Demographic stochasticity

Disease spread in populations of individuals

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US-Africa Advanced Study Institute on Mathematical Modeling of Infectious Diseases in Africa June 2007

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  - In fact, I have no idea what they say

Introduction

Modelling individual events

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- Discrete events are fundamentally stochastic
  - Even in theory we don't know when the next event will occur, nor even what it will be

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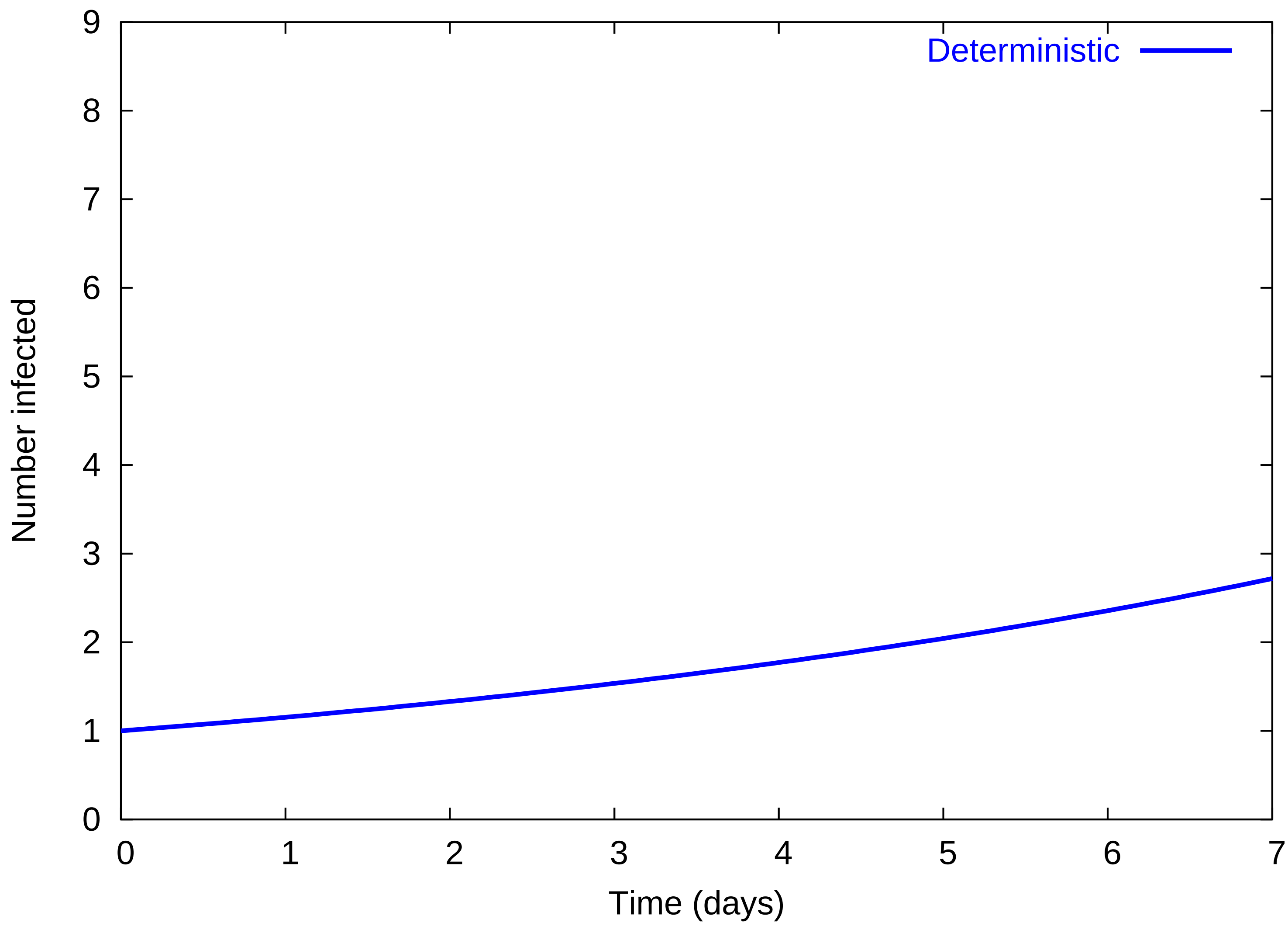
Modelling individual events

A stochastic potential-transmission event

Deterministic spread



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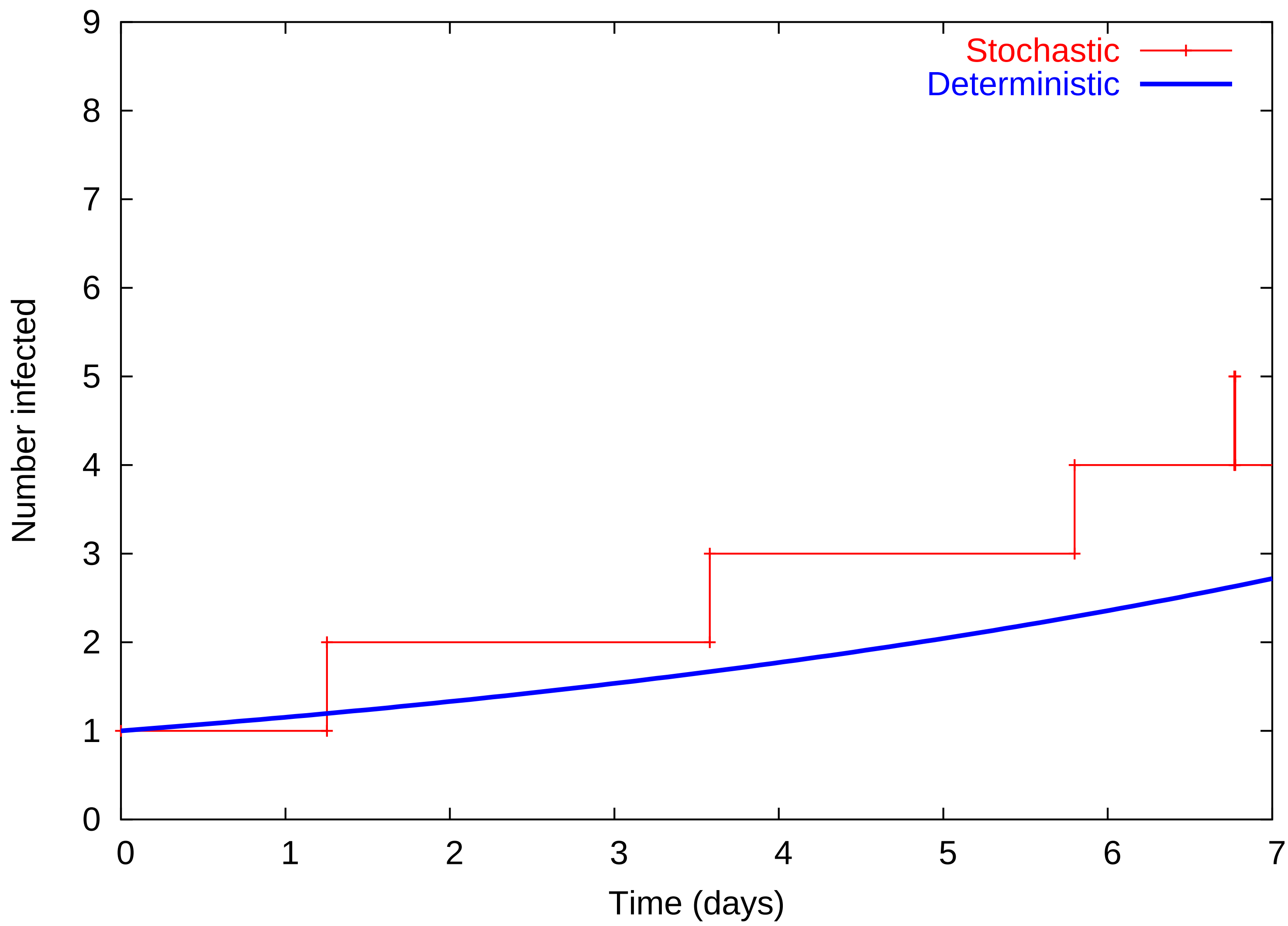

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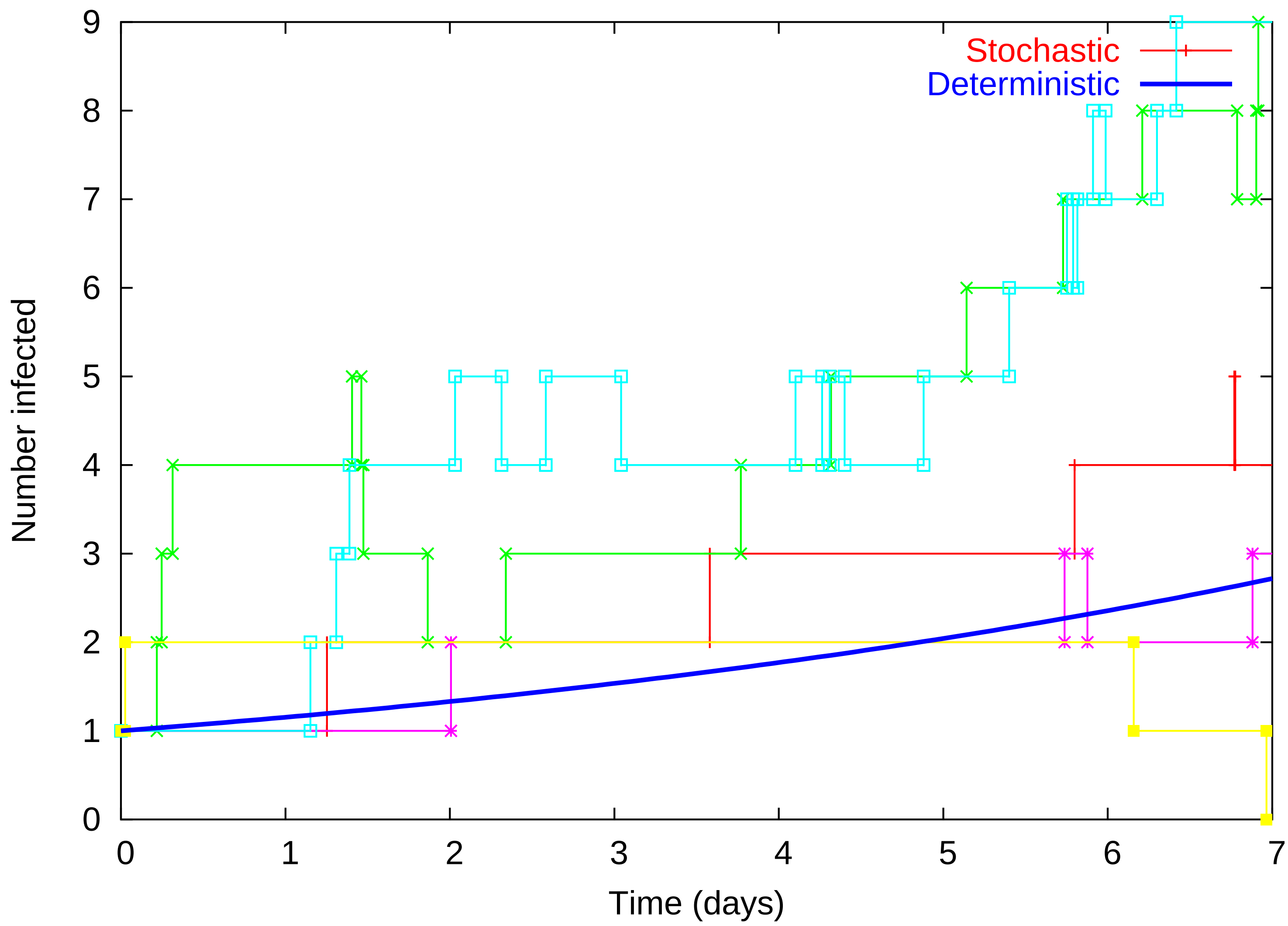

Describing a stochastic process

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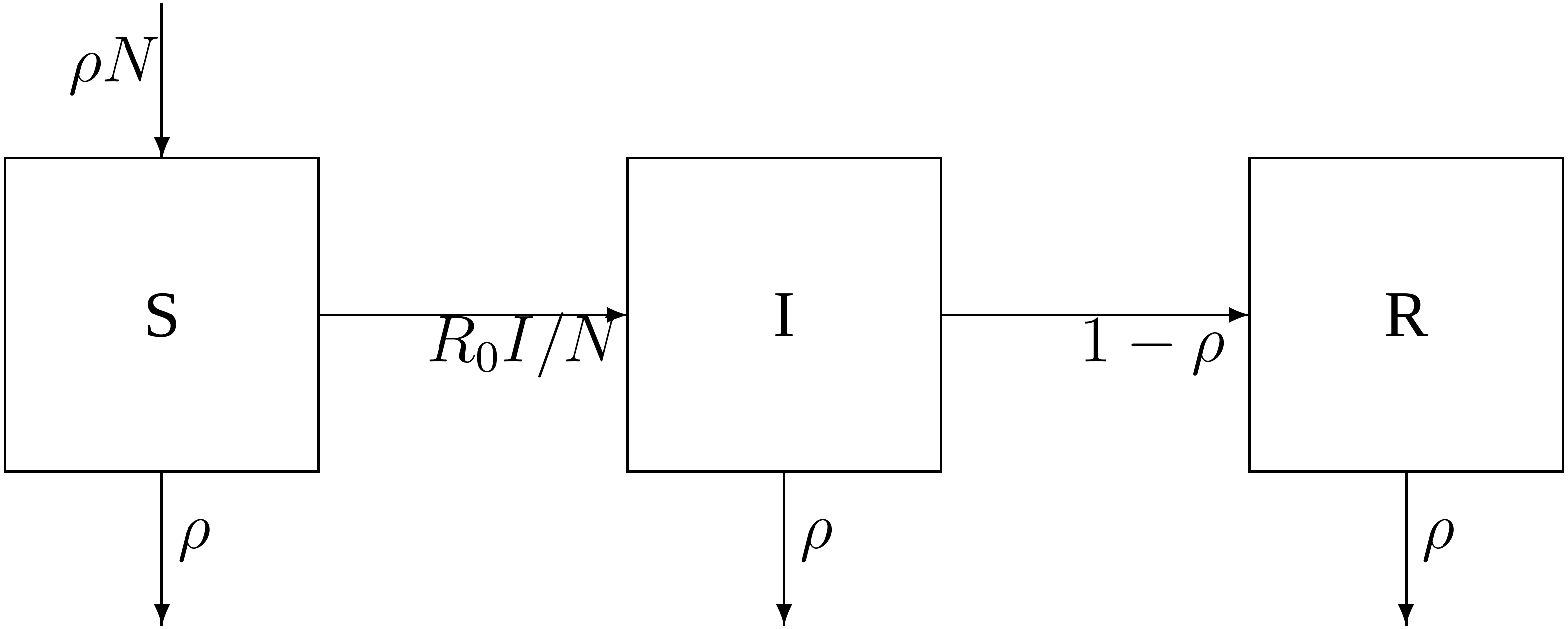
- If the system is *non-Markovian* $r_E(t)$ can depend (for example) on how long the system has been in a particular state

States and rates (Deterministic)

SIR model with birth and death



$$\begin{aligned}\dot{S} &= \rho(N-S) - R_0 SI/N \\ \dot{I} &= R_0 SI/N - I\end{aligned}$$

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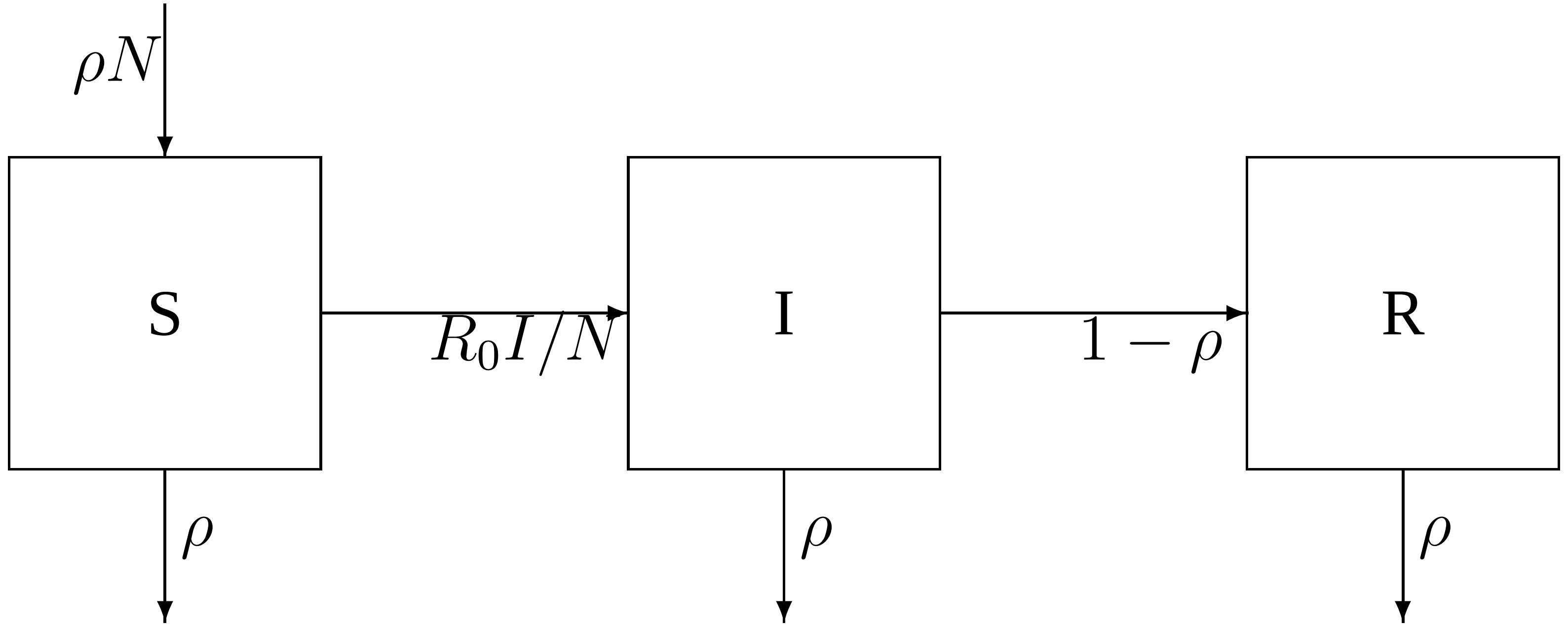

Event	**transition**	**rate**
Infection	$S \rightarrow I$	$R_0 SI/N$
Recovery	$I \rightarrow R$	$(1-\rho)I$
Rebirth	$R \rightarrow S$	ρR
Rebirth	$I \rightarrow S$	ρI

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  - Should we include stochasticity in population size?

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    - Knowing how the ensemble distribution evolves is not the same as understanding the whole ensemble

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- Analytic approximations to the ensemble distribution

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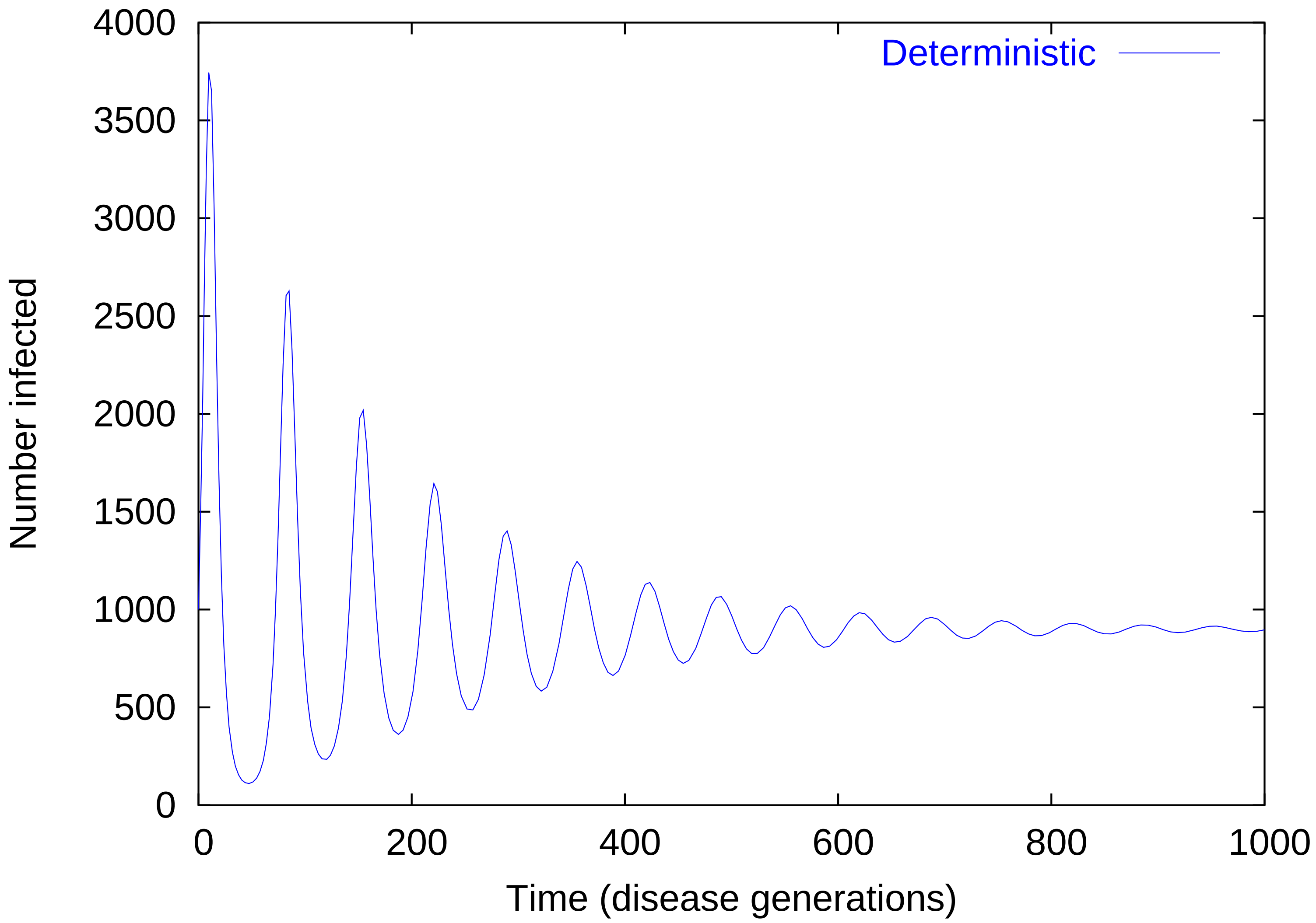

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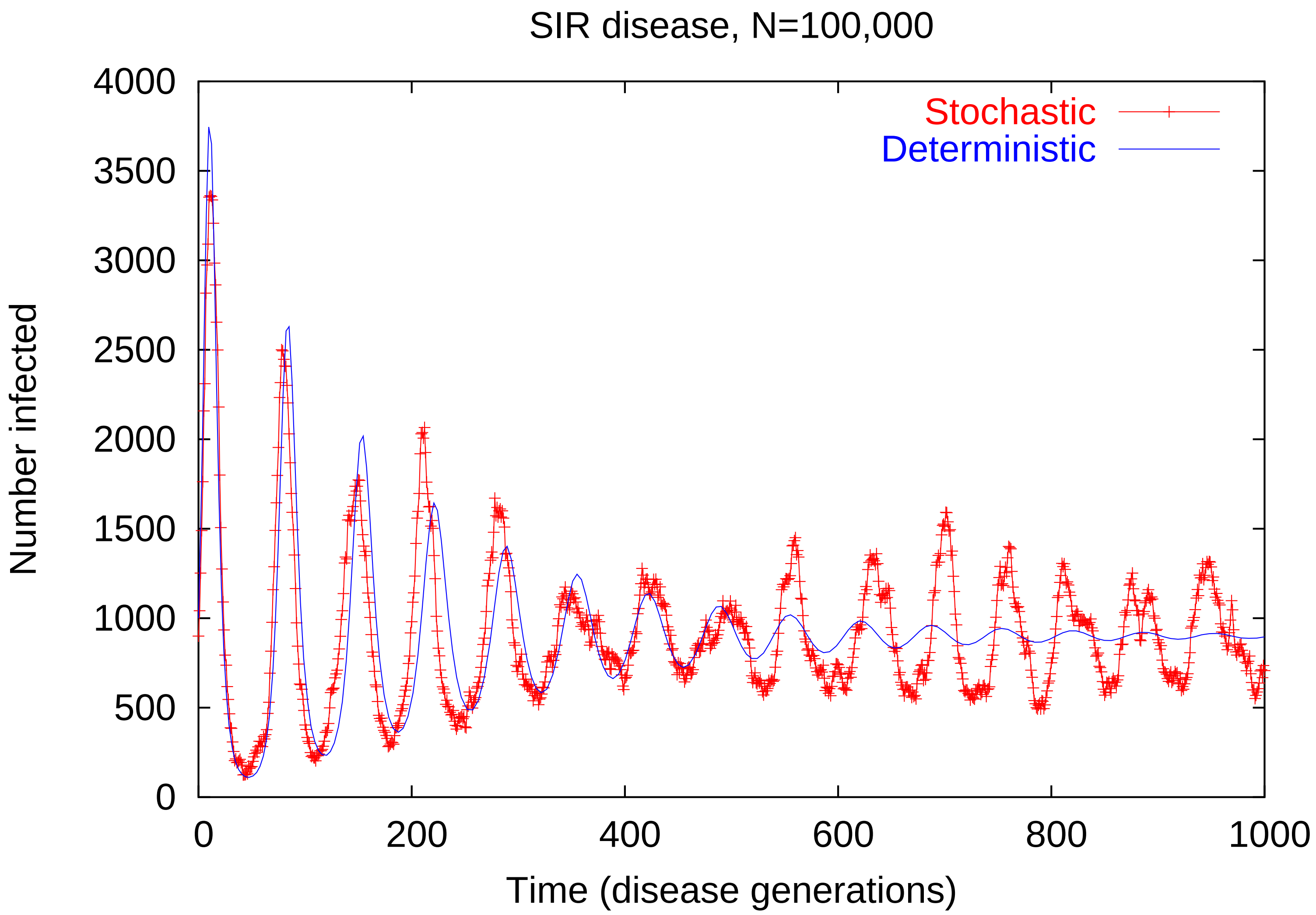

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  - Under what circumstances can we eliminate or eradicate a disease?

Equilibrium and quasi-equilibrium

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    - System will converge to a stable *relative* distribution of probabilities of being in each non-extinct box

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  - The expected persistence time (looking forward) approaches $-1/\lambda_q$ as the system continues to persist

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Analytic methods

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  - **Endemic equilibrium:** tendency to cycle, damping or persistence of cycles
- Both of these methods have analogues in demographic models

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  - Probability of extinction given persistence goes to zero, as expected number of infectious individuals goes to ∞

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- We can define this as the 'fizzle' probability: the disease would have gone extinct even without depleting any susceptibles.

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  - $\dot{\mu} = \sum_I I \dot{p}_I$
  - $= \sum_I (b_I - m_I) p_I$, where $b(I) = R_0 I$ is the 'birth' rate, and $m(I) = I$ is the 'death' rate

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- The expected value (average over realizations) grows exponentially!

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- Moment-closure techniques, and linearization approximations, can help.

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  - Negative binomial with mean $I_0 \exp((R_0 - 1)t$ and parameter I_0

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- Diffusion index $\approx \rho^2 N$. If ρ is small, demographic stochasticity can be important even for very large populations.

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- Reference: Nasell 1999 (on ASI site)

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Simulating realizations

Lagrangian and Eulerian views

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  - We can also use simple Eulerian models to *approximate* more realistic Lagrangian simulations

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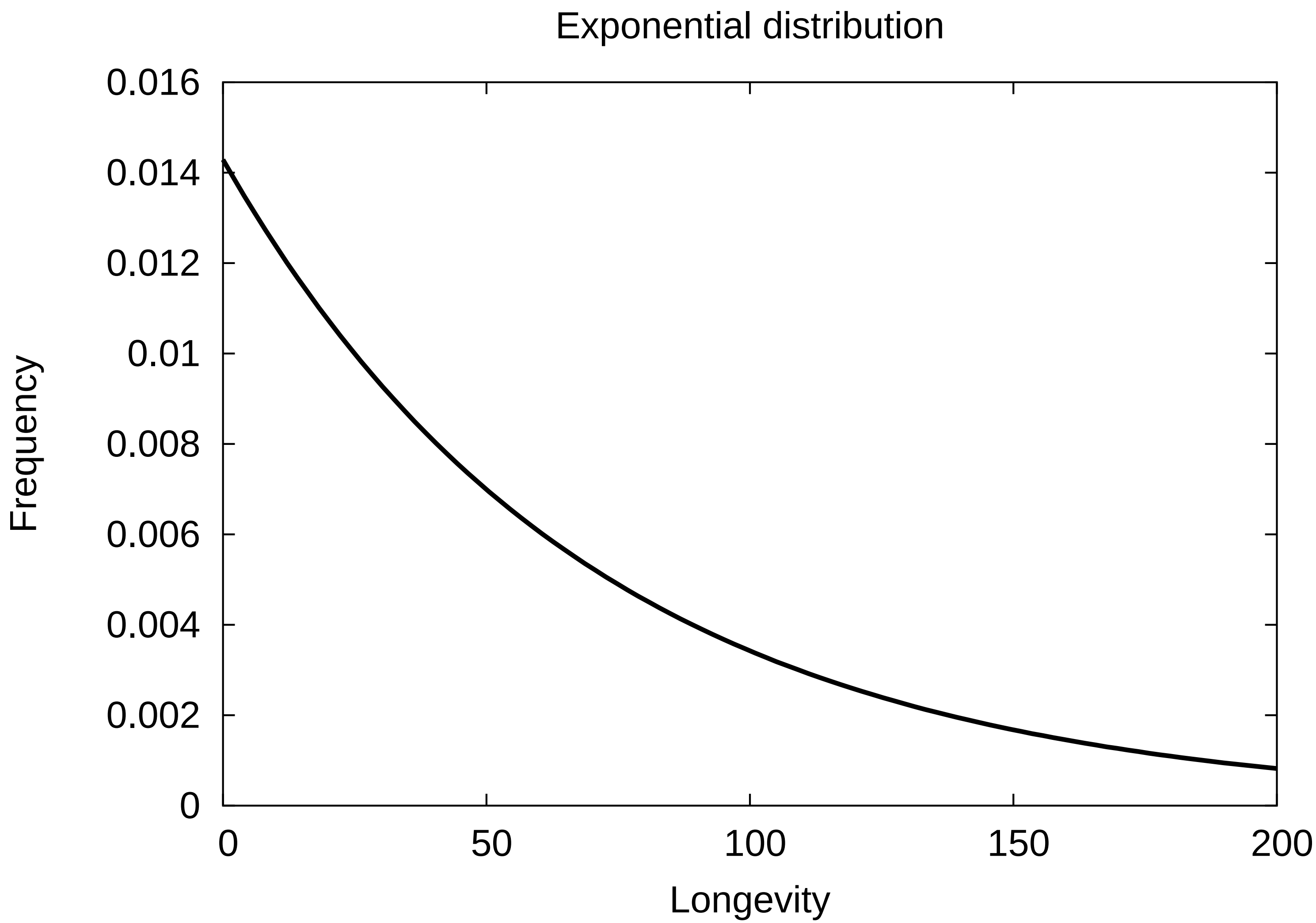
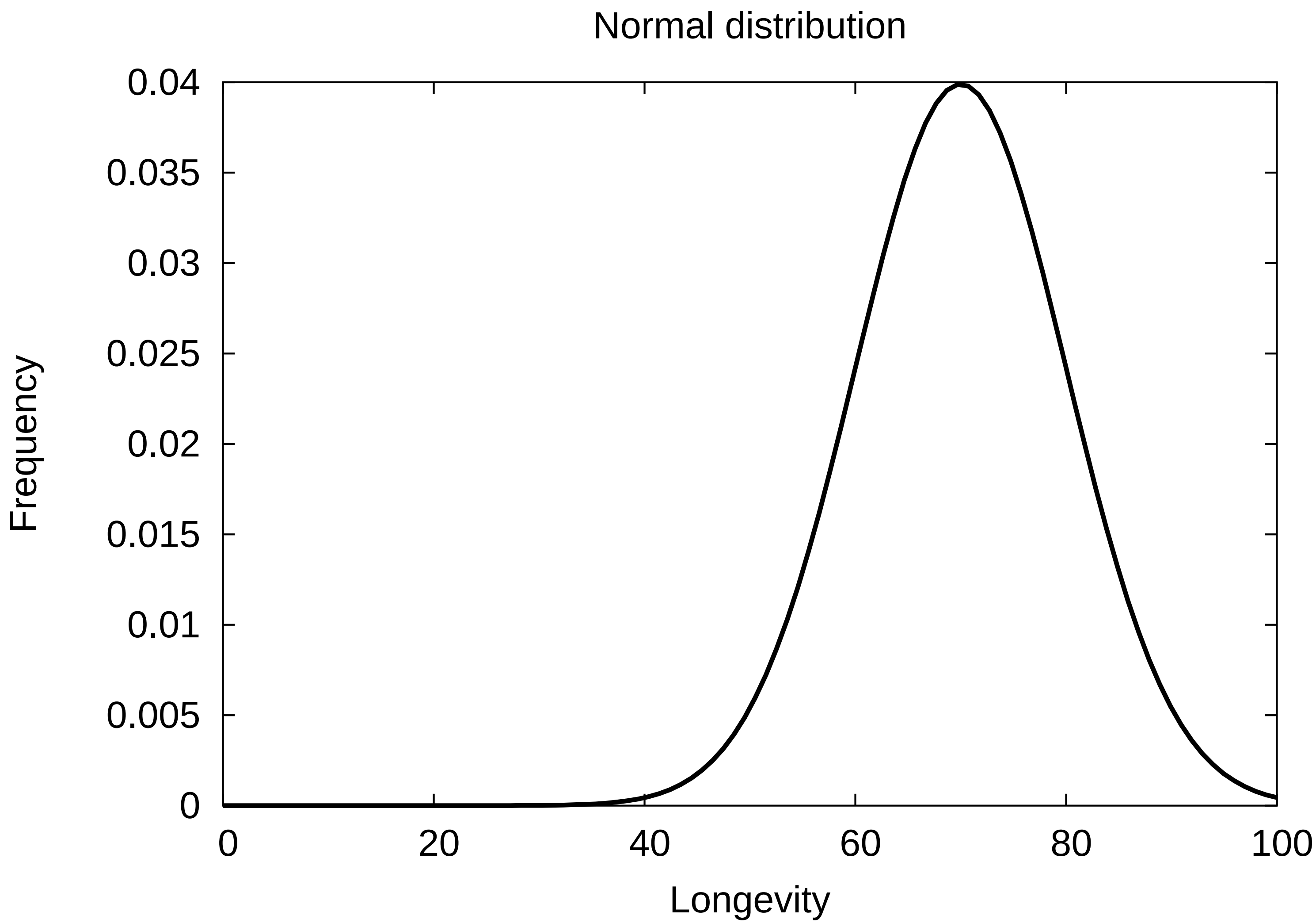
    - Does this mean that classical models are stupid?
    - The assumption needs to be evaluated in terms of the question the model is asking: how much does the assumption effect the dynamics?

An exponential distribution



A normal distribution



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Queueing

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    - When somebody is infected, add their recovery, potentially infectious contacts to the queue

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  - If the infector is infectious, choose a potential infectee at random and update their state

Strain interactions

Multiple strains

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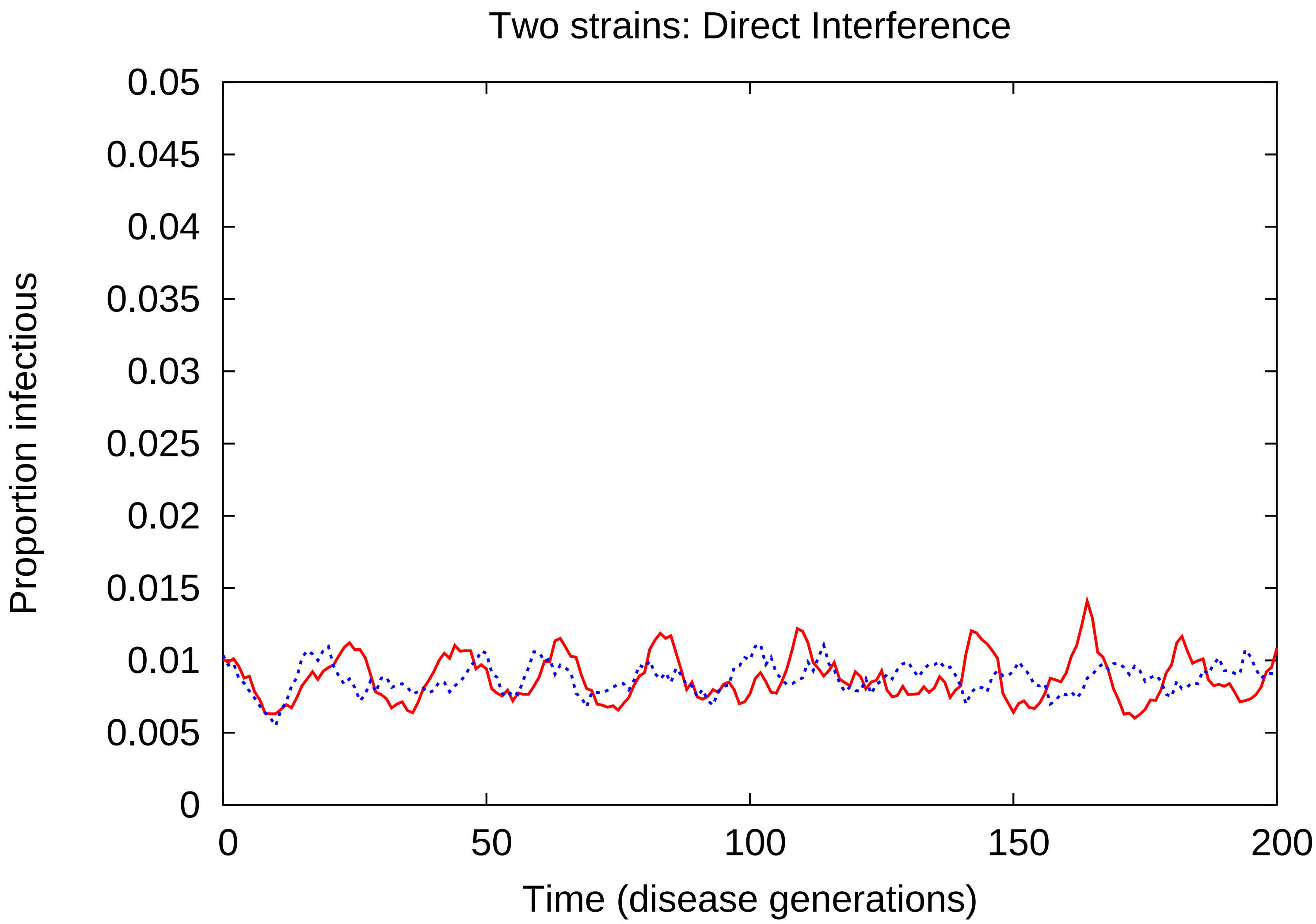
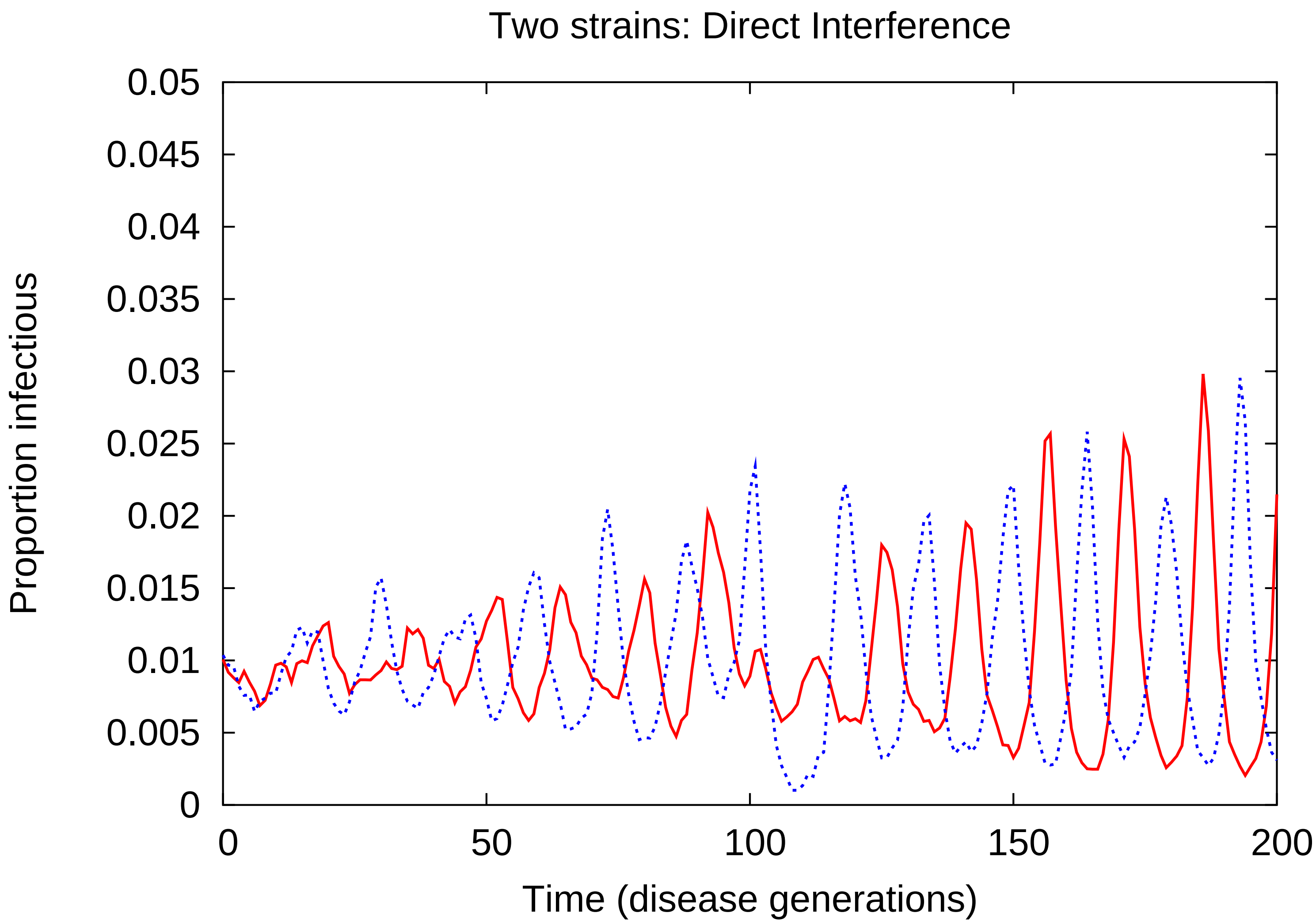
- This is another area where including demographic stochasticity has important effects

Markov (exponential) waiting times



Realistic (Gaussian) waiting times



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  - Simulating realizations
- Remember: demographic stochasticity is real!