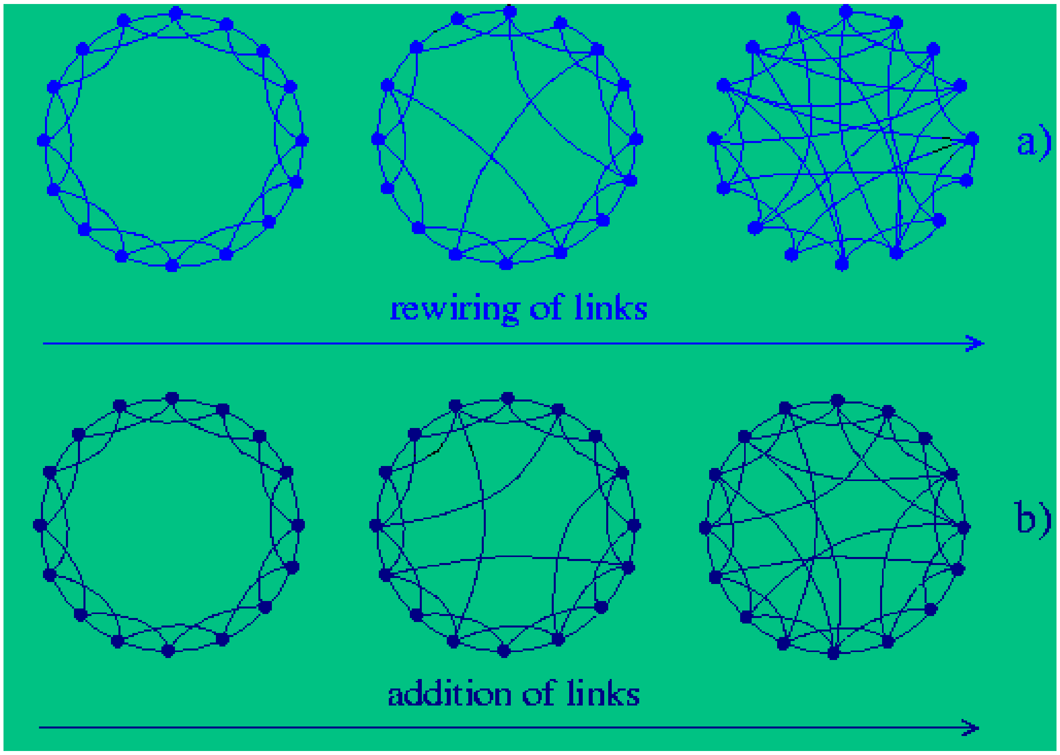


Fig. 8. Small-world models. (a) The original Watts–Strogatz model with the rewiring of links. (b) The Newmann–Watts model with the addition of links.

not show clustering. It is not surprising that the regular lattice model and the ER random model fail to reproduce some important features of real networks such as the human friendship network or the World Wide Web. After all, these real networks are neither entirely regular nor entirely random. People generally know their neighbors, but their circle of acquaintances may not be confined to those who live in the next door, as the lattice model would imply. Conversely, links between pages on the Web are certainly not created at random, as the Erdös–Rényi process requires.

Aiming to describe a transition from a regular lattice to a random graph, Watts and Strogatz [1998] introduced an interesting small-world network model, now referred to as WS small-world model, as already mentioned. The original WS model can be described as follows [Fig. 8(a)].

(i) *Start with order*: Start with a nearest-neighbor coupled network consisting of N nodes arranged in a ring and each node i is adjacent to its neighbor nodes $i \pm 1$, $i \pm 2, \ldots, i \pm K/2$, where K is even. In order to have a sparse but connected network at all times, consider $N \gg K \gg \ln(N) \gg 1$ in general.

(ii) *Randomize*: Randomly rewire each edge of the network with probability p. Rewiring in this context means shifting one end of the connection to a new node chosen at random from the whole network, with the constraint that no two different nodes can have more than one connection in between, and no node can have a connection with itself. This process introduces $pNK/2$ long-range edges, which connect nodes that otherwise would be part of different neighborhoods. Varying p the transition between order ($p = 0$) and randomness ($p = 1$) can be closely monitored.

The pioneering work of Watts and Strogatz started an avalanche of research on variants of the WS small-world models and their properties. A much studied variant of the WS model was the one proposed by Newman and Watts [1999a, 1999b], which is referred to as the NW small-world model lately. In the NW model, one does not break any connection between any two nearest neighbors, but adds with probability p a connection between a pair of nodes [Fig. 8(b)]. Likewise, one does not allow a node to be coupled to another node for more than once, or to couple a node with itself. For $p = 0$, the NW model reduces to the originally nearest-neighbor coupled network; for $p = 1$, it becomes a globally coupled network. The NW model is somewhat easier to analyze than the original WS model because it does not lead to the formation of isolated clusters, whereas this can happen in the WS model. For sufficiently small p and large N, the NW model is equivalent to the WS model. These two models are commonly referred to as small-world models today.

The small-world models have their roots in social networks, where most people are friends with their immediate neighbors, for example neighbors on the same street or colleagues in the same office. On the other hand, many people have a few friends who are far away, such as friends in other countries, which are represented by the long-range edges obtained by a rewiring procedure in the WS model or by an adding procedure in the NW model.

Many analytical and numerical results have been carried out on small-world models. For a summary of the most important results, see [Newman, 2000]. In particular, Watts and Strogatz [1998] investigated the behavior of clustering coefficient $C(p)$ and the average path length $L(p)$ of the WS small-world model as a function of the rewiring probability p. A regular ring lattice ($p = 0$) is highly clustered ($C(0) \simeq 3/4$) but has a large average path length ($L(0) \simeq N/2K \gg 1$). Watts and Strogatz found that, for the small probability of rewiring, when the local properties of the network are still nearly the same as those for the original regular network, and when the clustering coefficient does not differ subsequently from its initial value ($C(p) \sim C(0)$), the average path length drops rapidly and is in the same order as the one for random networks (i.e. $L(p) \gg L(0)$) (Fig. 9). This result is in fact quite natural. Intuitively, it is sufficient to make several random rewirings to decrease average path length significantly. On the other hand, several rewired links cannot crucially change the local clustering property of the network. It is believed that the average path length $L_{\rm sw}$ of a small-world network obeys the general scaling form [Barrat & Weigt, 2000]

$$L_{\rm sw}(N, p) \sim \frac{N}{K} f(pKN^d)\,, \qquad (12)$$

where d is the dimension of the original regular lattice and $f(u)$ is a universal scaling function

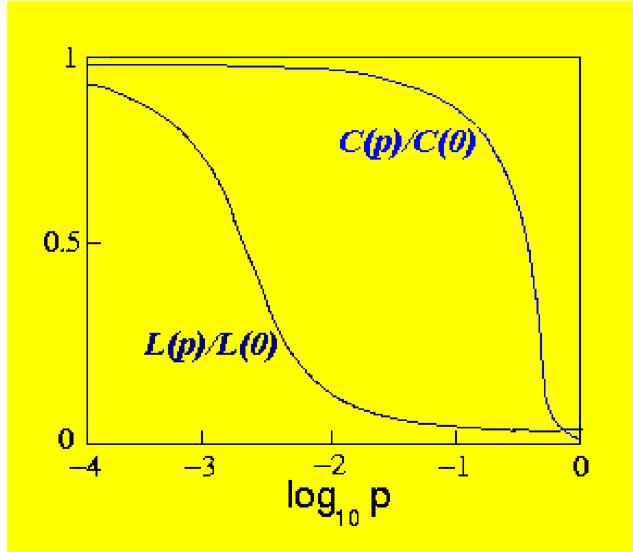


Fig. 9. Average path length and clustering coefficient of the Watts–Strogatz small-world model as a function of the rewiring probability p. Both are normalized to their values for the original regular lattice ($p = 0$) (after [Watts & Strogatz, 1998]).

satisfying

$$f(u) = \begin{cases} \text{constant} & \text{if} \quad u \ll 1 \\ \ln(u)/u & \text{if} \quad u \gg 1 \end{cases}$$

In the WS model, for $p = 0$ each node has the same degree K, thus the degree distribution is a delta function centered at K. A nonzero p introduces disorder into the network, broadening the degree distribution while maintaining the average degree to be equal to K. It has been found that, for large N, the degree distribution of small-world models is

$$P(k) = \sum_{n=0}^{f(k,K)} C_{K/2}^{n}(1-p)^n p^{K/2-n} \times \frac{(pK/2)^{k-K/2-n}}{(k-K/2-n)!} e^{-pK/2}\,, \qquad (13)$$

for $k \geq K/2$, where $f(k, K) = \min(k - K/2, K/2)$. The shape of the degree distribution of a small-world network is similar to that of a Possion distribution: it has a peak at $\langle k \rangle = K$ and decays exponentially for large k (Fig. 10). This means that the small-world model can also be viewed as a homogeneous network, in which all nodes have approximately the same number of edges.

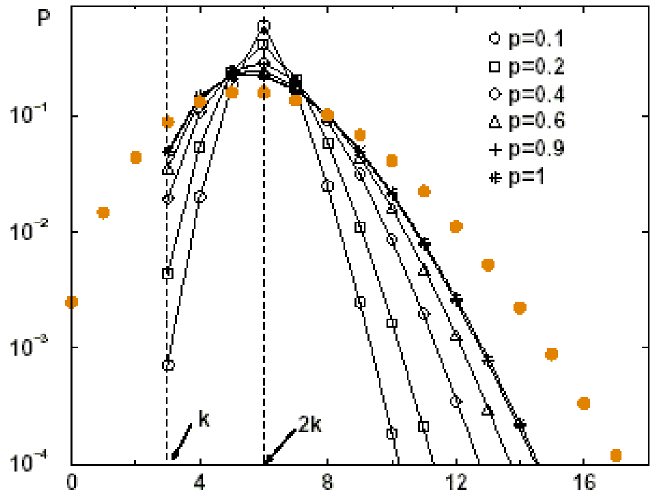


Fig. 10. Degree distribution of the WS small-world model for $K = 3$ and various rewiring probability p (after [Barrat & Weigt, 2000]).

3.4. *Scale-free networks*

A common feature of the ER random graph and the WS small-world model is that the connectivity distribution of the network peaks at an average value and decays exponentially. Such networks are called *exponential networks*. A significant recent discovery in the field of complex networks is the observation that a number of large-scale complex networks, including the Internet, WWW and metabolic networks, are scale-free, that is, their connectivity distributions have the power-law form [Barabási & Albert, 1999; Barabási *et al.*, 1999].

To explain the origin of power-law degree distribution, Barabási and Albert [1999] proposed a new network model. They argued that other models fail to take into account two important attributes of real networks. In the first place, most real networks are open and they form by the continuous addition of new nodes to the system, but most models are static: Although edges can be added or rearranged, the number of nodes is never changed. For example, the Web is continually sprouting new webpages, and the research literature constantly grows since new papers are continuously being published. Second, both the random graphs and the small-world models assume uniform probabilities when creating new edges, but this is not realistic either. Intuitively, webpages that already have many links (such as the homepage of Yahoo! or CNN) are more likely to acquire even more links. A new manuscript is more likely to cite a well-known and thus much cited paper than many other less known ones. One example in point is that, according to the Science Citation Index (SCI), until August 20, 2001 there has been a total of 2392 papers in the field of chaos control and synchronization, and two well-known papers — [Ott *et al.*, 1990] and [Pecora & Carroll, 1990] — have been cited 1209 and 869 times, respectively. This is the so-called "rich gets richer" phenomenon.

Barabási and Albert suggested that two ingredients of self-organization of a network in a scale-free structure are growth and preferential attachment [Barabási & Albert, 1999]. These refer to the facts that networks continuously grow by the addition of new nodes and new nodes are preferentially attached to existing nodes with large numbers of connections (again, "rich gets richer"). The algorithm of the BA scale-free model is the following (Fig. 11):

(i) *Growth*: Starting with a small number (m_0) of nodes, at every time step a new node is introduced and is connected to m ($\leq m_0$) already-existing nodes.
(ii) *Preferential attachment*: When choosing the nodes to which the new node connects, it is assumed that the probability Π_i that a new node will be connected to node i depends on the degree k_i of node i, in such a way that

$$\Pi_i = \frac{k_i}{\sum_j k_j}\,. \tag{14}$$

After t time steps, this algorithm results in a network with $N = t + m_0$ nodes and mt edges. Growing according to these two rules, the network evolves into a scale-invariant state: The shape of the degree distribution does not change over time. The degree distribution is described by a power law with exponent 3:

$$P(k) \sim 2m^{1/\beta}k^{-\gamma}, \quad \text{with} \quad \gamma = \frac{1}{\beta} + 1 = 3\,. \tag{15}$$

In other words, the probability of finding a node with k edges is proportional to the power term k^{-3}.

Numerical results have indicated that, compared to a random graph with the same size and average degree, the average path length of the scale-free model is smaller, and yet the clustering coefficient is much higher (Fig. 12). This implies that the existence of a few "big" nodes with very large degrees plays a key role in bringing the nodes of the network close to each other. However, there is

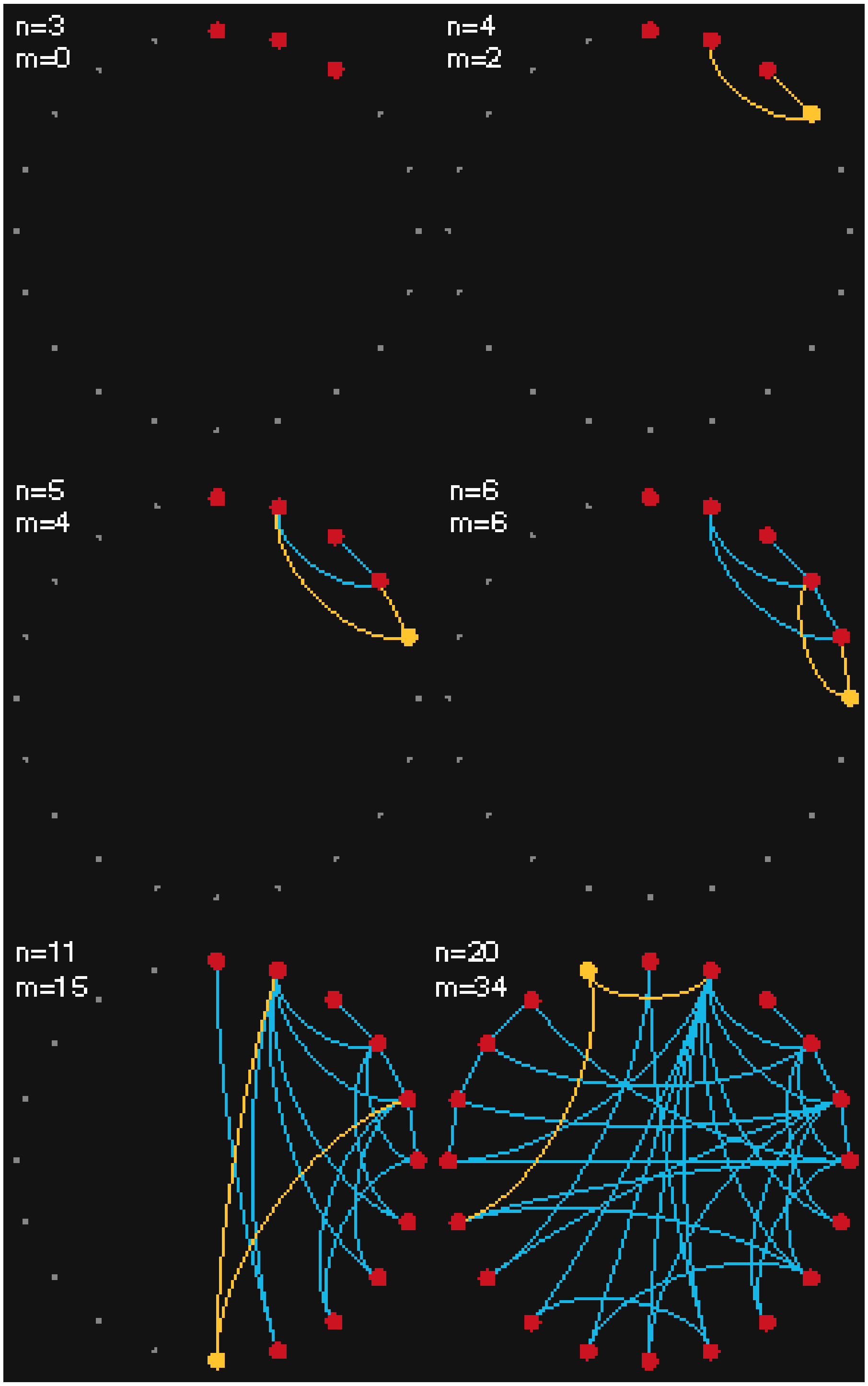


Fig. 11. Evolution of the Barabasi–Albert scale-free model. $m_0 = 3$; $m = 2$ (after [Hayes, 2000]).

no analytical prediction formula for average path length and clustering coefficient for the scale-free model today.

The BA scale-free model is based on linear growth and linear preferential attachment, and it has a power-law degree distribution with a fixed exponent, 3. However, for a number of real networks, the degree distribution is a power-law distribution with an exponent between 1 and 3. Also, the degree distribution of many real networks can have non-power-law features, like exponential cutoffs or a saturation for a small k [Amaral *et al.*, 2000].

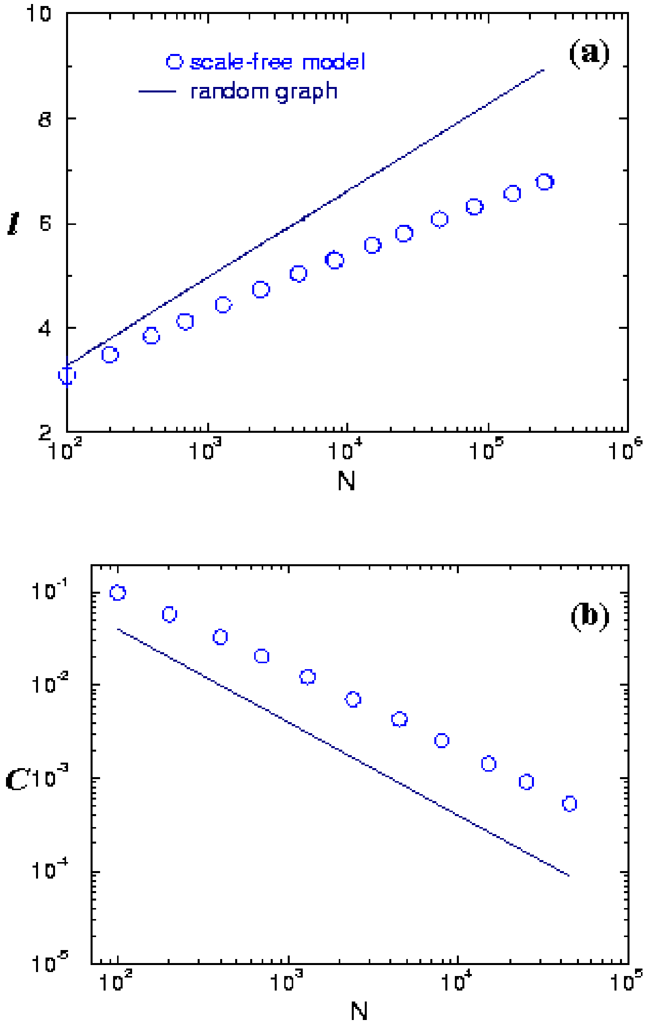


Fig. 12. Average path length and clustering coefficient of a Barabasi–Albert scale-free model and a random graph with the same size and same average degree $\langle k \rangle = 4$ (after [Albert & Barabási, 2001]).

Recently, variants of scale-free models have been proposed by taking into account more features of real networks, such as nonlinear growth and preferential attachment [Krapivsky *et al.*, 2000], the addition or rewiring of new edges or the removal of nodes and edges [Albert & Barabási, 2000]. These models are called *evolving networks*. A summary of the current evolving network models is given in Table III of [Albert & Barabási, 2001].

3.5. *Other network models*

3.5.1. *Weighted evolving network models*

Many biological, ecological, and economic systems are best described by weighted networks, as the nodes interact with each other through varying coupling strength. Recently, Yook *et al.* [2001a] introduced a class of weighted evolving network models. The weighted scale-free (WSF) model can be described as follows. Starting from a small number (m_0) of nodes, at every time step a new node is introduced and is connected to m ($\leq m_0$) already-existing nodes. The probability that a new node j will connect to an existing node i is given by (14). In assigning a weight to the newly established link $j \to i$, assume that the weight w_{ji} ($= w_{ij}$) is proportional to the degree k_i of node i, that is, a more connected node gains a larger weight. Also, one can assume that all new nodes have fairly uniform total "resources" for linking to other nodes in the system. Therefore, one may normalize w_{ij} such that the sum of the weights for the m new links is $\sum_{\{i'\}} w_{ji'} = 1$, where $\{i'\}$ represents a sum over the m existing nodes to which the new node j is connected. Under the two above assumptions, each link $i \leftrightarrow j$ of the newly added node is assigned a weight as

$$w_{ji} = \frac{k_i}{\sum\limits_{\{i'\}} k_{i'}} \,. \tag{16}$$

Then, the weighted exponential (WE) model is defined as follows: at each time step, a new node with m links is added, which is connected with equal probability to the already-existing nodes in the system. The weights of the links are assigned again by (16).

3.5.2. *Deterministic network models*

Some degree of randomness is a common feature of the above small-world networks and scale-free models. It would be interesting to construct models that lead to small-world or scale-free networks in a deterministic fashion [Comellas *et al.*, 2000]. Recently, Barabási and Ravasz [2001] presented a simple model that can generate a deterministic scale-free network using a hierarchical construction. The network is built in an iterative fashion, as shown in Fig. 13. At step n, two units of 3^{n-1} nodes are added, which are identical to the network created in step $n-1$, and each node of these two units is connected to the root of the network. The degree distribution of this deterministic network is

$$P(k) \sim k^{-\frac{ln2}{ln3}} \,. \tag{17}$$

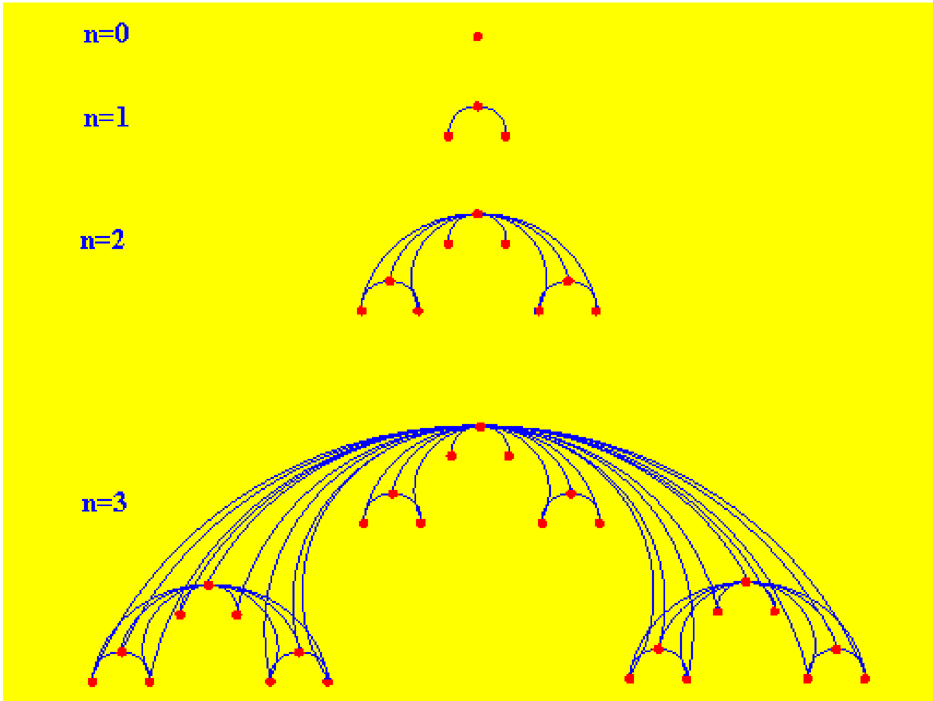


Fig. 13. A hierarchical construction of the deterministic scale-free network (after [Barabási & Ravasz, 2001]).

4. Connectivity of Complex Networks: Robustness and Fragility

The predecessor of the Internet — the ARPANET — was created by the US Department of Defense — by their Advanced Research Projects Agency (ARPA) in the late 1960s. The goal of the ARPANET was that it should be able to continuously supply communications services, even in cases when some subnetworks and gateways were failing [Clark, 1988]. Today, the Internet has grown to be a huge network and has played a crucial role in the world. One may wonder if we can continue to maintain the functionality of the network under inevitable failures or frequent attacks by computer hackers. The good news is that so far the Internet has proven rather resilient against failures: even if about 3% of the routers are down at any moment, one rarely observes major disruptions to the entire network. Where does this robustness come from? While there is a significant error tolerance built into the protocols that govern the switching of data packets, recent discoveries in the field of complex networks indicate that the topology of the network may also play a crucial role.

Let us start from a connected network, and at each time step remove a node. The disappearance of the node implies the removal of all of its connections, disrupting some of the paths between the remaining nodes. If there were multiple paths between two nodes i and j, the disruption of one of them may mean that the distance d_{ij} between them increases, and that may cause the increase of the average path length L of the whole network. In a more severe case, when initially there was a single path between i and j, the disruption of that particular path means that the two nodes become isolated from each other (Fig. 14).

The connectivity of a network is *robust* (or error tolerant) if it contains a giant cluster comprised of most of the nodes even after a removal of a fraction of its nodes. Albert *et al.* [2000] compared the robustness of ER random graph, which has an exponential degree distribution, and the BA scale-free model, which has a power-law degree distribution. Two types of node removal were discussed. Failures (random damages) were modeled by the removal of a fraction of randomly chosen nodes. Attacks (intentional damages) were described by the deletion of a fraction of nodes that have the highest numbers of connections. To monitor the disruption of an initially connected network, two quantities — the average path length l and the relative size S of the largest connected component (corresponds to the giant connected component

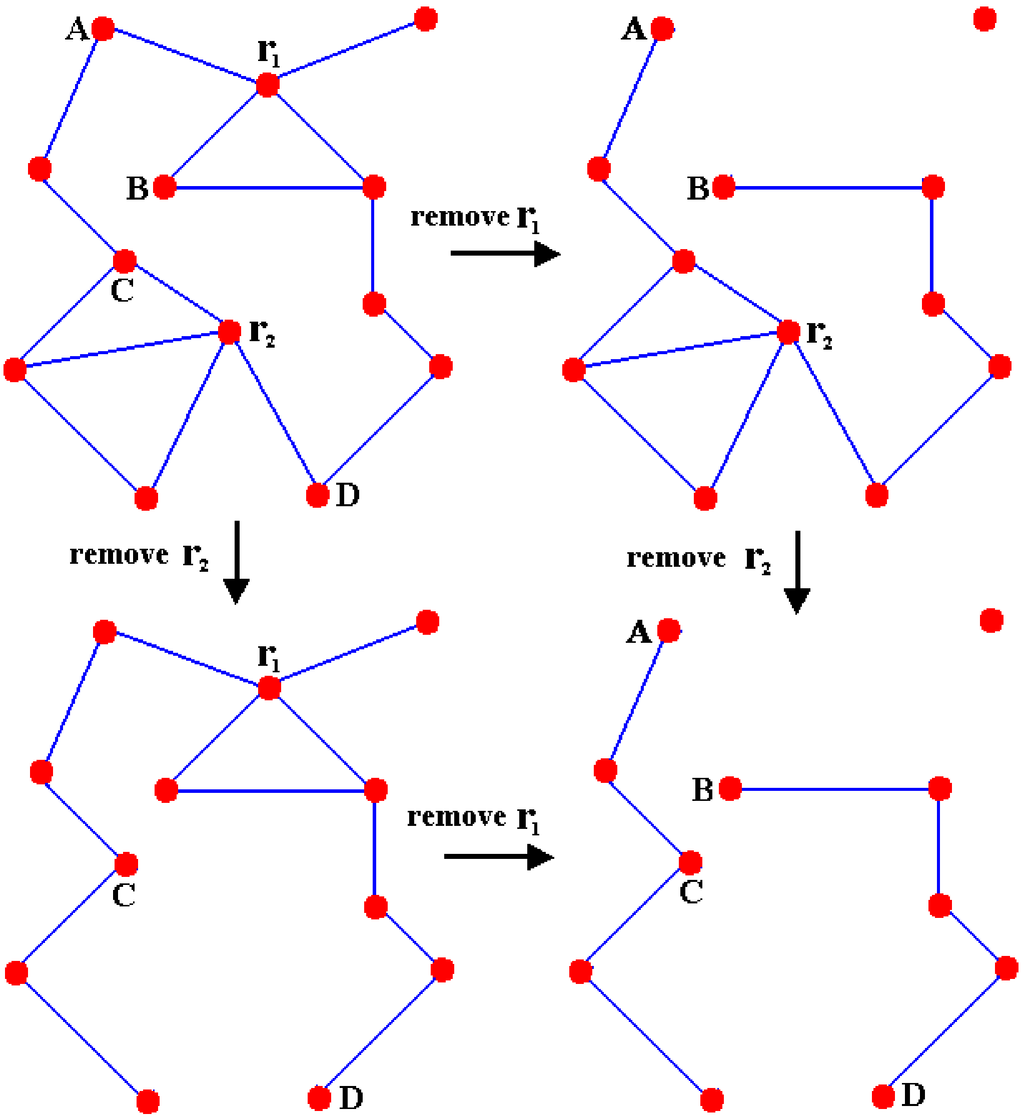


Fig. 14. Illustration of the effects of node removal on an initial connected network: In the unperturbed state $d_{AB} = d_{CD} = 2$. After the removal of node r_1 from the original network, $d_{AB} = 8$ and after the removal of node r_2 from the original network, $d_{CD} = 7$. After the removal of nodes r_1 and r_2, the network breaks into three isolated clusters and $d_{AB} = d_{CD} = \infty$.

if it exists) — were measured as functions of the fraction f of deleted nodes.

Albert *et al.* observed a striking difference between the scale-free networks and the random ones (Fig. 15). Scale-free networks display an exceptional robustness against random node failures: the size of the largest cluster decreases to 0 at a higher f and its average path length increases much slowly than that in the random case. This robustness of scale-free networks lies in their extremely inhomogeneous degree distribution: because the power-law distribution implies that the majority of nodes have only a few links, nodes with small connectivity will be selected with a much higher probability. The removal of these "small" nodes has a minor influence on the path structure of the remaining nodes. However, it is the same heterogeneity that makes scale-free networks fragile to intentional attacks: when the nodes with highest connections are removed, the network breaks down much faster than in the random case. This is not surprising, since the connectivity of a scale-free network is maintained by a few highly connected nodes, so removal of these nodes will have a dramatic disruptive effect on the topology of the network. Albert *et al.* also simulated the effect of random damage and attack on the Internet and WWW, as shown in Fig. 16. These numerical simulations were confirmed by

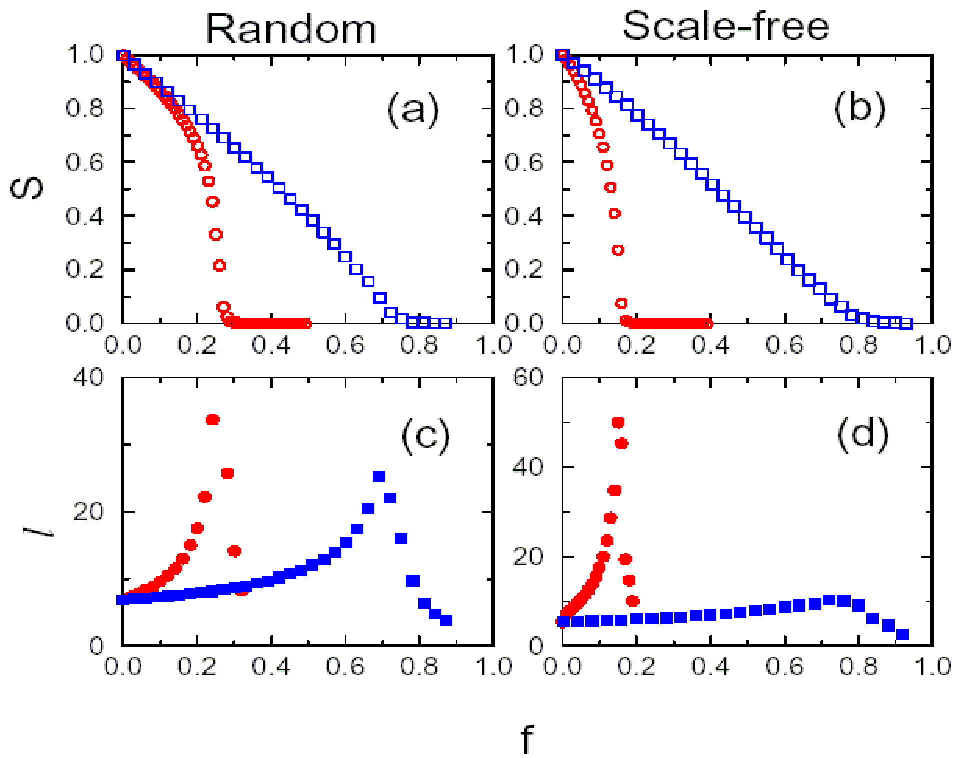


Fig. 15. The size $S(a, b)$ and average path length $l(c, d)$ of the largest cluster in an initially connected network when a fraction f of the nodes are removed. (a, c) ER random network with $N = 10\,000$ and $\langle k \rangle = 4$. (b, d) BA scale-free network with $N = 10\,000$ and $\langle k \rangle = 4$. Blue symbols indicate random node removal and red symbols mean preferential node removal (after [Albert *et al.*, 2000]).

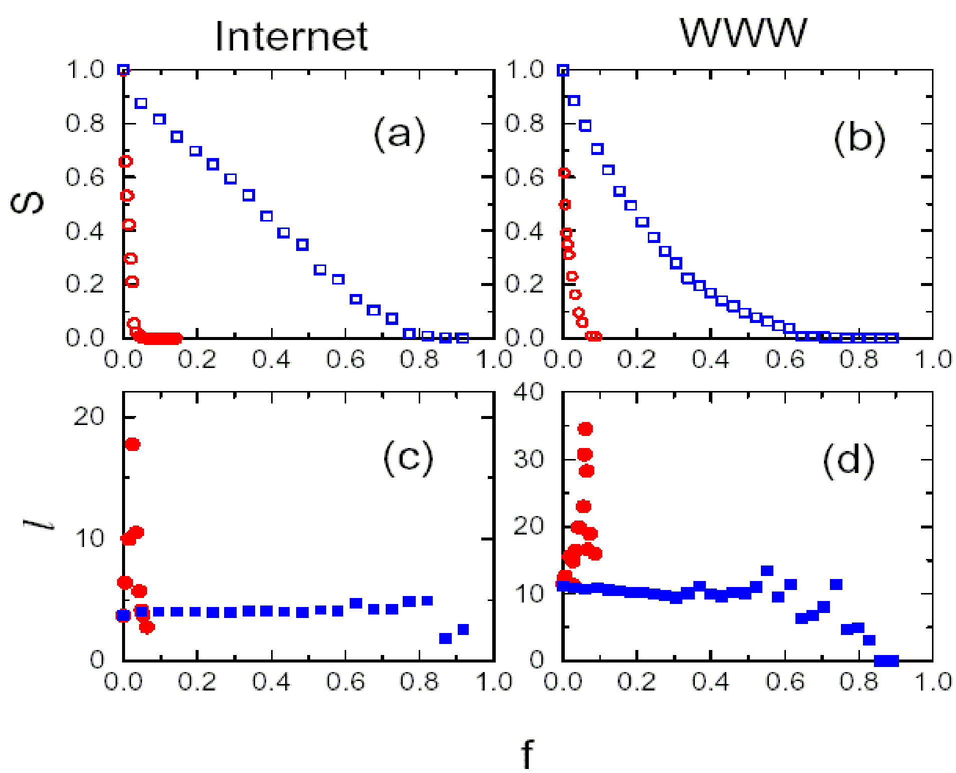


Fig. 16. Simulation results on the effect of intentional attacks and random failures on the topological characteristics of the Internet and WWW (after [Albert *et al.*, 2000]).

analytical results obtained by using the percolation theory [Callaway *et al.*, 2000; Cohen *et al.*, 2000, 2001].

5. Epidemic Dynamics in Complex Networks

When AIDS first emerged as a disease about twenty years ago, few people could have predicted how the epidemic would evolve, and even fewer could have been able to describe with certainty the best way of combating it. Unfortunately, according to estimates from the Joint United Nations Programme on HIV/AIDS (UNAIDS) and the World Health Organization (WHO), 21.8 million people around the world had died of AIDS up to the end of 2000 and 36.1 million people were living with the human immunodeficiency virus (HIV) by that time. In August 1996, as another example, a cascading series of failures in power lines led to blackouts in 11 US states and two Canadian provinces, leaving about 7 million customers without power for up to 16 hours. Still, the ILOVEYOU computer virus spread over the Internet in May 2000 and caused losses of nearly $7 billion in damage and computer down-time. How do diseases, jokes, and fashions spread on social networks? How do cascading failures propagate through large power grids? How do computer viruses spread on the Internet? Recently, some researchers have already been on their way to study such spreading phenomena, first on small-world networks [Newman & Watts, 1999b; Moore & Newman, 2000; Newman *et al.*, 2000; Watts, 2000; Svenson & Johnson, 2001] and, also, scale-free networks [Pastor-Satorras & Vespignani, 2001a, 2001b, 2001c; Moreno *et al.*, 2001]. These studies may help understand computer virus epidemics and other spreading phenomena on communication and social networks, among others.

A significant result was offered by Pastor-Satorras and Vespignani [2001a, 2001b], who studied by both analytical methods and large-scale simulations a dynamical model for the spreading of epidemics in complex networks. They considered the standard susceptible-infected-susceptible (SIS) epidemiological model. Each node of the network represents an individual and each link is a connection along which the infection can spread to other individuals. Individuals can only exist in two discrete states — susceptible and infected. At each time step, each susceptible node is infected with

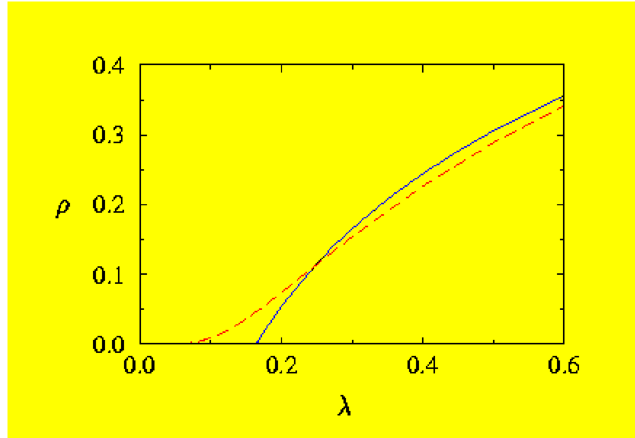


Fig. 17. Density of infected nodes as a function of λ in the WS network (full line) and the BA model (dashed line) (after [Pastor-Satorras & Vespignani, 2001b]).

probability ν if it is connected to at least one infected node. At the same time, infected nodes are cured and become susceptible again with probability δ, defining an effective spreading rate $\lambda = \nu/\delta$. The updating can be performed with both parallel and sequential dynamics. In models with regular connectivity (lattices and mean-field models), the most significant result is the general prediction of a nonzero epidemic threshold $\lambda_c > 0$. If $\lambda \geq \lambda_c$, the infection spreads and becomes persistent in time. On the other hand, if $\lambda < \lambda_c$, the infection dies out exponentially fast.

Pastor-Satorras and Vespignani found that, while for exponential networks the epidemic threshold is a positive constant, for a wide range of scale-free networks the critical spreading rate reduces to zero (Fig. 17). In other words, scale-free networks are prone to the spreading and the persistence of infections whatever spreading rate the epidemic agents might possess. Fortunately, this piece of bad news is balanced by the exponentially small prevalence for a wide range of spreading rates ($\lambda \ll 1$). This implies that even though computer viruses spread far and wide on the Internet, they typically do so by infecting only a tiny fraction of the entire network. These results have been generalized to the susceptible–infected–removed (SIR) model [Moreno *et al.*, 2001] and has also suggested new immunization strategies [Pastor-Satorras & Vespignani, 2001c; Dezso & Barabasi, 2001]. In particular, Park *et al.* [2001] have shown that there is an intimate relationship between the effectiveness of distributed packet filtering (DPF) at mitigating distributed denial of service (DDoS) attack on the Internet and the scale-free topology of the network.

6. Dynamical Networks and Their Synchronization

6.1. *Dynamical network model*

Several different kinds of complex network models have been described above. From a nonlinear dynamics point of view, one is particularly interested in the role network topology plays in the dynamical behavior of a network. Now, consider a dynamical network consisting of N identical linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system. The state equations of the network are

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + c\sum_{j=1}^{N} a_{ij}\Gamma\mathbf{x}_j\,, \quad i = 1,\, 2, \ldots,\, N\,. \tag{18}$$

where $\mathbf{x}_i = [x_{i1},\, x_{i2}, \ldots,\, x_{\text{in}}]^T \in \Re^n$ are the state variables of node i, the constant $c > 0$ represents the coupling strength of the network, and $\Gamma \in \Re^{n\times n}$ is a constant 0-1 matrix linking coupled variables. For simplicity, assume that $\Gamma = \text{diag}(r_1,\, r_2, \ldots,\, r_n)$ is a diagonal matrix with $r_i = 1$ for a particular i and $r_j = 0$ for $j \neq i$. This means that two coupled nodes are linked through their ith state variables. Coupling matrix $\mathbf{A} = (a_{ij}) \in \Re^{N\times N}$ represents the coupling configuration of the network. If there is a connection between node i and node j, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$ $(i \neq j)$. The diagonal elements of $\mathbf{A}$ is defined as

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij} = -\sum_{j=1, j\neq i}^{N} a_{ji}\,,\ \ i = 1, 2, \ldots, N. \tag{19}$$

Clearly, if the degree of node i is k_i, then

$$a_{ii} = -k_i\,, \quad i = 1,\, 2, \ldots,\, N\,. \tag{20}$$

Next, suppose that the network is connected in the sense that there are no isolated clusters. Then the coupling matrix $\mathbf{A}$ is a symmetric irreducible matrix. In this case, it can be shown that zero is an eigenvalue of $\mathbf{A}$ with multiplicity 1 and all the other eigenvalues of $\mathbf{A}$ are strictly negative.

6.2. *Synchronization stability analysis*

Dynamical network (18) is said to achieve (*asymptotically*) *synchronization* if

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_N(t)\,, \quad \text{as} \quad t \to \infty\,. \tag{21}$$

The diffusive coupling condition (19) guarantees that the synchronization state is a solution, $\mathbf{s}(t) \in \Re^n$, of an isolated node, i.e.

$$\dot{\mathbf{s}}(t) = f(\mathbf{s}(t))\,. \tag{22}$$

Here, $\mathbf{s}(t)$ can be an equilibrium point, a periodic orbit, or a chaotic attractor. Clearly, stability of the synchronization state,

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_N(t) = \mathbf{s}(t)\,, \tag{23}$$

of network (18) is determined by the dynamics of an isolated node (i.e. function f and solution $\mathbf{s}(t)$), the coupling strength c, the inner linking matrix Γ and the coupling matrix $\mathbf{A}$.

Given the dynamics of an isolated node and the inner linking structure, the synchronizability of network (18) with respect to a specific coupling configuration is said to be *strong* if the network can synchronize with a small coupling strength, c. Based on the Lyapunov stability theory, the following result can be derived.

Theorem 1 [Wang & Chen, 2002a, 2002b]. *Consider dynamical network* (*18*). *Let*

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N \tag{24}$$

be the eigenvalues of its coupling matrix $\mathbf{A}$. *Suppose that there exist an* $n\times n$ *diagonal matrix* $\mathbf{D} > \mathbf{0}$ *and two constants* $\bar{d} < 0$ *and* $\tau > 0$, *such that*

$$[Df(\mathbf{s}(t)) + d\Gamma]^\top\mathbf{D} + \mathbf{D}[Df(\mathbf{s}(t)) + d\Gamma] \leq -\tau\mathbf{I}_n \tag{25}$$

for all $d \leq \bar{d}$, *where* $I_n \in \Re^{n\times n}$ *is an unit matrix. If, moreover,*

$$c\lambda_2 \leq \bar{d}\,, \tag{26}$$

then the synchronization state (*23*) *is exponentially stable.*

Since $\lambda_2 < 0$ and $\bar{d} < 0$, inequality (26) is equivalent to

$$c \geq \left|\frac{\bar{d}}{\lambda_2}\right| \tag{27}$$

A small value of λ_2 corresponds to a large value of $|\lambda_2|$, which implies that network (18) can synchronize with a small coupling strength c. Therefore, synchronizability of network (18) with respect to a specific coupling configuration can be characterized by the second-largest eigenvalue of the corresponding coupling matrix $\mathbf{A}$.

6.3. *An example: Synchronization in a network of Chua's oscillators*

Chua's circuit is a physical system for which the presence of chaos in the sense of Shil'nikov has been established experimentally confirmed numerically, and proven mathematically. Therefore, in recent years, Chua's circuit has become a standard model for studying chaos in systems described by finite-dimensional ordinary differential equations [Chua *et al.*, 1993]. The state equation of Chua's oscillator in the dimensionless form is

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - f(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\beta x_2 - \gamma x_3 \,, \end{cases} \tag{28}$$

where

$$f(x_1) = bx + \frac{1}{2}(a-b)[|x+1| - |x-1|] \,, \tag{29}$$

in which $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $a < b < 0$. If the system parameters are chosen to be

$$\alpha = 10,\ \beta = 24.5,\ \gamma = 0,\ a = -1.27,\ b = -0.68 \,, \tag{30}$$

then the Chua's oscillator (28) has a chaotic attractor, as shown in Fig. 18.

Suppose that two coupled Chua's oscillators are linked through the first state variable, i.e. $\Gamma = \mathrm{diag}(1,\ 0,\ 0)$. The state equations of the entire network are

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} \alpha(x_{i2} - x_{i1} + f(x_{i1}) + c\sum_{j=1}^{N} a_{ij}x_{j1} \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} - \gamma x_{i3} \,. \end{bmatrix} \tag{31}$$

For network (31), the constant $\overline{d}$ in Theorem 1 can be taken as $\overline{d} = a$, which guarantees its synchronized states to be stable.

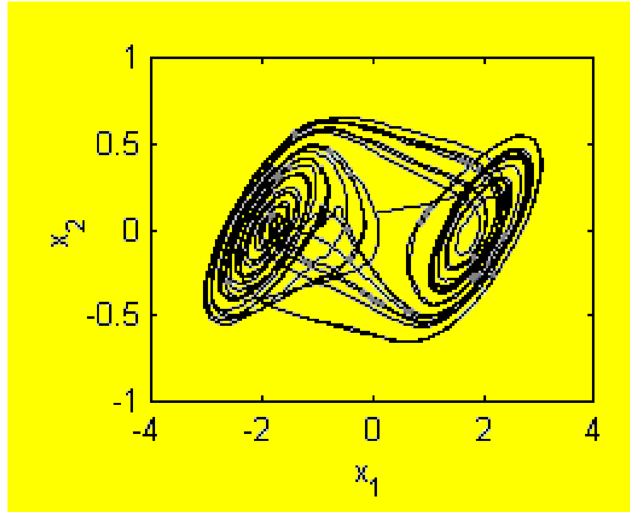


Fig. 18. The chaotic attractor of Chua's oscillator (28), with parameters given in (30).

7. Synchronization in Complex Dynamical Networks

7.1. *Synchronization in regular networks*

Over the past decade, much work on synchronization in coupled oscillator arrays have been devoted to regular coupling networks. Here, three commonly studied coupling configurations are discussed.

Global coupled network. A globally coupling configuration means that any two different nodes are connected directly. The corresponding coupling matrix is

$$\mathbf{A}_{\mathrm{gc}} = \begin{bmatrix} -N+1 & 1 & 1 & \cdots & 1 \\ 1 & -N+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & -N+1 \end{bmatrix} . \tag{32}$$

Matrix $\mathbf{A}_{\mathrm{gc}}$ has a single eigenvalue at 0 and all the others equal to $-N$. Hence, its second-largest eigenvalue $\lambda_{2\mathrm{gc}} = -N$ decreases to $-\infty$ as $N \to \infty$, i.e.

$$\lim_{N\to\infty} \lambda_{2\mathrm{gc}} = -\infty \,. \tag{33}$$

Nearest-neighbor coupled network. A nearest-neighbor coupled dynamical network with a periodic boundary condition $\mathbf{x}_{N+j} = \mathbf{x}_j$ consists of N nodes arranged in a ring and each node i is adjacent to its neighbor nodes $i \pm 1$, $i \pm 2, \ldots, i \pm K/2$, where K is even. The state equations of the network are

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + c \sum_{j=1}^{K/2} \Gamma(\mathbf{x}_{i+j} + \mathbf{x}_{i-j} - 2\mathbf{x}_i)\,, \quad i = 1,\, 2, \ldots, N\,. \tag{34}$$

The coupling matrix $\mathbf{A}_{\mathrm{nc}}$ of network (34) is a circulant matrix and its second-largest eigenvalue can be found to be

$$\lambda_{2\mathrm{nc}} = -4 \sum_{j=1}^{K/2} \sin^2 \left(\frac{j\pi}{N}\right)\,. \tag{35}$$

Clearly, for any fixed value of K, $\lambda_{2\mathrm{nc}}$ increases to zero as N tends to infinity, i.e.

$$\lim_{N\to\infty} \lambda_{2\mathrm{nc}} = 0\,. \tag{36}$$

Star coupled network. The coupling matrix of a dynamical network (18) with a star coupling configuration is

$$\mathbf{A}_{\mathrm{sc}} = \begin{bmatrix} -N+1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix} \tag{37}$$

The eigenvalues of $\mathbf{A}_{\mathrm{sc}}$ are $\{0, -N, -1, \ldots, -1\}$. Therefore, the second-largest eigenvalue of $\mathbf{A}_{\mathrm{sc}}$ is $\lambda_{2\mathrm{sc}} = -1$, which is unrelated with the size of the network.

In summary, for any given coupling strength $c > 0$, the globally coupled network can synchronize as long as the size of the network, N, is large enough. On the other hand, for a given coupling strength, no matter how large it is, the nearest-neighbor coupled network (34) cannot synchronize if N is sufficiently large. For the star coupled network, there exists a critical coupling strength $\bar{c} = |\bar{d}\,| > 0$, so that the network will synchronize if $c > \bar{c}$.

7.2. *Synchronization in small-world networks*

One can construct the coupling matrix of dynamical network (18) with NW small-world connections as follows [Wang & Chen, 2001a]. In the nearest-neighbor coupling matrix A_{nc}, if $a_{ij} = a_{ji} = 0$, set $a_{ij} = a_{ji} = 1$ with probability p, and then recompute the diagonal elements according to formula (19). Denote the new small-world coupling matrix by $\mathbf{A}_{\mathrm{sw}}$ and let $\lambda_{2\mathrm{sw}}$ be its second-largest eigenvalue. According to Theorem 1, if

$$c \geq \left| \frac{\bar{d}}{\lambda_{2\mathrm{sw}}} \right|\,,$$

then the corresponding small-world dynamical network will synchronize.

Figures 19 and 20 show the numerical values of $\lambda_{2\mathrm{sw}}$ as a function of the probability p and the

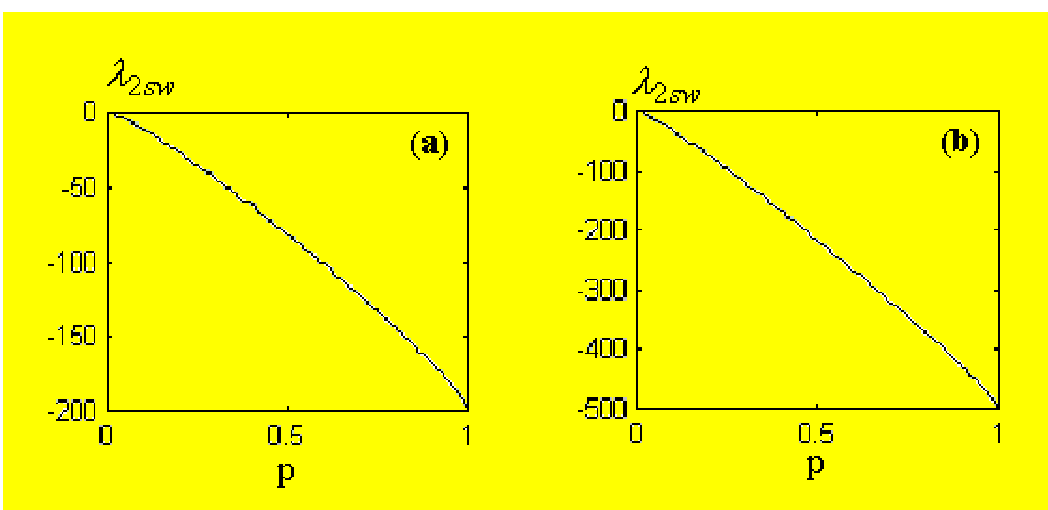


Fig. 19. Numerical values of $\lambda_{2\mathrm{sw}}(p, N)$ as a function of the probability p: (a) $N = 200$; (b) $N = 500$.

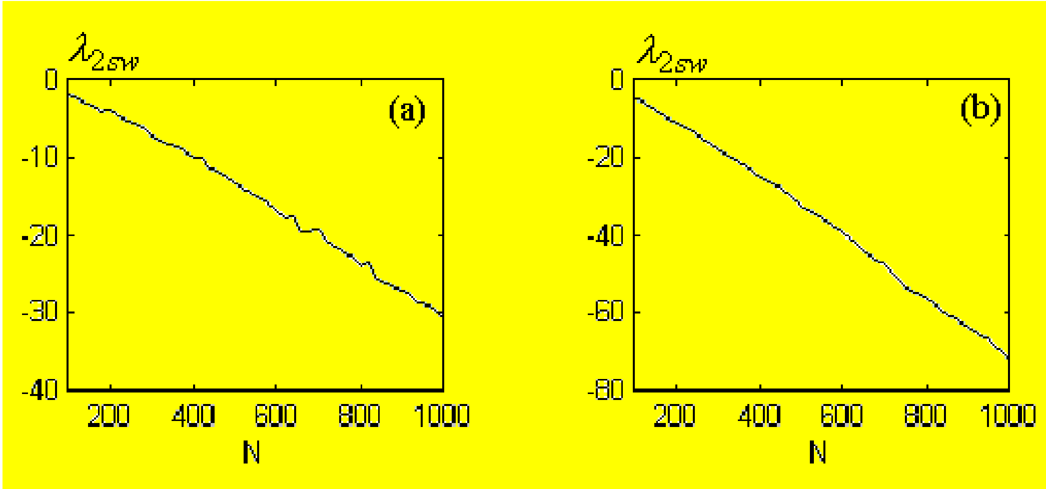


Fig. 20. Numerical values of $\lambda_{2\text{sw}}(p,\,N)$ as a function of N: (a) $p = 0.05$; (b) $p = 0.1$.

number of nodes, N. In these figures, for each pair of values of p and N, $\lambda_{2\text{sw}}$ is obtained by averaging the results of 20 runs. It can be seen that

(i) For any given value of N, $\lambda_{2\text{sw}}(p,\,N)$ decreases to $-N$ as p increases from 0 to 1.
(ii) For any given value of $p \in (0,\,1]$, $\lambda_{2\text{sw}}(p,\,N)$ decreases to $-\infty$ as N increases to ∞.

The above results imply that, for any given coupling strength $c > 0$,

(i) For any given $N > |\bar{d}\,|/c$, there exists a critical value, $\overline{p}$, so that if $\overline{p} \le p \le 1$, then the small-world dynamical network will synchronize.
(ii) For any given $p \in (0,\,1]$, there exists a critical value, $\overline{N}$, so that if $N > \overline{N}$, then the small-world dynamical network will synchronize.

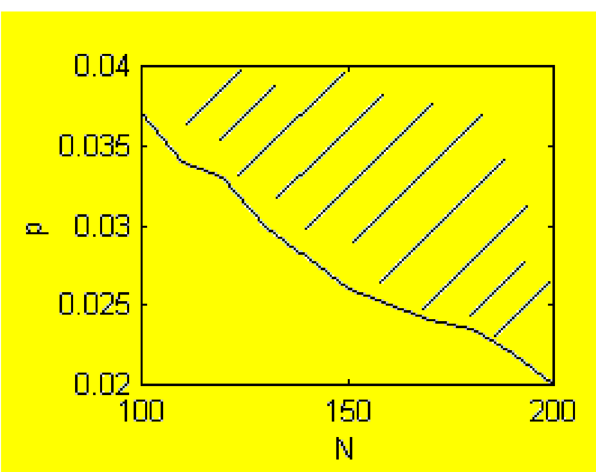


Fig. 21. Values of p and N achieving synchronization in a small-world network of Chua's oscillators.

Figure 21 shows the values of p and N that can achieve synchronization in a network of the form (31) consisting of Chua's oscillators with small-world connections. These results imply that the ability to achieve synchronization in a nearest-neighbor coupled network can be greatly enhanced by just adding a tiny fraction of long-range connections, revealing an advantage of small-world networks for chaos synchronization.

7.3. *Synchronization in scale-free networks*

Let $\mathbf{A}_{\text{sf}}(\overline{m},\,N)$ be the coupling matrix of a dynamical network of the form (18) with scale-free connections derived from the BA scale-free algorithm with $m_0 = m = \overline{m}$. The second-largest eigenvalue of $\mathbf{A}_{\text{sf}}(\overline{m},\,N)$ is denoted as $\lambda_{2\text{sf}}(\overline{m},\,N)$. It was found [Wang & Chen, 2001b] that, for a fixed value of $\overline{m}$, $\lambda_{2\text{sf}}(\overline{m},\,N)$ is a decreasing and bounded function of N. Therefore,

$$\lim_{N\to\infty} \lambda_{2\text{sf}}(\overline{m},\,N) = \widehat{\lambda}_{2\text{sf}}(\overline{m}) < 0\,, \qquad (38)$$

which means $\lambda_{2\text{sf}}(\overline{m},\,N)$ decreases to a negative constant, $\widehat{\lambda}_{2\text{sf}}(\overline{m})$, as N increases to infinity. In particular, for $\overline{m} = 3$, 5 and 7, one has $\widehat{\lambda}_{2\text{sf}}(\overline{m}) \approx -0.94$, -0.97 and -0.98, respectively (Fig. 22).

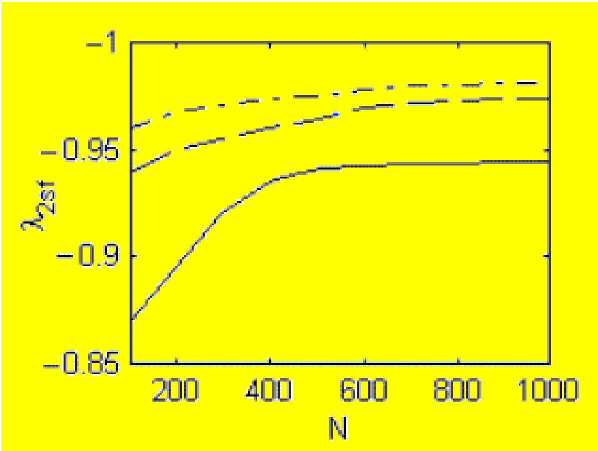


Fig. 22. The second-largest eigenvalue of the coupling matrix of the scale-free dynamical network (18) for $m_0 = m = 3$ (—); $m_0 = m = 5$ (- -); $m_0 = m = 7$ (- ·).

This implies that the adding of new nodes in a scale-free network cannot decrease the synchronizability of the network.

Due to the self-organization process of a scale-free network, the synchronizability of a large size scale-free dynamical network will remain almost unchanged by the constant adding of new nodes. Recall that $\lambda_{2sc} = -1$ for a star coupled network. The synchronizability of a scale-free network is about the same as that of a star network. This may be due to the extremely inhomogeneous connectivity distribution of a scale-free network: a few "hubs" in a scale-free network play a similar role as a single center in a star network.

For sufficiently large number of nodes, synchronization in network (31), consisting of Chua's oscillators with a scale-free coupling structure, can be achieved if

$$c > \left| \frac{a}{\overline{\lambda}_{2sf}(\overline{m})} \right| . \tag{39}$$

For instance, for $\overline{m} = 3, 5, 7$, synchronization can be achieved with $c > 1.35$, $c > 1.31$ and $c > 1.30$, respectively [Wang & Chen, 2001b].

7.4. *Robustness and fragility of synchronization in scale-free dynamical networks*

In Sec. 4, the robustness and fragility of connectivity of scale-free networks were discussed. Now, consider the robustness of synchronization in scale-free dynamical networks against either random or specifical removal of a small fraction of nodes from the network.

Clearly, the removal of some nodes from network (18) can only change the coupling matrix. According to Theorem 1, if the second-largest eigenvalue of the coupling matrix remains unchanged, then the synchronizability of the network will also remain unchanged after the removal of some nodes.

For simplicity, use $\mathbf{A}_{sf} \in \Re^{N\times N}$ and $\overline{\mathbf{A}}_{sf} \in \Re^{(N-[fN])\times(N-[fN])}$ to denote the coupling matrix of the original network with N nodes and the new network after removal of $[fN]$ nodes, respectively. Here, $[fN]$ stands for the smaller but nearest integer to the real number fN. Let λ_{2sf} and $\overline{\lambda}_{2sf}$ be the average values of the second-largest eigenvalues of $\mathbf{A}_{sf}$ and $\overline{\mathbf{A}}_{sf}$, respectively. Suppose that nodes $i_1, i_2, \ldots, i_{[\mathrm{fN}]}$ have been removed from the network. One can construct the new coupling matrix $\overline{\mathbf{A}}_{sf}$ from the original coupling matrix $\mathbf{A}_{sf}$ as follows:

(i) Form the minor matrix $\mathbf{B} \in \Re^{(N-[fN])\times(N-[fN])}$ of $\mathbf{A}_{sf}$ by removing the i_1th, i_2th, …, $i_{[fN]}$th row-column pairs of $\mathbf{A}_{sf}$.
(ii) Obtain $\overline{\mathbf{A}}_{sf} = (\overline{a}_{ij})$ by recomputing the diagonal elements of the minor matrix $\mathbf{B} = (b_{ij})$ according to the diffusive coupling condition (19), with N being replaced by $N - [fN]$, i.e.

$$\overline{a}_{ij} = b_{ij}, \quad i \neq j; \quad \overline{a}_{ii} = -\sum_{j=1, j\neq i}^{N-[fN]} \overline{a}_{ij} . \tag{40}$$

In simulation, $N = 3000$ and $m_0 = m = \overline{m} = 3$, namely, the original network contains 3000 nodes and about 9000 connections. It was found [Wang & Chen, 2001b] that even when as many as 5% randomly chosen nodes are removed, the second-largest eigenvalue of the coupling matrix remains almost unchanged (Fig. 23), i.e.

$$\overline{\lambda}_{2sf} \cong \lambda_{2sf} , \tag{41}$$

which means that the synchronizability of the network is almost unaffected.

Although the scale-free structure is particularly well-suited to tolerate random errors, it is also particularly vulnerable to deliberate attacks. To simulate the influence of an attack on the synchronizability of the network, one may first remove the node with highest degree, and then continue to select and remove other nodes in their decreasing order of degrees. In doing so, it was found (Fig. 24)

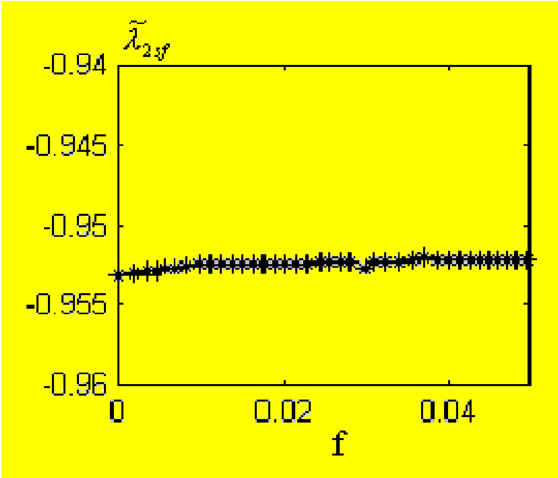


Fig. 23. Second-largest eigenvalues of the coupling matrix of the scale-free dynamical network (18) when a fraction f of the randomly selected nodes is removed.

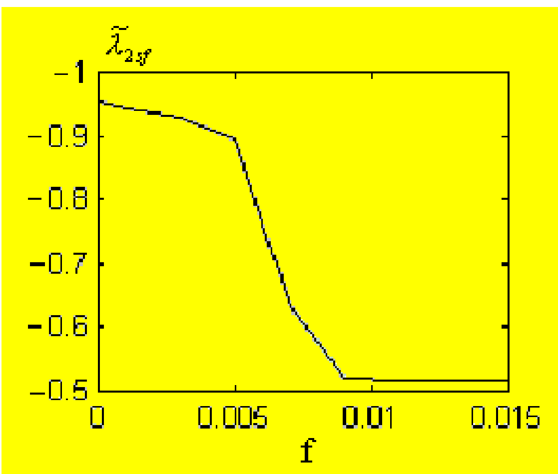


Fig. 24. Second-largest eigenvalues of the coupling matrix of the scale-free dynamical network (18) when a fraction f of the specifically selected nodes is removed.

that the magnitude of the second-largest eigenvalue of the coupling matrix decreases rapidly — almost decreases to half of its original value in magnitude (from the original $|\lambda_{2\rm sf}| = 0.953$ to $|\overline{\lambda}_{2\rm sf}| = 0.519$), when only $f \approx 1\%$ fraction of the most connected nodes were removed. At a low critical threshold, $f \approx 1.6\%$, $\overline{\lambda}_{2\rm sf}$ abruptly changes to zero, implying that the whole network was broken into isolated clusters.

The error tolerance and attack vulnerability of synchronizability of scale-free networks are rooted to their extremely inhomogeneous connectivity distributions. In fact, if $N \gg 1$ and $f \ll 1$, then no matter whether these removed nodes are randomly selected or specifically selected, one has

$$\lambda_2(\mathbf{B}) \approx \lambda_{2\rm sf}\,, \tag{42}$$

where $\lambda_2(\mathbf{B})$ is the second-largest eigenvalue of the minor matrix $\mathbf{B}$.

On the other hand, from the construction of the new coupling matrix $\overline{\mathbf{A}}_{\rm sf}$, it is easy to see that

$$\overline{\mathbf{A}}_{\rm sf} = \mathbf{B} + \Gamma\,, \tag{43}$$

where $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{N-[fN]})$ is a nonnegative definite diagonal matrix, with

$$\sum_{j=1}^{N-[fN]} \gamma_j = \sum_{j=1}^{[fN]} k_{i_j}\,. \tag{44}$$

Due to the extremely inhomogeneous connectivity distribution of a scale-free network, if the $[fN]$ removed nodes are randomly selected, then it is quite possible that $[fN]$ "small" nodes are selected. This means that the right-head side of (44) is small. Thus,

$$\overline{\lambda}_{2\rm sf} \approx \lambda_2(\mathbf{B}) \approx \lambda_{2\rm sf}\,. \tag{45}$$

However, if $[fN]$ most connected nodes are removed, then the new coupling matrix $\overline{\mathbf{A}}$ is derived from the minor matrix $\mathbf{B}$ by adding a nonnegative definite diagonal matrix Γ with a large sum of the diagonal elements. This would result in a drastic increase in the second-largest eigenvalue of the matrix, i.e. $\overline{\lambda}_{2\rm sf}$ is much greater than $\lambda_2(\mathbf{B})$ and $\lambda_{2\rm sf}$.

In summary, the connectivity and synchronizability of scale-free networks are robust against random failures but fragile to intentional attacks. These results, together with some recent findings about the *robust, yet fragile* feature and the power law in complex systems [Carlson & Doyle, 1999, 2000; Doyle & Carlson, 2000] indicate that *robust, yet fragile* seems to be an essential and common property of complex networks.

8. Synchronization in Other Dynamical Network Models

8.1. *Synchronization in a network of phase oscillators*

Watts [1999] investigated synchronization in a network of Kuramoto oscillators with random or small-world connections. The Kuramoto model was originally established based on the phenomenon of collective synchronization, in which an enormous number of oscillators spontaneously were locked to a common frequency, despite the inevitable differences in the natural frequencies of the individual oscillators. Biological examples of this kind of networks include networks of pacemaker cells in the heart; circadian pacemaker cells in the suprachiasmatic nucleus of the brain; metabolic synchrony in yeast cell suspensions; congregations of synchronously flashing fireflies; and crickets that chirp in unison.

More precisely, the Kuramoto model [1984] of globally coupled phase oscillators is described by

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\,, \tag{46}$$

where θ_i denotes the phase of oscillator i and ω_i is its natural frequency, chosen at random from a Lorentzian probability density,

$$g(\omega) = \frac{\gamma}{\pi|\gamma^2 + (\omega - \omega_0)^2|} \tag{47}$$

of width γ and mean ω_0, K represents the coupling strength, and the factor $1/N$ ensures that the model is well-behaved as $N \to \infty$.

If the population of oscillators is visualized as a swarm of points running around the unit circle in the complex plane, the position of the centroid of the population can be expressed as a complex order parameter

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}\,, \tag{48}$$

which can be interpreted as the collective rhythm produced by the whole population. Here, the radius r measures the phase coherence, and ψ is the average phase (Fig. 25). A uniform, random distribution of the oscillators around the ring corresponds to $r = 0$, and $r = 1$ indicates a perfect synchronization.

Kuramoto showed that there exists a critical coupling strength K_c. For $K < K_c$, no synchronization is possible and $r(t)$ decays rapidly to a small, residual value of size $O(N^{1/2})$. But, when K exceeds K_c, synchronization emerges spontaneously and r increases to $r_\infty < 1$, though still with $O(N^{1/2})$ fluctuations (Fig. 26). With a further increase in K, more and more oscillators are recruited into the synchronized cluster, and r_∞ grows as shown in Fig. 27.

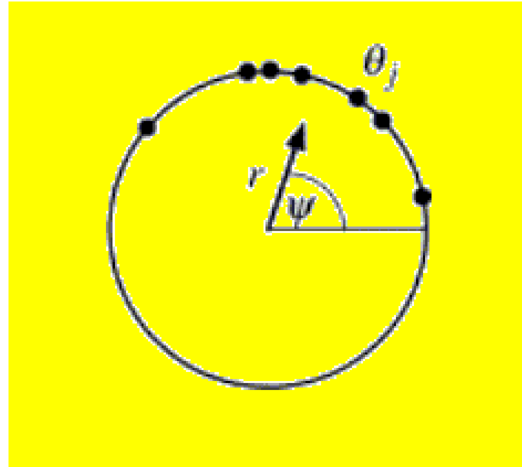


Fig. 25. Geometric interpretation of the order parameter (after [Strogatz, 2001]).

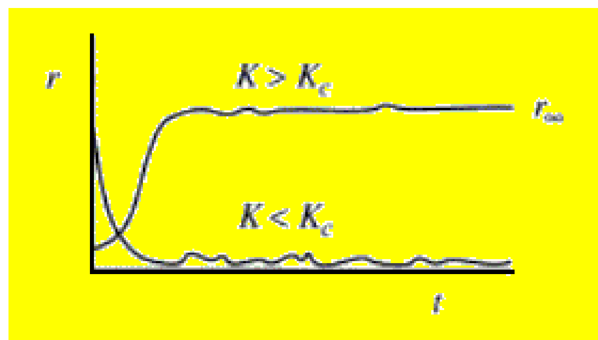


Fig. 26. Schematic illustration of the typical evolution of $r(t)$ in the Kuramoto model.

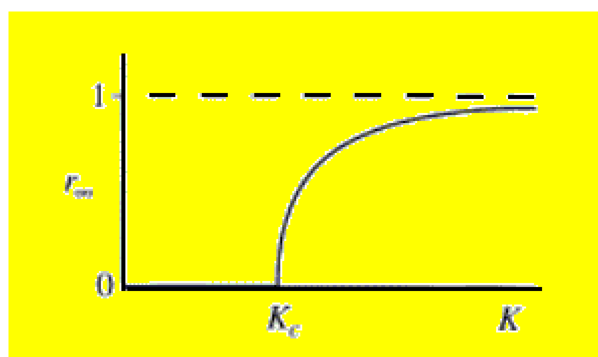


Fig. 27. Dependence of the steady-state coherence r_∞ on the coupling strength K.

Watts [1999] found that a similar phenomenon occurs when the oscillators are coupled according to a random graph or small-world connection, except that the value of K_c is shifted.

8.2. *Synchronization of chaos in coupled map lattices with small-world connections*

Consider a coupled map lattice (CML) of the form

$$x_i(t+1) = \sum_{j=1}^{N} a_{i,j} f(x_j(t)), \quad i = 1, 2, \ldots, n, \tag{49}$$

where $x_i \in \Re$ is the state variable of node i, f is a chaotic map, $\mathbf{A} = (a_{i,j})$ is the coupling matrix which satisfies

$$\sum_{j=1}^{N} a_{i,j} = 1, \quad i = 1,\, 2, \ldots,\, n\,. \tag{50}$$

Gade and Hu [2000] studied synchronization of chaos in a CML lattice with NW small-world interactions, where they allowed a node to be coupled to another node more than once. The small-world CML is defined as

$$x_i(t+1) = \frac{1}{k(i)} \sum_{j \in nbr(i)} f(x_j(t))\,, \qquad i = 1,\, 2, \ldots,\, n\,, \tag{51}$$

where $k(i)$ and $nbr(i)$ are the degree and neighbor of node i, respectively. They found that there is an invariable gap between the largest eigenvalue and the second-largest eigenvalue of the coupling matrix $\mathbf{A}$, which indicates the possibility of synchronization. In particular, for a fixed adding probability $p > 0$, synchronous chaos is always a stable attractor in the thermodynamic limit ($N \to \infty$). This result is in good accordance with the present author's result on the synchronization in a continuous-time dynamical network of the form (18) with small-world connections.

8.3. *Fast response and temporal coherent oscillations in small-world networks*

Recently, Lago-Fernandez *et al.* [2000] investigated the dynamical behavior of a neural network of Hodgkin–Huxley neurons coupled by excitatory synapses. The unit dynamics is described by the following set of coupled ordinary differential equations:

$$\begin{cases} C_m \dot{V}_i = I^e - g_L \hat{V}_L - g_{Na} m^3 h \hat{V}_{Na} - g_k n^4 \hat{V}_k + I^s(t) \\ \dot{m} = \alpha_m(V)(1-m) - \beta_m(V) m \\ \dot{h} = -\alpha_h(V)(1-h) - \beta_h(V) h \\ \dot{n} = \alpha_n(V)(1-n) - \beta_n(V) n, \end{cases} \tag{52}$$

where V_i represents the membrance potential of unit i, C_m is the membrance capacitance per unit area, $I^e(t)$ is the external current, which occurs as a pulse of amplitude I_0, $\hat{V}_r = V_i - V_r$, with V_r being the equilibrium potentials for the different ionic contributions ($r = Na, k$) and passive channel ($r = L$), g_r are the corresponding maximum conductances per unit area, h, m and n are the voltage dependent activating and inactivating variables, α and β are functions of V adjusted to physiological data by voltage clamp techniques.

The synaptic current $I^s(t)$ is given by

$$I_i^s(t) = g_{ij} r_j(t)[V_s(t) - E_s]\,, \tag{53}$$

where i stands for the postsynaptic and j for the presynaptic neuron, and g_{ij} is the maximum conductance, which determines the degree of coupling between the two connected neurons, V_s is the postsynaptic potential, E_s is the synaptic reversal potential, and $r_j(t)$ is the fraction of bound receptors given by

$$\dot{r} = \alpha(t)(1-r) - \beta r\,, \tag{54}$$

where the rise constant $\alpha(t) = 0.94\theta(t_0 + \tau - t)\theta\cdot(t - t_0)$ msec^{-1} and decay constant $\beta = 0.18$ msec^{-1}, where $\theta(\cdot)$ is the Heaviside function, t_0 is the time when the presynaptic neuronfires (membrane potential over 27 mV), and $\tau = 1.5$ msec.

Lago-Fernandez *et al.* [2000] studied three different kinds of connectivity patterns: regular lattices, random graphs and small-world models. They found that different connectivity topologies exhibit the following features (Fig. 28): regular topologies give rise to coherent oscillations, but in a temporal scale that is not in accordance with fast signal processing; on the other hand, random topologies give rise to fast system response yet are unable to produce coherent oscillations in the average activity of the network; finally, small-world topology, which falls in between the random and the regular ones, takes advantage of the best features of both, giving rise to fast system responses with coherent oscillations.

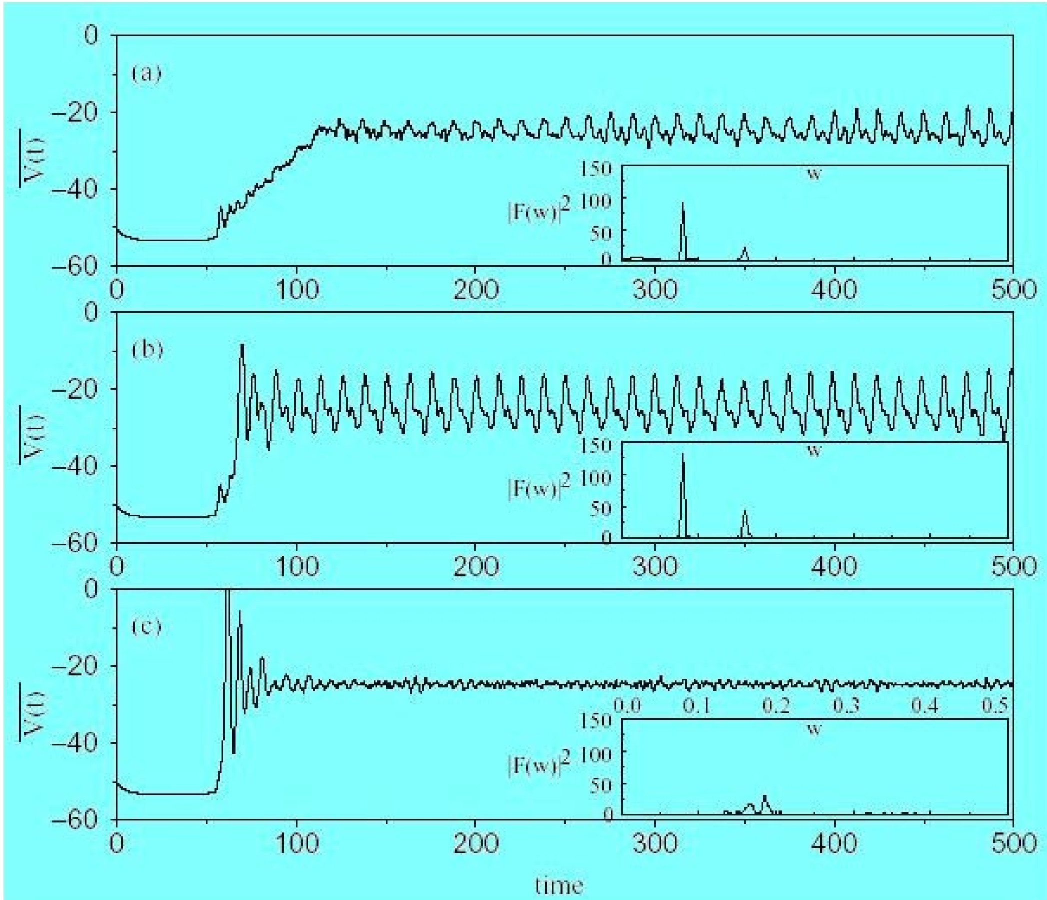


Fig. 28. Average activity and power spectrum (inset) in a network of 797 neurons. (a) Regular network ($p = 0$). (b) Small-world network ($p = 0.032$). (c) Random network ($p = 1$). p is the rewiring probability. The input onset occurs at $t = 50$ and is offset at $t = 350$.

More recently, Bohland and Minai [2001] showed that associative memory networks with small-world architectures can provide the same retrieval performance as random networks, while only a fraction of the total connection length is used.

9. Conclusions

In the past few years, the discovery of small-world and scale-free properties of complex networks has stimulated a great deal of interest in the underlying organizing principles of complex networks and has led to dramatic advances in this active research direction.

First, the small-world, scale-free, and other properties of many real networks have been investigated. Second, a considerable number of network models have been proposed, which can be viewed as generalizations of the now well-known ER random graph, WS small-world network, and BA scale-free model. Various properties of these network models have been studied by numerical simulation as well as theoretical analysis. Third, there is increasing interest in trying to understand the relationship between the structural properties of such networks and their behaviors. In particular, significant progress has been made on the effect of network topology on network robustness and epidemic spreading. From the perspective of nonlinear dynamics, it is particularly interesting to find out the relationship between the dynamical and topological properties of large-scale networks of coupled complex dynamical systems, given their individual dynamics and the coupling configuration.

Recent studies on synchronization of complex dynamical networks according to small-world or scale-free connections may shed new light on the synchronization behavior of real complex networks. However, this is seen as only the tip of the iceberg and there are plenty of challenges about the modeling, analysis and control of such complex dynamical networks.

Worth mentioning is that most biological systems can be described by complex networks and the focus in biology is beginning to shift to understanding the vast networks that biological molecules create to regulate and to control life. It is expected that viewing biology in terms of complex feedback networks will become as essential as the role of molecular biology has become over the last few decades. Today, we are building increasing integrated and interconnected networks for information, energy, transportation and commerce. Many of these networks have become "critical infrastructure," meaning that they have gradually been integrated into the basic functioning of our society. It has also been noticed that their failures would have widespread and immediate serious consequences. The critical nature of these networks raises concerns about the risks and impacts of system failures and make it imperative to better understand the nature of these systems. This calls for better understanding and greater effort in the design of all kinds of large-scale networks, thereby better anticipating and addressing potential issues and better maximizing their net potentials and benefits to the society. Achieving such understanding and effect requires intensive advanced research — research that will develop a solid scientific foundation for the study of large-scale networks and new engineering methodologies for their construction and utilization. Given the important role that large-scale complex networks have been playing and will continuously play in our society, the twenty-first century has become a century of complexity for the new generation of scientists and engineers.

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