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A Comprehensive View of Discrete Mathematics: Chapter 14 of the New Jersey Mathematics Curriculum Framework

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Introduction

This article contains the chapter of the *New Jersey Mathematics Curriculum Framework*[1] which deals with discrete mathematics. The first three pages of the article describes what this document is and why it was written.

On May 1, 1996, the New Jersey Board of Education adopted core curriculum content standards in seven content areas, including mathematics.

These standards describe what all New Jersey students need to know and be able to do at the end of grades 4, 8, and 12. Statewide assessments reflecting these standards are being developed at these grade levels, and students will be expected to demonstrate that they meet these standards in order to graduate from high school.

The standards for mathematics includes a discrete mathematics standard; thus all New Jersey students will be expected to demonstrate understanding and proficiency in discrete mathematics.

The development and adoption of standards extended over a period of three years, and, as Director of the New Jersey Mathematics Coalition, I was very much involved at every step along the way. The mathematics standards represent what New Jersey mathematics educators believe are high achievable standards for all students in the state.

How will New Jersey teachers ensure that their students can meet these standards? During the past four years, the New Jersey Mathematics Coalition, working in collaboration with the New Jersey Department of Education and with an Eisenhower grant from the United States Department of Education, has developed a resource book, the *New Jersey Mathematics Curriculum Framework*; this 688-page document was developed to assist teachers

1991 *Mathematics Subject Classification.* Primary 00A05, 00A35.

[1]Rosenstein, Joseph G., Janet H. Caldwell, and Warren D. Crown, *New Jersey Mathematics Curriculum Framework*, New Jersey Mathematics Coalition, 1996.

and administrators in implementing the mathematics standards at both the classroom and the district level. The preliminary version was published in Spring 1995, and a revised version in December 1996. The preliminary version included the contributions of many New Jersey educators; the revised version incorporated the suggestions of many reviewers and reflected the standards adopted by the Board.

I am pleased to have spearheaded and directed this effort; we have produced a valuable guide for New Jersey teachers and, through its availability on the World Wide Web,[2] for those of other states. The *New Jersey Mathematics Curriculum Framework* is not intended to be a curriculum; rather it is intended to be a structure (i.e., a "framework") around which a district can build its own curriculum (or curricula). This particular framework, however, provides much more detail about the content of K-12 mathematics than any other state framework of which I am aware; for that reason, it should be a valuable resource to all teachers of mathematics.

What should students be expected to know and be able to do? The discrete mathematics standard, like the other mathematics standards (and those in other content areas), consists of a general statement about discrete mathematics followed by five or six statements, called "cumulative progress indicators", which describe what students should be able to do at each of the three grade levels. The discrete mathematics standard and cumulative progress indicators appear at the end of this Introduction.

How will teachers be able to reflect these indicators in their curricula? The discrete mathematics chapter of the *Framework* (like each of the other chapters) is intended to respond to this question. The chapter consists of a K-12 overview of discrete mathematics, followed by sections addressing five different grade levels; for each grade level there is a (self-contained) overview of discrete mathematics for that grade level, followed by a number of classroom activities that illustrate how each indicator could be addressed at that grade level. These materials are arranged in this article in the following sections:

1. Grades K-12 Overview
2. Grades K–2 Overview
3. Grades K–2 Indicators and Activities
4. Grades 3–4 Overview
5. Grades 3–4 Indicators and Activities
6. Grades 5–6 Overview
7. Grades 5–6 Indicators and Activities
8. Grades 7–8 Overview
9. Grades 7–8 Indicators and Activities
10. Grades 9–12 Overview
11. Grades 9–12 Indicators and Activities

[2] `http://dimacs.rutgers.edu/nj_math_coalition/framework.html/`

Note that because the materials for each grade level are self-contained, there is considerable overlap between the overviews (even numbered sections). Note also that all references for each grade level are provided at the end of the odd numbered sections.

The activities in this chapter are based on activities used by teachers in the DIMACS-sponsored and NSF-funded Leadership Program in Discrete Mathematics, which I have directed since its inception in 1989 (see article in this volume). The organization of discrete mathematics into five areas and the list of indicators, one for each area at each grade level, emerged from a series of discussions in 1993 by Rutgers University faculty associated with DIMACS. Although I have been responsible for the selection and writing of the activities, as well as the overall organization of the material, I would like to acknowledge the assistance I received from a number of people, including many participants in the Leadership Program, who reviewed and commented on drafts of this chapter. The expectation is that, through the wonders of the Web, the entire *Framework* and this chapter on discrete mathematics in particular will continue to evolve.

And now, Chapter 14 of the *New Jersey Mathematics Curriculum Framework*, which addresses the following standard and cumulative progress indicators of the *New Jersey Core Curriculum Content Standards:*

> All students will apply the concepts and methods of discrete mathematics to model and explore a variety of practical situations.

Cumulative Progress Indicators

By the end of Grade 4, students:

1. Explore a variety of puzzles, games, and counting problems.
2. Use networks and tree diagrams to represent everyday situations.
3. Identify and investigate sequences and patterns found in nature, art, and music.
4. Investigate ways to represent and classify data according to attributes, such as shape or color, and relationships, and discuss the purpose and usefulness of such classification.
5. Follow, devise, and describe practical lists of instructions.

Building upon knowledge and skills gained in the preceding grades, by the end of Grade 8, students:

6. Use systematic listing, counting, and reasoning in a variety of contexts.
7. Recognize common discrete mathematical models, explore their properties, and design them for specific situations.
8. Experiment with iterative and recursive processes, with the aid of calculators and computers.
9. Explore methods for storing, processing, and communicating information.

10. Devise, describe, and test algorithms for solving optimization and search problems.

Building upon knowledge and skills gained in the preceding grades, by the end of Grade 12, students:

11. Understand the basic principlies of iteration, recursion, and mathematical induction.
12. Use basic principles to solve combinatorial and algorithmic problems.
13. Use discrete models to represent and solve problems.
14. Analyze iterative processes with the aid of calculators and computers.
15. Apply discrete methods to storing, processing, and communicating information.
16. Apply discrete methods to problems of voting, apportionment, and allocations, and use fundamental strategies of optimizaion to solve problems.

1. Grades K-12 Overview

Descriptive Statement. Discrete mathematics is the branch of mathematics that deals with arrangements of distinct objects. It includes a wide variety of topics and techniques that arise in everyday life, such as how to find the best route from one city to another, where the objects are cities arranged on a map. It also includes how to count the number of different combinations of toppings for pizzas, how best to schedule a list of tasks to be done, and how computers store and retrieve arrangements of information on a screen. Discrete mathematics is the mathematics used by decision-makers in our society, from workers in government to those in health care, transportation, and telecommunications. Its various applications help students see the relevance of mathematics in the real world.

Meaning and Importance. During the past 30 years, discrete mathematics has grown rapidly and has evolved into a significant area of mathematics. It is the language of a large body of science and provides a framework for decisions that individuals will need to make in their own lives, in their professions, and in their roles as citizens. Its many practical applications can help students see the relevance of mathematics to the real world. It does not have extensive prerequisites, yet it poses challenges to all students. It is fun to do, is often geometry based, and can stimulate an interest in mathematics on the part of students at all levels and of all abilities.

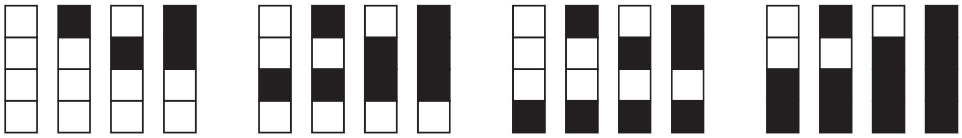
K-12 Development and Emphases. Although the term "discrete mathematics" may seem unfamiliar, many of its themes are already present in the classroom. Whenever objects are counted, ordered, or listed, whenever instructions are presented and followed, whenever games are played and analyzed, teachers are introducing themes of discrete mathematics. Through understanding these themes, teachers will be able to recognize and introduce them regularly in classroom situations. For example, when calling three students to work at the three segments of the chalkboard, the teacher might ask *In how many different orders can these three students work at the board?* Another version of the same question is *How many different ways, such as ABC, can you name a triangle whose vertices are labeled A, B, and C?* A similar, but slightly different question is *In how many different orders can three numbers be multiplied?*

Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools.* The material in this chapter is drawn from activities that have been reviewed and classroom-tested by the K-12 teachers in the Rutgers University Leadership Program in Discrete Mathematics over the past nine years; this program is funded by the National Science Foundation.

Students should learn to recognize examples of discrete mathematics in familiar settings, and explore and solve a variety of problems for which discrete techniques have proved useful. These ideas should be pursued throughout the school years. Students can start with many of the basic ideas in concrete settings, including games and general play, and progressively develop these ideas in more complicated settings and more abstract forms. Five major themes of discrete mathematics should be addressed at all K-12 grade levels — **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using algorithms to find the best solution to real-world problems.** These five themes are discussed in the paragraphs below.

Students should use a variety of strategies to **systematically list and count** the number of ways there are to complete a particular task. For example, elementary school students should be able to make a list of all possible outcomes of a simple situation such as the number of outfits that can be worn using two coats and three hats. Middle school students should be able to systematically list and count the number of different four-block-high towers that can be built using blue and red blocks (see example below), or the number of possible routes from one location on a map to another, or the number of different "words" that can be made using five letters. High school students should be able to determine the number of possible orderings of an arbitrary number of objects and to describe procedures for listing and counting all such orderings. These strategies for listing and counting should be applied by both middle school and high school students to solve problems in probability.

Following is a list of all four-block-high towers that can be built using clear blocks and solid blocks. The 16 towers are presented in a systematic list — the first 8 towers have a clear block at the bottom and the second 8 towers have a solid block at the bottom; within each of these two groups, the first 4 towers have the second block clear, and the second 4 towers have the second block solid; etc.

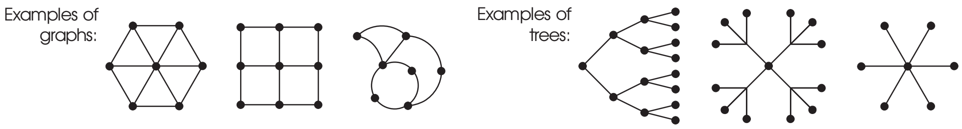


If each tower is described alphabetically as a sequence of C's and S's, representing "clear" and "solid" — the tower at the left, for example, would be `C-C-C-C`, and the third tower from the left would be `C-C-S-C`, reading from the bottom up — then the sixteen towers would be in alphabetical order:

```
C-C-C-C  C-S-C-C  S-C-C-C  S-S-C-C
C-C-C-S  C-S-C-S  S-C-C-S  S-S-C-S
C-C-S-C  C-S-S-C  S-C-S-C  S-S-S-C
C-C-S-S  C-S-S-S  S-C-S-S  S-S-S-S
```

There are other ways of systematically listing the 16 towers; for example, the list could contain first the one tower with no solid blocks, then the four towers with one solid block, then the six towers with two solid blocks, then the four towers with three solid blocks, and finally the one tower with four solid blocks.

Discrete mathematical models such as graphs (networks) and trees (such as those pictured below) can be used to represent and solve a variety of problems based on real-world situations.



In the left-most graph of the figures above, all seven dots are linked into a network consisting of the six line segments emerging from the center dot; these six line segments form the tree at the far right which is said to "span" the original graph since it reaches all of its points. Another example: if we think of the second graph as a street map and we make the streets one way, we can represent the situation using a directed graph where the line segments are replaced by arrows.

Elementary school students should recognize that a street map can be represented by a graph and that routes can be represented by paths in the graph; middle school students should be able to find cost-effective ways of linking sites into a network using spanning trees; and high school students should be able to use efficient methods to organize the performance of individual tasks in a larger project using directed graphs.

Iterative patterns and processes are used both for describing the world and in solving problems. An iterative pattern or process is one which involves repeating a single step or sequence of steps many times. For example, elementary school students should understand that multiplication corresponds to repeatedly adding the same number a specified number of times. They should investigate how decorative floor tilings can often be described as the repeated use of a small pattern, and how the patterns of rows in pine cones follow a simple mathematical rule. Middle school students should explore how simple repetitive rules can generate interesting patterns by using spirolaterals or Logo commands, or how they can result in extremely complex behavior by generating the beginning stages of fractal curves. They should investigate the ways that the plane can be covered by repeating patterns, called tessellations. High school students should understand how many processes describing the change of physical, biological, and economic systems over time can be modeled by simple equations applied repetitively, and use these models to predict the long-term behavior of such systems.

Students should explore different methods of **arranging, organizing, analyzing, transforming, and communicating information**, and understand how these methods are used in a variety of settings. Elementary school students should investigate ways to represent and classify data according to attributes such as color or shape, and to organize data into structures like tables or tree diagrams or Venn diagrams. Middle school students should be able to read, construct, and analyze tables, matrices, maps and other data structures. High school students should understand the application of discrete methods to problems of information processing and computing such as sorting, codes, and error correction.

Students should be able to **follow and devise lists of instructions, called "algorithms," and use them to find the best solution to real-world problems** — where "best" may be defined, for example, as most cost-effective or as most equitable. For example, elementary school students should be able to carry out instructions for getting from one location to another, should discuss different ways of dividing a pile of snacks, and should determine the shortest path from one site to another on a map laid out on the classroom floor. Middle school students should be able to plan an optimal route for a class trip (see the vignette in the Introduction to this *Framework* entitled *Short-circuiting Trenton*), write precise instructions for adding two two-digit numbers, and, pretending to be the manager of a fast-food restaurant, devise work schedules for employees which meet specified conditions yet minimize the cost. High school students should be conversant with fundamental strategies of optimization, be able to use flow charts to describe algorithms, and recognize both the power and limitations of computers in solving algorithmic problems.

IN SUMMARY, discrete mathematics is an exciting and appropriate vehicle for working toward and achieving the goal of educating informed citizens who are better able to function in our increasingly technological society; have better reasoning power and problem-solving skills; are aware of the importance of mathematics in our society; and are prepared for future careers which will require new and more sophisticated analytical and technical tools. It is an excellent tool for improving reasoning and problem-solving abilities.

Note: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

References.

- Kenny, M. J., Ed. *Discrete Mathematics Across the Curriculum K-12.* 1991 Yearbook of the National Council of Teachers of Mathematics (NCTM). Reston, VA, 1991.
- Rosenstein, J. G., D. Franzblau, and F. Roberts, Eds. *Discrete Mathematics in the Schools.* Proceedings of a 1992 DIMACS Conference on "Discrete Mathematics in the Schools." DIMACS Series on Discrete Mathematics and Theoretical Computer Science. Providence, RI: American Mathematical Society (AMS), 1997.

On-Line Resources.

`http://dimacs.rutgers.edu/nj_math_coalition/framework.html/`

> The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

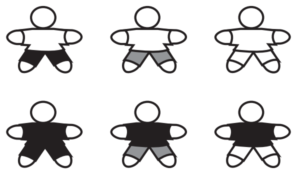
2. Grades K-2 Overview

The five major themes of discrete mathematics, as discussed in the K-12 Overview,[3] are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

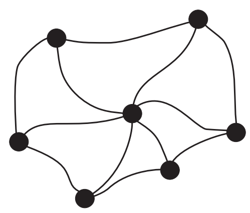
Despite their formidable titles, these five themes can be addressed with activities at the K-2 grade level which involve purposeful play and simple analysis. Indeed, teachers will discover that many activities they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

[3]Since K-2 grade level teachers may not read the K-12 Overview, and, more generally, teachers at other grade levels will begin their review of this chapter of the *Framework* by turning to the section addressing their own grade levels, the grade level overviews have significant overlap.

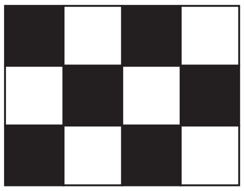
Activities involving **systematic listing, counting, and reasoning** can be done very concretely at the K-2 grade level. For example, dressing cardboard teddy bears with different outfits becomes a mathematical activity when the task is to make a list of all possible outfits and count them; pictured below are the six outfits that can be arranged using one of two types of shirts and one of three types of shorts. Similarly, playing any game involving choices becomes a mathematical activity when children reflect on the moves they make in the game.



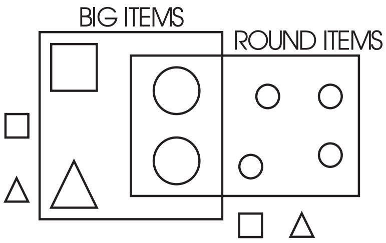
An important **discrete mathematical model** is that of a **network or graph**, which consists of dots and lines joining the dots; the dots are often called *vertices* (*vertex* is the singular) and the lines are often called *edges*. (This is different from other mathematical uses of the term "graph.") The two terms "network" and "graph" are used interchangeably for this concept. An example of a graph with seven vertices and twelve edges is given below. You can think of the vertices of this graph as islands in a river and the edges as bridges. You can also think of them as buildings and roads, or houses and telephone cables, or people and handshakes; wherever a collection of things are joined by connectors, the mathematical model used is that of a network or graph. At the K-2 level, children can recognize graphs and use life-size models of graphs in various ways. For example, a large version of this graph, or any other graph, can be "drawn" on the floor using paper plates as vertices and masking tape as edges. Children might select two "islands" and find a way to go from one island to the other island by crossing exactly four "bridges." (This can be done for any two islands in this graph, but not necessarily in another graph.)



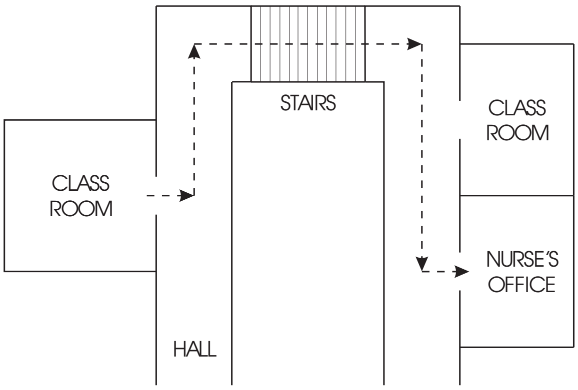
Children can recognize and work with **repetitive patterns and processes** involving numbers and shapes, using objects in the classroom and in the world around them. For example, children at the K-2 level can create (and decorate) a pattern of triangles or squares (as pictured here) that cover a section of the floor (this is called a "tessellation"), or start with a number and repeatedly add three, or use clapping and movement to simulate rhythmic patterns.



Children at the K-2 grade levels should investigate ways of **sorting items** according to attributes like color, shape, or size, and ways of **arranging data** into charts, tables, and family trees. For example, they can sort attribute blocks or stuffed animals by color or kind, as in the diagram, and can count the number of children who have birthdays in each month by organizing themselves into birthday-month groups.



Finally, at the K-2 grade levels, children should be able to **follow and describe simple procedures** and determine and discuss **what is the best solution** to a problem. For example, they should be able to follow a prescribed route from the classroom to another room in the school (as pictured below) and to compare various alternate routes, and in the second grade should determine the shortest path from one site to another on a map laid out on the classroom floor.



Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools*. Another important resource for K-2 teachers is *This Is MEGA-Mathematics!*

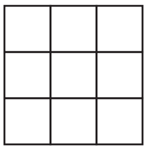
3. Grades K-2 Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Explore a variety of puzzles, games, and counting problems.**

- Students use teddy bear cut-outs with, for example, shirts of two colors and shorts of three colors, and decide how many different outfits can be made by making a list of all possibilities and arranging them systematically. (See illustration in K-2 Overview.)
- Students use paper faces or Mr. Potato Head type models to create a "regular face" given a nose, mouth, and a pair of eyes. Then they use another pair of eyes, then another nose, and then another mouth (or other parts) and explore and record the number of faces that can be made after each additional part has been included.
- Students read *A Three Hat Day* and then try to create as many different hats as possible with three hats, a feather, a flower, and a ribbon as decoration. Students count the different hats they've made and discuss their answers.
- Students count the number of squares of each size (1×1, 2×2, 3×3) that they can find on the square grid below. They can be challenged to find the numbers of small squares of each size on a larger square or rectangular grid.



- Students work in groups to figure out the rules of addition and placement that are used to pass from one row to the next in the diagram below, and use these rules to find the numbers in the next few rows.

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \end{array}$$

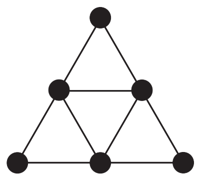
  In this diagram, called Pascal's triangle, each number is the sum of the two numbers that are above it, to its left and right; the numbers on the left and right edges are all 1.
- Students cut out five "coins" labeled 1¢, 2¢, 4¢, 8¢, and 16¢. For each number in the counting sequence $1, 2, 3, 4, 5, \ldots$ (as far as is appropriate for a particular group of students), students determine

how to obtain that amount of money using a combination of different coins.

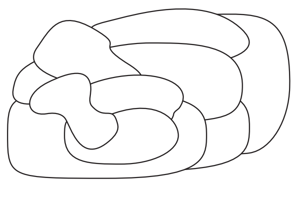
- Students play simple games and discuss why they make the moves they do. For example, two students divide a six-piece domino set (with 0-0, 0-1, 0-2, 1-1, 1-2, and 2-2) and take turns placing dominoes so that dominoes which touch have the same numbers and so that all six dominoes are used in the chain.

2. **Use networks and tree diagrams to represent everyday situations.**

- Students find a way of getting from one island to another, in the graph described in the K-2 Overview laid out on the classroom floor with masking tape, by crossing exactly four bridges. They make their own graphs, naming each of the islands, and make a "from-to" list of islands for which they have found a four-bridge-route. (Note: it may not always be possible to find four-bridge-routes.)
- Students count the number of edges at each vertex (called the **degree** of the vertex) of a network and construct graphs where all vertices have the same degree, or where all the vertices have one of two specified degrees.
- On a pattern of islands and bridges laid out on the floor, students try to find a way of visiting each island exactly once; they can leave colored markers to keep track of islands already visited. Note that for some patterns this may not be possible! Students can be challenged to find a way of visiting each island exactly once which returns them to their starting point. Similar activities can be found in *Inside, Outside, Loops, and Lines* by Herbert Kohl.

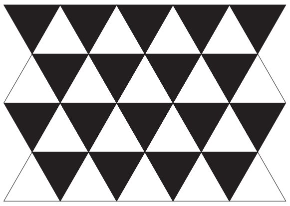


- Students create a map with make-believe countries (see example below), and color the maps so that countries which are next to each other have different colors. *How many colors were used? Could it be done with fewer colors? with four colors? with three colors? with two colors?* A number of interesting map coloring ideas can be found in *Inside, Outside, Loops and Lines* by Herbert Kohl.



3. **Identify and investigate sequences and patterns found in nature, art, and music.**

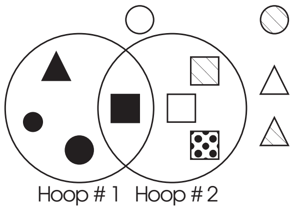
   - Students use a calculator to create a sequence of ten numbers starting with zero, each of which is three more than the previous one; on some calculators, this can be done by pressing $0 + 3 = = = \ldots$, where $=$ is pressed ten times. As they proceed, they count one 3, two 3s, three 3s, etc.
   - Students "tessellate" the plane, by using groups of squares or triangles (for example, from sets of pattern blocks) to completely cover a sheet of paper without overlapping; they record their patterns by tracing around the blocks on a sheet of paper and coloring the shapes.



   - Students listen to or read *Grandfather Tang's Story* by Ann Tompert and then use tangrams to make the shape-changing fox fairies as the story progresses. Students are then encouraged to do a retelling of the story with tangrams or to invent their own tangram characters and stories.
   - Students read *The Cat in the Hat* or *Green Eggs and Ham* by Dr. Seuss and identify the pattern of events in the book. Students could create their own books with similar patterns.
   - Students collect leaves and note the patterns of the veins. They look at how the veins branch off on each side of the center vein and observe that their branches are smaller copies of the original vein pattern. Students collect feathers, ferns, Queen Anne's lace, broccoli, or cauliflower and note in each case how the pattern of the original is repeated in miniature in each of its branches or clusters.
   - Students listen for rhythmic patterns in musical selections and use clapping, instruments, and movement to simulate those patterns.
   - Students take a "patterns walk" through the school, searching for patterns in the bricks, the play equipment, the shapes in the classrooms, the number sequences of classrooms, the floors and ceilings, etc.; the purpose of this activity is to create an awareness of all the patterns around them.

4. **Investigate ways to represent and classify data according to attributes, such as shape or color, and relationships, and discuss the purpose and usefulness of such classification.**

- Students sort themselves by month of birth, and then within each group by height or birth date. (Other sorting activities can be found in *Mathematics Their Way*, by Mary Baratta-Lorton.)
- Each student is given a card with a different number on it. Students line up in a row and put the numbers in numerical order by exchanging cards, one at a time, with adjacent children. (After practice, this can be accomplished without talking.)
- Students draw stick figures of members of their family and arrange them in order of size.
- Students sort stuffed animals in various ways and explain why they sorted them as they did. Students can use *Tabletop, Jr.* software to sort characters according to a variety of attributes.
- Using attribute blocks, buttons, or other objects with clearly distinguishable attributes such as color, size, and shape, students develop a sequence of objects where each differs from the previous one in only one attribute. *Tabletop, Jr.* software can also be used to create such sequences of objects.
- Students use two Hula Hoops (or large circles drawn on paper so that a part of their interiors overlap) to assist in sorting attribute blocks or other objects according to two characteristics. For example, given a collection of objects of different colors and shapes, students are asked to place them so that all red items go inside hoop #1 and all others go on the outside, and so that all square items go inside hoop #2 and all others go on the outside. *What items should be placed in the overlap of the two hoops? What is inside only the first hoop? What is outside both hoops?*

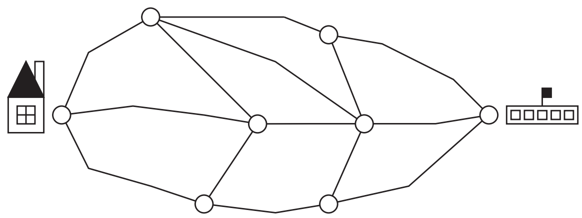


This is an example of a Venn diagram. Students can also use Venn diagrams to organize the similarities and differences between the information in two stories by placing all features of the first story in hoop #1 and all features of the second story in hoop #2, with common features in the overlap of the two hoops. A similar activity can be found in the *Shapetown* lesson that is described in the First Four Standards of this *Framework*. *Tabletop, Jr.* software allows students to arrange and sort data, and to explore these concepts easily.

5. **Follow, devise, and describe practical lists of instructions.**

- Students follow directions for a trip within the classroom — for example, students are asked where they would end up if they started at a given spot facing in a certain direction, took three steps forward, turned left, took two steps forward, turned right, and moved forward three more steps.

- Students follow oral directions for going from the classroom to the lunchroom, and represent these directions with a diagram. (See K-2 Overview for a sample diagram.)
- Students agree on a procedure for filling a box with rectangular blocks. For example, a box with dimensions 4"×4"×5" can be filled with 10 blocks of dimensions 1"×2"×4". (Linking cubes can be used to create the rectangular blocks.)
- Students explore the question of finding the shortest route from school to home on a diagram like the one pictured below, laid out on the floor using masking tape, where students place a number of counters on each line segment to represent the length of that segment. (The shortest route will depend on the placement of the counters; what appears to be the most direct route may not be the shortest.)



- Students find a way through a simple maze. They discuss the different paths they took and their reasons for doing so.
- Students use Logo software to give the turtle precise instructions for movement in specified directions.

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Software.

- *Logo.* Many versions of Logo are commercially available.
- *Tabletop, Jr.* Broderbund Software. TERC.

4. Grades 3-4 Overview

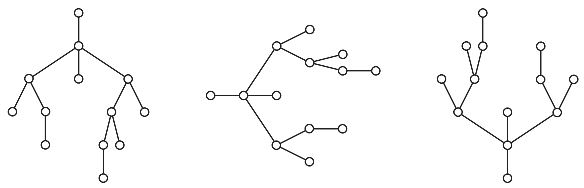
The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

Despite their formidable titles, these five themes can be addressed with activities at the 3-4 grade level which involve purposeful play and simple analysis. Indeed, teachers will discover that many activities that they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

The following discussion of activities at the 3-4 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-2 grade levels. Hence 3-4 grade teachers should review the K-2 grade level discussion of discrete mathematics and might use activities similar to those described there before introducing the activities for this grade level.

Activities involving **systematic listing, counting, and reasoning** should be done very concretely at the 3-4 grade levels, building on similar activities at the K-2 grade levels. For example, the children could systematically list and count the total number of possible combinations of dessert and beverage that can be selected from pictures of those two types of foods they have cut out of magazines or that can be selected from a restaurant menu. Similarly, playing games like Nim, dots and boxes, and dominoes becomes a mathematical activity when children systematically reflect on the moves they make in the game and use those reflections to decide on the next move.

An important **discrete mathematical model** is that of a **graph**, which is used whenever a collection of things are joined by connectors — such as buildings and roads, islands and bridges, or houses and telephone cables — or, more abstractly, whenever the objects have some defined relationship to each other; this kind of model is described in the K-2 Overview. At the 3-4 grade levels, children can recognize and use models of graphs in various ways, for example, by finding a way to get from one island to another by crossing exactly four bridges, or by finding a route for a city mail carrier which uses each street once, or by constructing a collaboration graph for the class which describes who has worked with whom during the past week. A special kind of graph is called a "tree." Three views of the same tree are pictured in the diagram below; the first suggests a family tree, the second a tree diagram, and the third a "real" tree.



At the 3-4 grade levels, students can use a tree diagram to organize the six ways that three people can be arranged in order. (See the Grades 3-4 Indicators and Activities for an example.)

Students can recognize and work with **repetitive patterns and processes** involving numbers and shapes, with classroom objects and in the world around them. Children at the 3-4 grade levels are fascinated with the Fibonacci sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... where every number is the sum of the previous two numbers. This sequence of numbers turns up in petals of flowers, in the growth of populations (see the activity involving rabbits), in pineapples and pine cones, and in lots of other places in nature. Another important sequence to introduce at this age is the doubling sequence 1, 2, 4, 8, 16, 32, ... and to discuss different situations in which it appears.

Students at the 3-4 grade levels should investigate ways of **sorting items** according to attributes like color or shape, or by quantitative information like size, **arranging data** using tree diagrams and building charts and tables, and **recovering hidden information** in games and encoded messages. For example, they can sort letters into zip code order or sort the class alphabetically, create bar charts based on information obtained experimentally (such as soda drink preferences of the class), and play games like hangman to discover hidden messages.

Students at the 3-4 grade levels should **describe and discuss simple algorithmic procedures** such as providing and following directions from one location to another, and should in simple cases determine and discuss **what is the best solution** to a problem. For example, they might follow a recipe to make a cake or to assemble a simple toy from its component parts.

Or they might find the best way of playing tic-tac-toe or the shortest route that can be used to get from one location to another.

Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools*. Another important resource for 3-4 teachers is *This Is MEGA-Mathematics!*

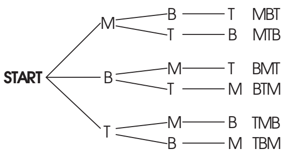
5. Grades 3-4 Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 3-4:

1. **Explore a variety of puzzles, games, and counting problems.**

- Students read *One Hundred Hungry Ants* by Elinor Pinczes and then illustrate and write their own story books (perhaps titled *18 Ailing Alligators* or *24 Furry Ferrets*) in a style similar to the book using as many different arrangements of the animals as possible in creating their books. They read their books to students in the lower grades.
- Students count the number of squares of each size (1×1, 2×2, 3×3, 4×4, 5×5) that they can find on a geoboard, and in larger square or rectangular grids.
- Students determine the number of possible combinations of dessert and beverage that could be selected from pictures of those two types of foods they have cut out of magazines. Subsequently, they determine the number of possible combinations of dessert and beverage that could be chosen from a restaurant menu, and how many of those combinations could be ordered if they only have $4.
- Students find the number of different ways to make a row of four flowers each of which could be red or yellow. They can model this with Unfix cubes and explain how they know that all combinations have been obtained.
- Students determine the number of different ways any three people can be arranged in order, and use a tree diagram to organize the information. The tree diagram below represents the six ways that Barbara (B), Maria (M), and Tarvanda (T), can be arranged in order. The three branches emerging from the "start" position represent the three people who could be first; each path from left to right represents the arrangement of the three people listed to the right.



- Each student uses four squares to make designs where each square shares an entire side with at least one of the other three squares. Geoboards, attribute blocks or Linker cubes can be used. *How many different shapes can be made?* These shapes are called "tetrominoes."
- Each group of students receives a bag containing four colored beads. One group may be given 1 red, 1 black and 2 green beads; other groups may have the same four beads or different ones. Students take turns drawing a bead from the bag, recording its color, and replacing it in the bag. After 20 beads are drawn, each group makes a bar graph illustrating the number of beads drawn of each color. They make another bar graph illustrating the number of beads of each color actually in the bag, and compare the two bar graphs. As a follow-up activity, students should draw 20 or more times from a bag containing an unknown mixture of beads and try to guess, and justify, how many beads of each color are in the container.
- Students determine what amounts of postage can and cannot be made using only 3¢ and 5¢ stamps.
- Students generate additional rows of Pascal's triangle (below). They color all odd entries one color and all even entries another color. They examine the patterns that result, and try to explain what they see. They discuss whether their conclusions apply to a larger version of Pascal's triangle.

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \end{array}$$

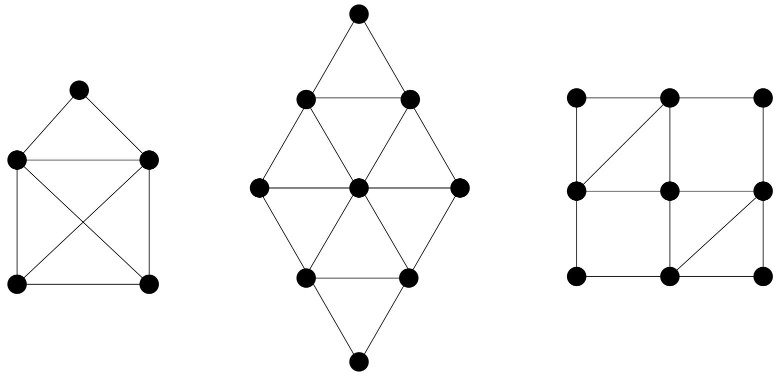
- Students make a table indicating which stamps of the denominations 1¢, 2¢, 4¢, 8¢, 16¢, 32¢ would be used (with no repeats) to obtain each amount of postage from 1¢ to 63¢. For the table, they list the available denominations across the top and the postage amounts from 1¢ to 63¢ at the left; they put a checkmark in the appropriate spot if they need the stamp for that amount, and leave it blank otherwise. They try to find a pattern which could be used to decide which amounts of postage could be made if additional stamps (like 64¢ and 128¢) were used.
- Students play games like Nim and reflect on the moves they make in the game. (See *Math for Girls and Other Problem Solvers*, by D. Downie et al., for other games for this grade level.) In Nim, you start

with a number of piles of objects — for example, you could start with two piles, one with five buttons, the other with seven buttons. Two students alternate moves, and each move consists of taking some or all of the buttons from a single pile; the child who takes the last button off the table wins the game. Once they master this game, students can try Nim with three piles, starting with three piles which have respectively 1, 2, and 3 buttons.

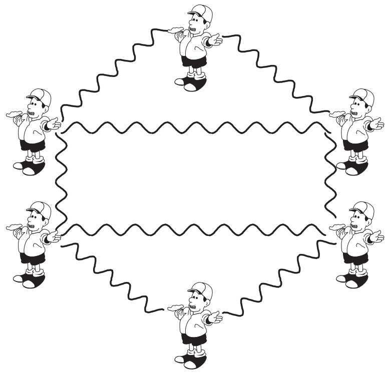
- Students play games like *dots and boxes* and systematically think about the moves they make in the game. In dots and boxes, you start with a square (or rectangular) array of dots, and two students alternate drawing a line which joins two adjacent dots. Whenever all four sides of a square have been drawn, the student puts her or his initial in the square and draws another line; the person with initials in more squares wins the game.

2. **Use networks and tree diagrams to represent everyday situations.**

- Students make a collaboration graph for the members of the class which describes who has worked with whom during the past week.
- Students draw specified patterns on the chalkboard without retracing, such as those below. Alternatively, they may trace these patterns in a small box of sand, as done historically in African cultures. (See *Ethnomathematics, Drawing Pictures With One Line*, or *Insides, Outsides, Loops, and Lines.*) Alternatively, on a pattern of islands and bridges laid out on the floor with masking tape, students might try to take a walk which involves crossing each bridge exactly once (leaving colored markers on bridges already crossed); note that for some patterns this may not be possible. The patterns given here can be used, but students can develop their own patterns and try to take such a walk for each pattern that they create.



- Students create "human graphs" where they themselves are the vertices and they use pieces of yarn (several feet long) as edges; each piece of yarn is held by two students, one at each end. They might create graphs with specified properties; for example, they might create a human graph with four vertices of degree 2, or, as in the figure below, with six vertices of which four have degree 3 and two have degree 2. (The *degree* of a vertex is the total number of edges that meet at the vertex.) They might count the number of different shapes of human graphs they can form with four students (or five, or six).

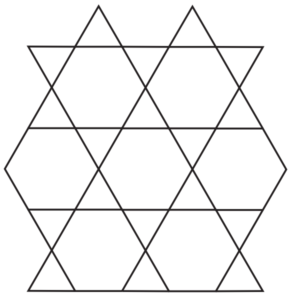


- Students use a floor plan of their school to map out alternate routes from their classroom to the school's exits, and discuss whether the fire drill route is in fact the shortest route to an exit.
- Students draw graphs of their own neighborhoods, with edges representing streets and vertices representing locations where roads meet. *Can you find a route for the mail carrier in your neighborhood which enables her to walk down each street, without repeating any streets, and which ends where it begins? Can you find such a route if she needs to walk up and down each street in order to deliver mail on both sides of the street?*
- Students color maps (e.g., the 21 counties of New Jersey) so that adjacent counties (or countries) have different colors, using as few colors as possible. The class could then share a NJ cake frosted accordingly. (See *The Mathematician's Coloring Book.*)
- Students recognize and understand family trees in social and historical studies, and in stories that they read. Where appropriate, they create their own family trees.

3. **Identify and investigate sequences and patterns found in nature, art, and music.**

- Students read *A Cloak for a Dreamer* by A. Friedman, and make outlines of cloaks or coats like those worn by the sons of the tailor in the book by tracing their upper bodies on large pieces of paper. Students could use pattern blocks or pre-cut geometric shapes to cover (tessellate) the paper cloaks with patterns like those in the book or try to make their own cloth designs.

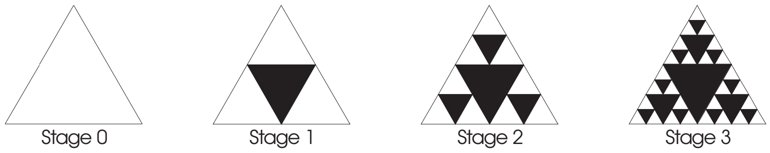
- Students read *Sam and the Blue Ribbon Quilt* by Lisa Ernst, and by rotating, flipping, or sliding cut-out squares, rectangles, triangles, etc., create their own symmetrical designs on quilt squares similar to those found in the book. The designs from all the members of the class are put together to make a patchwork class quilt or to form the frame for a math bulletin board.
- Students take a "pattern walk" through the neighborhood, searching for patterns in the trees, the houses, the buildings, the manhole covers (by the way, *why are they always round?*), the cars, etc.; the purpose of this activity is to create an awareness of the patterns around us. *By Nature's Design* is a photographic journey with an eye for many of these natural patterns.
- Students "tessellate" the plane using squares, triangles, or hexagons to completely cover a sheet of paper without overlapping. They also tessellate the plane using groups of shapes, like hexagons and triangles as in the figure below.



- Students might ask if their parents would be willing to give them a penny for the first time they do a particular chore, two pennies for the second time they do the chore, four pennies for the third time, eight pennies for the fourth time, and so on. Before asking, they should investigate, perhaps using towers of Unifix cubes that keep doubling in height, how long their parents could actually afford to pay them for doing the chore.
- Students cut a sheet of paper into two halves, cut the resulting two pieces into halves, cut the resulting four pieces into halves, etc. *If they do this a number of times, say 12 times, and stacked all the pieces of paper on top of each other, how high would the pile of paper be?* Students estimate the height before performing any calculations.
- Students color half a large square, then half of the remaining portion with another color, then half of the remaining portion with a third color, etc. *Will the entire area ever get colored? Why, or why not?*
- Students count the number of rows of bracts on a pineapple or pine cone, or rows of petals on an artichoke, or rows of seeds on a sunflower, and verify that these numbers all appear in the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ of Fibonacci numbers, where each number is

the sum of the two previous numbers on the list. Students find other pictures depicting Fibonacci numbers as they arise in nature, referring, for example, to *Fibonacci Numbers in Nature*. In *Mathematical Mystery Tour* by Mark Wahl, an elementary school teacher provides a year's worth of Fibonacci explorations and activities.

- Using a large equilateral triangle provided by the teacher, students find and connect the approximate midpoints of the three sides, and then color the triangle in the middle. (See Stage 1 picture.) They then repeat this procedure with each of the three uncolored triangles to get the Stage 2 picture, and then repeat this procedure again with each of the nine uncolored triangles to get the Stage 3 picture. These are the first three stages of the Sierpinski triangle; subsequent stages become increasingly intricate. *How many uncolored triangles are there in the Stage 3 picture? How many would there be in the Stage 4 picture if the procedure were repeated again?*



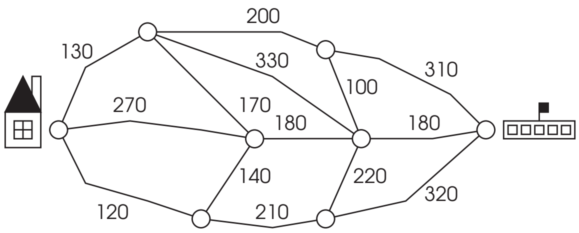
4. **Investigate ways to represent and classify data according to attributes, such as shape or color, and relationships, and discuss the purpose and usefulness of such classification.**

- Students are provided with a set of index cards on each of which is written a word (or a number). Working in groups, students put the cards in alphabetical (or numerical) order, explain the methods they used to do this, and then compare the various methods that were used.
- Students bring to class names of cities and their zip codes where their relatives and friends live, paste these at the appropriate locations on a map of the United States, and look for patterns which might explain how zip codes are assigned. Then they compare their conclusions with post office information to see whether they are consistent with the way that zip codes actually are assigned.
- Students send and decode messages in which each letter has been replaced by the letter which follows it in the alphabet (or occurs two letters later). Students explore other coding systems described in *Let's Investigate Codes and Sequences* by Marion Smoothey.
- Students collect information about the soft drinks they prefer and discuss various ways of presenting the resulting information, such as tables, bar graphs, and pie charts, displayed both on paper and on a computer.

- Students play the game of *Set* in which participants try to identify three cards from those on display which, for each of four attributes (number, shape, color, and shading), all share the attribute or are all different. Similar ideas can be explored using *Tabletop, Jr.* software.

5. **Follow, devise, and describe practical lists of instructions.**

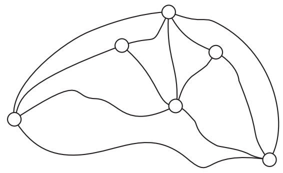
- Students follow a recipe to make a cake or to assemble a simple toy from its component parts, and then write their own versions of those instructions.
- Students give written and oral directions for going from the classroom to another room in the school, and represent these directions with a diagram drawn approximately to scale.
- Students read *Anno's Mysterious Multiplying Jar* by Mitsumasa Anno. During a second reading they devise a method to record and keep track of the increasing number of items in the book and predict how that number will continue to grow. Each group explains its method to the class.
- Students write step-by-step directions for a simple task like making a peanut butter and jelly sandwich, and follow them to prove that they work.
- Students find and describe the shortest path from the computer to the door or from one location in the school building to another.
- Students find the shortest route from school to home on a map (see figure below), where each edge has a specified numerical length in meters; students modify lengths to obtain a different shortest route.



- Students write a program which will create specified pictures or patterns, such as a house or a clown face or a symmetrical design. Logo software is well-suited to this activity. In *Turtle Math,* students use Logo commands to go on a treasure hunt, and look for the shortest route to complete the search.
- Working in groups, students create and explain a fair way of sharing a bagful of similar candies or cookies. (See also the vignette entitled *Sharing A Snack* in the Introduction to this *Framework.*) For example, if the bag has 30 brownies and there are 20 children, then they might suggest that each child gets one whole brownie and that the teacher divide each of the remaining brownies in half. Or they might suggest that each pair of children figure out how to share one

brownie. *What if there were 30 hard candies instead of brownies? What if there were 25 brownies? What if there were 15 brownies and 15 chocolate chip cookies?* The purpose of this activity is for students to brainstorm possible solutions in the situations where there may be no solution that *everyone* perceives as fair.

- Students devise a strategy for never losing at tic-tac-toe.
- Students find different ways of paving just enough streets of a "muddy city" (like the street map below, perhaps laid out on the floor) so that a child can walk from any one location to any other location along paved roadways. In "muddy city" none of the roads are paved, so that whenever it rains all streets turn to mud. The mayor has asked the class to propose different ways of paving the roads so that a person can get from any one location to any other location on paved roads, but so that the fewest number of roads possible are paved.



- Students divide a collection of Cuisenaire rods of different lengths into two or three groups whose total lengths are equal (or as close to equal as possible).

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Software.

- *Logo.* Many versions of Logo are commercially available.
- *Tabletop, Jr.* Broderbund Software. TERC.
- *Turtle Math.* LCSI.

6. Grades 5-6 Overview

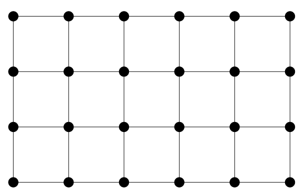
The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.** Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools.*

Despite their formidable titles, these five themes can be addressed with activities at the 5-6 grade level which involve both the purposeful play and simple analysis suggested for elementary school students and experimentation and abstraction appropriate at the middle grades. Indeed, teachers will discover that many activities that they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

The following discussion of activities at the 5-6 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-4 grade levels. Hence 5-6 grade teachers should review the K-2 and 3-4 grade level discussions of discrete mathematics and might use activities similar to those described there before introducing the activities for this grade level.

Activities involving **systematic listing, counting, and reasoning** at K-4 grade levels can be extended to the 5-6 grade level. For example, they might determine the number of possible license plates with two letters followed by three numbers followed by one letter, and decide whether this total number of license plates is adequate for all New Jersey drivers. They need to become familiar with the idea of permutations, that is, the different ways in which a group of items can be arranged. Thus, for example, if three children are standing by the blackboard, there are altogether six different ways, call permutations, in which this can be done; for example, if the three children are Amy (A), Bethany (B), and Coriander (C), the six different **permutations** can be described as ABC, ACB, BAC, BCA, CAB, and CBA. Similarly, the total number of different ways in which three students out of a class of thirty can be arranged at the blackboard is altogether $30 \times 29 \times 28$, or 24,360 ways, an amazing total!

An important **discrete mathematical model** is that of a **network or graph**, which consists of dots and lines joining the dots; the dots are often called *vertices* (*vertex* is the singular) and the lines are often called *edges*. (This is different from other mathematical uses of the term "graph"; the two terms "network" and "graph" are used interchangeably for this concept.) An example of a graph with 24 vertices and 38 edges is given below. Graphs can be used to represent islands and bridges, or buildings and roads, or houses and telephone cables; wherever a collection of things are joined by connectors, the mathematical model used is that of a graph. At the 5-6 level, students should be familiar with the notion of a graph and recognize situations in which graphs can be an appropriate model. For example, they should be familiar with problems involving routes for garbage pick-ups, school buses, mail deliveries, snow removal, etc.; they should be able to model such problems by using graphs, and be able to solve such problems by finding suitable paths in these graphs, such as in the town whose street map is the graph below.



Students should recognize and work with **repetitive patterns and processes** involving numbers and shapes, with objects found in the classroom and in the world around them. Building on these explorations, fifth- and sixth-graders should also recognize and work with **iterative and recursive processes.** They explore iteration using Logo software, where they recreate a variety of interesting patterns (such as a checkerboard) by iterating the construction of a simple component of the pattern (in this case a square). As with younger students, 5th and 6th graders are fascinated with the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$ where every number is the sum of the previous two numbers. Although the Fibonacci sequence starts with small numbers, the numbers in the sequence become large very quickly. Students can now also begin to understand the Fibonacci sequence and other sequences recursively — where each term of the sequence is described in terms of preceding terms.

Students in the 5th and 6th grade should investigate **sorting items** using Venn diagrams, and continue their explorations of **recovering hidden information** by decoding messages. They should begin to **explore how codes are used to communicate information**, by traditional methods such as Morse code or semaphore (flags used for ship-to-ship messages) and also by current methods such as zip codes, which describe a location in the United States by a five-digit (or nine-digit) number. Students should also explore modular arithmetic through applications involving clocks, calendars, and binary codes.

Finally, at grades 5-6, students should be able to **describe, devise, and test algorithms for solving a variety of problems**. These include finding the shortest route from one location to another, dividing a cake fairly, planning a tournament schedule, and planning layouts for a class newspaper.

Two important resources on discrete mathematics for teachers at all levels is the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools.* Another important resource for 5-6 teachers is *This Is MEGA-Mathematics!*

7. Grades 5-6 Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 5-6:

6. **Use systematic listing, counting, and reasoning in a variety of different contexts.**
   - Students determine the number of different sandwiches or hamburgers that can be created at local eateries using a combination of specific ingredients.

- Students find the number of different ways to make a row of flowers each of which is red or yellow, if the row has 1, 2, 3, 4, or 5 flowers. Modeling this with Unifix cubes, they discover that adding an additional flower to the row doubles the number of possible rows, provide explanations for this, and generalize to longer rows. Similar activities can be found in the *Pizza Possibilities* and *Two-Toned Towers* lessons that are described in the First Four Standards of this *Framework.*

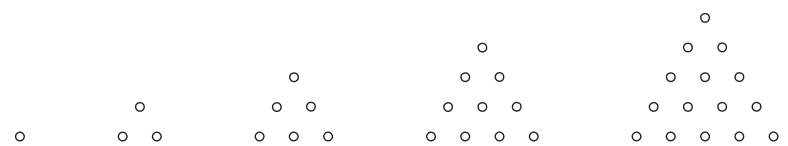
- Students find the number of ways of asking three different students in the class to write three homework problems on the blackboard.

- Students understand and use the concept of permutation. They determine the number of ways any five items can be arranged in order, justify their conclusion using a tree diagram, and use factorial notation, 5!, to summarize the result.

- Students find the number of possible telephone numbers with a given area code and investigate why several years ago the telephone company introduced a new area code (908) in New Jersey, and why additional area codes are being introduced in 1997. *Is the situation the same with zip codes?*

- Students estimate and then calculate the number of possible license plates with two letters followed by three numbers followed by one letter. They investigate why the state license bureau tried to introduce license plates with seven characters and why this attempt might have been unsuccessful.

- Students explore the sequence of triangular numbers 1, 1+2, 1+2+3, 1+2+3+4, ... which represent the number of dots in the triangular arrays below, and find the location of the triangular numbers in Pascal's triangle.



- Students look for patterns in the various diagonals of Pascal's triangle, and in the differences between consecutive terms in these diagonals. *Patterns in Pascal's Triangle Poster* is a nice resource for introducing these ideas.

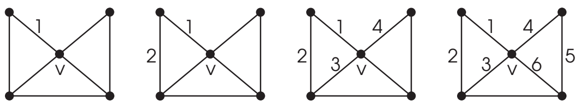
- Students analyze simple games like the following: Beth wins the game whenever the two dice give an even total, and Hobart wins whenever the two dice give an odd total. They play the game a number of times, and using experimental evidence, decide whether the game is fair, and, if not, which player is more likely to win. They then try to justify their conclusions theoretically, by counting the number of combinations of dice that would result in a win for each player.

- Students create a table in the form of a grid which indicates how many of each of the coins of the fictitious country "Ternamy" — in denominations of 1, 3, 9, 27, and 81 "terns" — are needed to make up any amount from 1 to 200. They list the denominations in the columns at the top of the table and the amounts they are trying to make in the rows at the left. They write the number of each coin needed to add up to the desired amount in the appropriate squares in that row. The only "rule" to be followed is that the least number of coins must be used; for example, three 1's should always be replaced by one 3. This table can be used to introduce base 3 ("ternary") numbers, and then numbers in other bases.

7. **Recognize common discrete mathematical models, explore their properties, and design them for specific situations.**

- Students experiment with drawing make-believe maps which can be colored with two, three, and four colors (where adjacent countries must have different colors), and explain why their fictitious maps, and real maps like the map of the 50 states, cannot be colored with fewer colors. Note that it was proven in 1976 that no map can be drawn on a flat surface which requires more than four colors. *The Mathematician's Coloring Book* contains a variety of map-coloring activities, as well as historical background on the map coloring problem.

- Students play games using graphs. For example, in the strolling game, two players stroll together on a path through the graph which never repeats itself; they alternate in selecting edges for the path, and the winner is the one who selects the last edge on the path. Who wins? In the game below, Charles and Diane both start at V, Charles picks the first edge (marked 1) and they both stroll down that edge. Then Diane picks the second edge (marked 2) and the game continues. Diane has won this play of the game since the path cannot be continued after the sixth edge without repeating itself. *Does Diane have a way of always winning this game, or does Charles have a winning strategy? What if there was a different starting point? What if a different graph was used? What if the path must not cross itself (instead of requiring that it not repeat itself)?* Students should try to explain in each case why a certain player has a winning strategy.

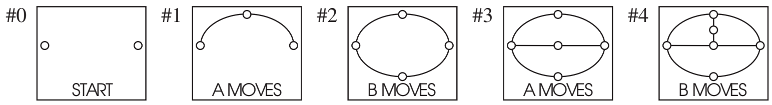


- Students find paths in graphs which utilize each edge exactly once; a path in a graph is a sequence of edges each of which begins where the previous one ends. They apply this idea by converting a street map to a graph where vertices on the graph correspond to intersections on the street map, and by using this graph to determine whether a garbage truck can complete its sector without repeating any streets. See the segment *Snowbound: Euler Circuits* on the videotape *Geometry: New Tools for New Technologies*; the module *Drawing Pictures With One Line* provides a strong background for problems of this kind.

- Students plan emergency evacuation routes at school or from home using graphs.

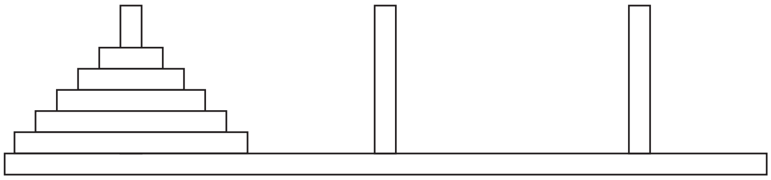
- All of the students together create a "human graph" where each child in the class is holding two strings, one in each hand. This can be accomplished by placing in the center of the room a number of pieces of yarn (each six feet long) equal to the number of students, and having each student take the ends of two strings. The children are asked to untangle themselves, and discuss or write about what happens.

- Students play the game of Sprouts, in which two students take turns in building a graph until one of them (the winner!) completes the graph. The rules are: start the game with two or three vertices; each person adds an edge (it can be a curved line!) joining two vertices, and then adds a new vertex at the center of that edge; no more than three edges can occur at a vertex; edges may not cross. In the sample game below, the second player (B) wins because the first player (A) cannot draw an edge connecting the only two vertices that have degree less than three without crossing an existing edge.



8. **Experiment with iterative and recursive processes, with the aid of calculators and computers.**

- Students develop a method for solving the Tower of Hanoi problem: There are three pegs, on the first of which is stacked five disks, each smaller than the ones underneath it (see diagram below); the problem is to move the entire stack to the third peg, moving disks, one at a time, from any peg to either of the other two pegs, with no disk ever placed upon a smaller one. *How many moves are required to do this?*



- Students use iteration in Logo software to draw checkerboards, stars, and other designs. For example, they iterate the construction of a simple component of a pattern, such as a square, to recreate an entire checkerboard design.

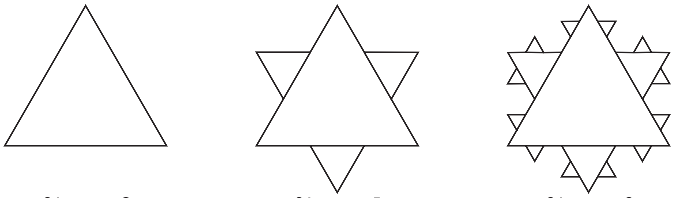
- Students use paper rabbits (prepared by the teacher) with which to simulate Fibonacci's 13th century investigation into the growth of rabbit populations: *If you start with one pair of baby rabbits, how many pairs of rabbits will there be a year later?* Fibonacci's assumption was that each pair of baby rabbits results in another pair of baby rabbits two months later — allowing a month for maturation and a month for gestation. Once mature, each pair has baby rabbits monthly. (Each pair of students should be provided with 18 cardboard pairs each of baby rabbits, not-yet-mature rabbits, and mature rabbits.) *The Fascinating Fibonaccis* by Trudi Garland illustrates the rabbit problem and a number of other interesting Fibonacci facts. In *Mathematics Mystery Tour* by Mark Wahl, an elementary school teacher provides a year's worth of Fibonacci explorations and activities.

- Students use calculators to compare the growth of various sequences, including counting by 4's $(4, 8, 12, 16, \ldots)$, doubling $(1, 2, 4, 8, 16, \ldots)$, squaring $(1, 4, 9, 16, 25, \ldots)$, and Fibonacci $(1, 1, 2, 3, 5, 8, 13, \ldots)$.

- Students explore their surroundings to find rectangular objects whose ratio of length to width is the "golden ratio." Since the golden ratio can be approximated by the ratio of two successive Fibonacci numbers, students should cut a rectangular peephole of dimensions 21mm x 34 mm out of a piece of cardboard, and use it to "frame" potential objects; when it "fits," the object is a golden rectangle. They describe these activities in their math journals.

- Students study the patterns of patchwork quilts, and make one of their own. They might first read *Eight Hands Round.*

- Students make equilateral triangles whose sides are 9", 3", and 1" (or other lengths in ratio 3:1), and use them to construct "Koch snowflakes of stage 2" (as shown below) by pasting the 9" triangle on a large sheet of paper, three 3" triangles at the middle of the three sides of the 9" triangle (pointing outward), and twelve 1" triangles at the middle of the exposed sides of the twelve 3" segments (pointing outward). To get Koch snowflakes of stage 3, add forty-eight 1/3" equilateral triangles. *How many 1/9" equilateral triangles would be needed for the Koch snowflake of stage 4? Fractals for the Classroom* is a valuable resource for these kinds of activities and explorations.



- Students mark one end of a long string and make another mark midway between the two ends. They then continue marking the string by following some simple rule such as "make a new mark midway between the last midway mark and the marked end" and then repeat this instruction. Students investigate the relationship of the lengths of the segments between marks. *How many marks are possible in this process if it is assumed that the marks take up no space on the string? What happens if the rule is changed to "make a new mark midway between the last two marks?"*

9. **Explore methods for storing, processing, and communicating information.**

- After discussing possible methods for communicating messages across a football field, teams of students devise methods for transmitting a short message (using flags, flashlights, arm signals, etc.). Each team receives a message of the same length and must transmit it to members of the team at the other end of the field as quickly and accurately as possible.

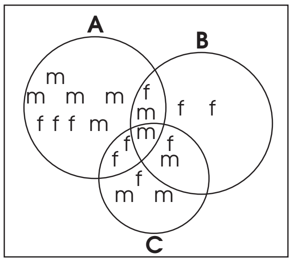
- Students devise rules so that arithmetic expressions without parentheses, such as $5 \times 8 - 2/7$, can be evaluated unambiguously. They then experiment with calculators to discover the calculators' built-in rules for evaluating these expressions.

- Students explore binary arithmetic and arithmetic for other bases through applications involving clocks (base 12), days of the week (base 7), and binary (base 2) codes.

- Students assign each letter in the alphabet a numerical value (possibly negative) and then look for words worth a specified number of points.

- Students send and decode messages in which letters of the message are systematically replaced by other letters. *The Secret Code Book* by Helen Huckle shows these coding systems as well as others.

- Students use Venn diagrams to sort and then report on their findings in a survey. For example, they can seek responses to the question, *When I grow up I want to be a) rich and famous, b) a parent, c) in a profession I love*, where respondents can choose more than one option. The results can be sorted into a Venn diagram like that below, where entries "m" and "f" are used for male and female students. The class can then determine answers to questions like *Are males or females in our class more likely to have a single focus? Tabletop, Jr.* software can be used to sort and explore data using Venn diagrams.



10. **Devise, describe, and test algorithms for solving optimization and search problems.**

- Students use a systematic procedure to find the total number of routes from one location in their town to another, and the shortest such route. (See *Problem Solving Using Graphs.*)

- In *Turtle Math*, students use Logo commands to go on a treasure hunt, and look for the shortest route to complete the search.

- Students discuss and write about various methods of dividing a cake fairly, such as the "divider/chooser method" for two people (one person divides, the other chooses) and the "lone chooser method" for three people (two people divide the cake using the divider/chooser method, then each cuts his/her half into thirds, and then the third person takes one piece from each of the others). *Fair Division: Getting Your Fair Share* can be used to explore methods of fairly dividing a cake or an estate.

- Students conduct a class survey for the top ten songs and discuss different ways to use the information to select the winners.

- Students devise a telephone tree for disseminating messages to all 6th grade students and their parents.

- Students schedule the matches of a volleyball tournament in which each team plays each other team once.

- Students use flowcharts to represent visually the instructions for carrying out a complex project, such as scheduling the production of the class newspaper.

- Students develop an algorithm to create an efficient layout for a class newspaper.

References.

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- Wahl, Mark. *Mathematical Mystery Tour: Higher-Thinking Math Tasks.* Tucson, AZ: Zephyr Press, 1988.

Software.

- *Logo.* Many versions of Logo are commercially available.
- *Tabletop, Jr.* Broderbund, TERC.
- *Turtle Math.* LCSI.

Video.

- *Geometry: New Tools for New Technologies*, videotape by the Consortium for Mathematics and Its Applications (COMAP). Lexington, MA, 1992.

8. Grades 7-8 Overview

The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

Despite their formidable titles, these five themes can be addressed with activities at the 7-8 grade level which involve both the purposeful play and simple analysis suggested for elementary school students and experimentation and abstraction appropriate at the middle grades. Indeed, teachers will discover that many activities that they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

The following discussion of activities at the 7-8 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-6 grade levels. Hence 7-8 grade teachers should review the K-2, 3-4, and 5-6 grade level discussions of discrete mathematics and might use activities similar to those described there before introducing the activities for this grade level.

Students in 7th and 8th grade should be able to **use permutations and combinations and other counting strategies in a wide variety of contexts**. In addition to working with permutations, where the order of the items is important (see Grades 5-6 Overview and Activities), they should also be able to work with combinations, where the order of the items is irrelevant. For example, the number of different three digit numbers that can be made using three different digits is $10 \times 9 \times 8$ because each different ordering of the three digits results in a different number. However, the number of different pizzas that can be made using three of ten available toppings is $(10 \times 9 \times 8)/(3 \times 2 \times 1)$ because the *order* in which the toppings are added is irrelevant; the division by $3 \times 2 \times 1$ eliminates the duplication.

An important **discrete mathematical model** is that of a **network or graph**, which consists of dots and lines joining the dots; the dots are often called *vertices* (*vertex* is the singular) and the lines are often called *edges*. (This is different from other mathematical uses of the term "graph.") Graphs can be used to represent islands and bridges, or buildings and roads, or houses and telephone cables; wherever a collection of things are joined by connectors, the mathematical model used is that of a graph. Students in the 7th and 8th grades should be able to **use graphs to model situations and solve problems using the model**. For example, students should be able to use graphs to schedule a school's extracurricular activities so that, if at all possible, no one is excluded because of conflicts. This can be done by creating a graph whose vertices are the activities, with two activities joined by an edge if they have a person in common, so that the activities should be scheduled for different times. Coloring the vertices of the graph so that adjacent vertices have different colors, using a minimum number of colors, then provides an efficient solution to the scheduling problem — a separate time slot is needed for each color, and two activities are scheduled for the same time slot if they have the same color.

Students can recognize and work with **iterative and recursive processes**, extending their earlier explorations of **repetitive patterns and procedures**. In the 7th and 8th grade, they can combine their understanding of exponents and iteration to solve problems involving compound interest with a calculator or spreadsheet. Topics which before were viewed iteratively — arriving at the present situation by repeating a procedure n times — can now be viewed recursively - arriving at the present situation by modifying the previous situation. They can apply this understanding to Fibonacci numbers, to the Tower of Hanoi puzzle, to programs in Logo, to permutations and to other areas.

Students in the 7th and 8th grades should **explore how codes are used to communicate information**, by traditional methods such as Morse code or semaphore (flags used for ship-to-ship messages) and also by current methods such as zip codes. Students should investigate and report about various codes that are commonly used, such as binary codes, UPCs (universal product codes) on grocery items, and ISBN numbers on books. They should also **explore how information is processed**. A useful metaphor is how a waiting line or queue is handled (or "processed") in various situations; at a bank, for example, the queue is usually processed in first-in-first-out (FIFO) order, but in a supermarket or restaurant there is usually a pre-sorting into smaller queues done by the shoppers themselves before the FIFO process is activated.

In the 7th and 8th grade, students should be able to **use algorithms to find the best solution in a number of situations** — including the shortest route from one city to another on a map, the cheapest way of connecting sites into a network, the fastest ways of alphabetizing a list of words, the optimal route for a class trip (see the *Short-Circuiting Trenton* lesson in the Introduction to this *Framework*), or optimal work schedules for employees at a fast-food restaurant.

Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools.* Teachers of grades 7-8 would also find useful the textbook *Discrete Mathematics Through Applications.*

9. Grades 7-8 Indicators and Activities

The cumulative progresses indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 7-8:

6. **Use systematic listing, counting, and reasoning in a variety of different contexts.**

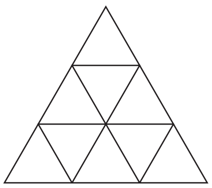
- Students determine the number of possible different sandwiches or hamburgers that can be created at local eateries using a combination of specified ingredients. They find the number of pizzas that can be made with three out of eight available toppings and relate the result to the numbers in Pascal's triangle.
- Students determine the number of dominoes in a set that goes up to 6:6 or 9:9, the number of candles used throughout Hannukah, and the number of gifts given in the song "The Twelve Days of Christmas," and connect the results through discussion of the triangular numbers. (Note that in a 6:6 set of dominoes there is exactly one domino with each combination of dots from 0 to 6.)
- Students determine the number of ways of spelling "Pascal" in the array below by following a path from top to bottom in which each letter is directly below, and just to the right or left of the previous letter.

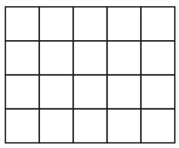
```
          P
         A A
        S S S
       C C C C
      A A A A A
     L L L L L L
```

- Students design different license plate systems for different population sizes; for example, *how large would the population be before you would run out of plates which had only three numbers, or only five numbers, or two letters followed by three numbers?*
- Students find the number of different ways of making a row of six red and yellow flowers, organize and tabulate the possibilities according to the number of flowers of the first color, and explain the connection with the numbers in the sixth row of Pascal's triangle. (See also *Visual Patterns in Pascal's Triangle.*)
- Students pose and act out problems involving the number of different ways a group of people can sit around a table, using as motivation the scene of the Mad Hatter at the tea party. (See *Mathematics, a Human Endeavor*, p. 394.)
- Students count the total number of different cubes that can be made using either red or green paper for each face. (To solve this problem, they will have to use a "break up the problem into cases" strategy.)

- Students determine the number of handshakes that take place if each person in a room shakes hands with every other person exactly once, and relate this total to the number of line segments joining the vertices in a polygon, to the number of two-flavor ice-cream cones, and to triangular numbers.

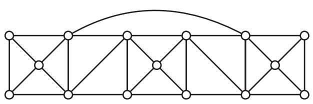
- Students count the number of triangles or rectangles in a geometric design. For example, they should be able to count systematically the number of triangles (and trapezoids) in the figure below to the left, noting that there are triangles of three sizes, and the number of rectangles in the 4×5 grid pictured below to the right, listing first all dimensions of rectangles that are present.

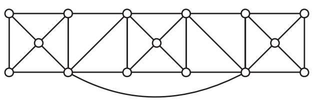




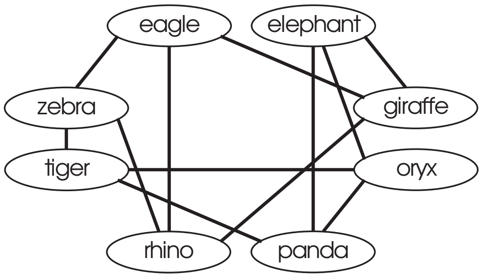
7. **Recognize common discrete mathematical models, explore their properties, and design them for specific situations.**

- Students find the minimum number of colors needed to assign colors to all vertices in a graph so that any two adjacent vertices are assigned different colors and justify their answers. For example, students can explain why one of the graphs below requires four colors while for the other, three colors are sufficient.





- Students use graph coloring to solve problems which involve avoiding conflicts such as: scheduling the school's extra curricular activities; scheduling referees for soccer games; determining the minimum number of aquariums needed for a specified collection of tropical fish; and assigning channels to radio stations to avoid interference. In the graph below an edge between two animals indicates that they cannot share a habitat. The videotape, *Geometry: New Tools for New Technologies* has a segment *Connecting the Dots: Vertex Coloring* which discusses the minimum number of habitats required for this situation.



- Students use tree diagrams to represent and analyze possible outcomes in counting problems, such as tossing two dice.

- Students determine whether or not a given group of dominoes can be arranged in a line (or in a rectangle) so that the number of dots on the ends of adjacent dominoes match. For example, the dominoes (03), (05), (12), (14), (15), (23), (34) can be arranged as (12), (23), (30), (05), (51), (14), (43); and if an eighth domino (13) is added, they can be formed into a rectangle. *What if instead the eighth domino was (24) — could they then be arranged in a rectangle or in a line?*

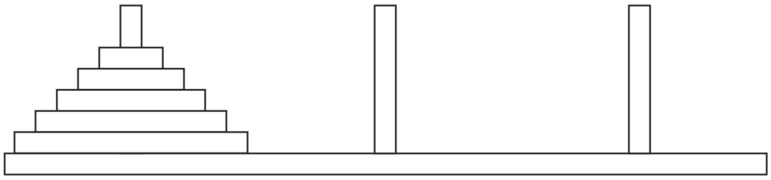
- Students determine the minimum number of blocks that a police car has to repeat if it must try to patrol each street exactly once on a given map. *Drawing Pictures With One Line* contains similar real-world problems and a number of related game activities.

- Students find the best route for collecting recyclable paper from all classrooms in the school, and discuss different ways of deciding what is the "best." (See *Drawing Pictures With One Line.*)

- Students make models of various polyhedra with straws and string, and explore the relationship between the number of edges, faces, and vertices.

8. **Experiment with iterative and recursive processes, with the aid of calculators and computers.**

- Students develop a method for solving the Tower of Hanoi problem: There are three pegs, on the first of which five disks are stacked, each smaller than the ones underneath it (see diagram below); the problem is to move the entire stack to the third peg, moving disks, one at a time, from any peg to either of the other two pegs, with no disk ever placed upon a smaller one. *How many moves are required to do this? What if there were 6 disks? How long would it take to do this with 64 disks?* (An ancient legend predicts that when this task is completed, the world will end; should we worry?)



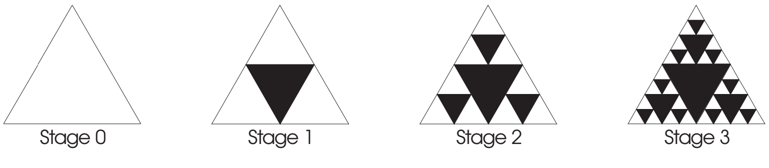
Students view recursively Tower of Hanoi puzzles with various numbers of disks so that they can express the number of moves needed to solve the puzzle with one more disk in terms of the number of moves needed for the puzzle with the current number of disks.

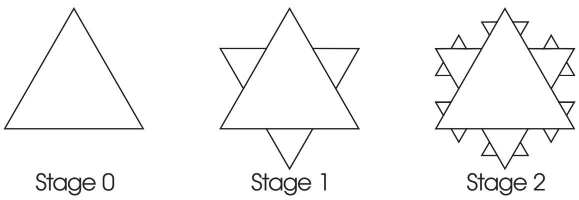
- Students attempt to list the different ways they could travel 10 feet in a straight line if they were a robot which moved only in one or two foot segments, and then thinking recursively determine the number of different ways this robot could travel n feet.

- Students develop arithmetic and geometric progressions on a calculator.

- Students find square roots using the following iterative procedure on a calculator. Make an estimate of the square root of a number B, divide the estimate into B, and average the result with the estimate to get a new estimate. Then repeat this procedure until an adequate estimate is obtained. *For example, if the first estimate of the square root of 10 is 3, then the second would be the average of 3 and 10/3, or 19/6 = 3.166. What is the next estimate of the square root of 10? How many repetitions are required to get the estimate to agree with the square root of 10 provided by the calculator?*

- Students develop the sequence of areas and perimeters of iterations of the constructions of the Sierpinski triangle (top figures) and the Koch snowflake (bottom figures), and discuss the outcome if the process were continued indefinitely. (These are discussed in more detail in the sections for earlier grade levels. See Unit 1 of *Fractals for the Classroom* for related activities.)





- Students recognize the computation of the number of permutations as a recursive process — that is, that the number of ways of arranging 10 students is 10 times the number of ways of arranging 9 students.

9. **Explore methods for storing, processing, and communicating information.**

- Students conjecture which of the following (and other) methods is the most efficient way of handing back corrected homework papers which are already sorted alphabetically: (1) the teacher walks around the room handing to each student individually; (2) students pass the papers around, each taking their own; (3) students line themselves up in alphabetical order. Students test their conjectures and discuss the results.

- Students investigate and report about various codes that are commonly used, such as zip codes, UPCs (universal product codes) on grocery items, and ISBN numbers on books. (A good source for information about these and other codes is *Codes Galore* by J. Malkevitch, G. Froelich, and D. Froelich.)

- Students write a Logo procedure for making a rectangle that uses variables, so that they can use their rectangle procedure to create a graphic scene which contains objects, such as buildings, of varying sizes.

- Students are challenged to guess a secret word chosen by the teacher from the dictionary, using at most 20 yes-no questions. *Is this always, or only sometimes possible?*

- Students use Venn diagrams to solve problems like the following one from the New Jersey Department of Education's *Mathematics Instruction Guide* (p. 7-13). *Suppose the school decided to add the springtime sport of lacrosse to its soccer and basketball offerings for its 120 students. A follow-up survey showed that: 35 played lacrosse, 70 played soccer, 40 played basketball, 20 played both soccer and basketball, 15 played both soccer and lacrosse, 15 played both basketball and lacrosse, and 10 played all three sports. Using this data, complete a Venn diagram and answer the following questions: How many students played none of the three sports? What percent of the students played lacrosse as their only sport? How many students played both basketball and lacrosse, but not soccer?*

- Students keep a scrapbook of different ways in which information is stored or processed. For example a list of events is usually stored by date, so the scrapbook might contain a picture of a pocket calendar; a queue of people at a bank is usually processed in first-in-first-out (FIFO) order, so the scrapbook could contain a picture of such a queue. *(How is this different from the waiting lines in a supermarket, or at a restaurant?)*

- Students determine whether it is possible to have a year in which there is no Friday the 13th, and the maximum number of Friday the 13th's that can occur in one calendar year.

- Students predict and then explore the frequency of letters in the alphabet through examination of sample texts, computer searches, and published materials.

- Students decode messages where letters are systematically replaced by other letters without knowing the system by which letters are replaced; newspapers and games magazines are good sources for "cryptograms" and students can create their own. They also explore the history of code-making and code-breaking. The videotape *Discrete Mathematics: Cracking the Code* provides a good introduction to the uses of cryptography and the mathematics behind it.

10. **Devise, describe, and test algorithms for solving optimization and search problems.**

- Students find the shortest route from one city to another on a New Jersey map, and discuss whether that is the best route. (See *Problem Solving Using Graphs.*)

- Students write and solve problems involving distances, times, and costs associated with going from towns on a map to other towns, so that different routes are "best" according to different criteria.

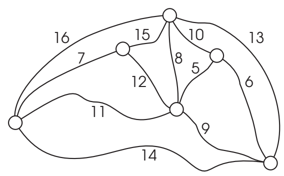
- Students use binary representations of numbers to find winning strategy for Nim. (See *Mathematical Investigations* for other mathematical games.)

- Students plan an optimal route for a class trip. (See the *Short-circuiting Trenton* lesson in the Introduction to this *Framework.*)

- Students devise work schedules for employees of a fast-food restaurant which meet specified conditions yet minimize the cost.

- Students compare strategies for alphabetizing a list of words, and test to see which strategies are more efficient.

- Students find a network of roads which connects a number of sites and involves the smallest cost. *In the example below, what roads should be built so as to minimize the total cost, where the number on each road reflects the cost of building that road (in hundreds of thousands of dollars)?*



- Students develop a precise description of the standard algorithm for adding two two-digit integers.
- Students devise strategies for dividing up the work of adding a long list of numbers among the members of the team.

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10. Grades 9-12 Overview

The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

The following discussion of activities at the 9-12 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-8 grade levels. Hence high school teachers should review the discussions of discrete mathematics at earlier grade levels and might use activities similar to those described there before introducing the activities for these grade levels.

At the high school level, students are becoming familiar with algebraic and functional notation, and their understanding of all of the themes of discrete mathematics and their ability to generalize earlier activities should be **enhanced by their algebraic skills and understandings**. Thus, for example, they should use formulas to express the results of problems involving permutations and combinations, relate Pascal's triangle to the coefficients of the binomial expansion of $(x + y)^n$, explore models of growth using various algebraic models, explore iterations of functions, and discuss methods for dividing an estate among several heirs.

At the high school level, students are particularly interested in applications; they ask *What is all of this good for?* In all five areas of discrete mathematics, students should **focus on how discrete mathematics is used to solve practical problems.** Thus, for example, they should be able to apply their understanding of counting techniques, to analyze lotteries; of graph coloring, to schedule traffic lights at a local intersection; of paths in graphs, to devise patrol routes for police cars; of iterative processes, to analyze and predict fish populations in a pond or concentration of medicine in the bloodstream; of codes, to understand how bar-code scanners detect errors and how CD's correct errors; and of optimization, to understand the 200 year old debates about apportionment and to find efficient ways of scheduling the components of a complex project.

Two important resources on discrete mathematics for teachers at all grade levels are the 1991 NCTM Yearbook, *Discrete Mathematics Across the Curriculum K-12* and the DIMACS Volume, *Discrete Mathematics in the Schools* edited by J. Rosenstein, D. Franzblau, and F. Roberts. Useful resources at the high school level are *Discrete Mathematics Through Applications* by N. Crisler, P. Fisher, and G. Froelich; *For All Practical Purposes: Introduction to Contemporary Mathematics*, by the Consortium for Mathematics and its Applications; and *Excursions in Modern Mathematics* by P. Tannenbaum and R. Arnold.

11. Grades 9-12 Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 9-12:

11. **Understand the basic principles of iteration, recursion, and mathematical induction.**

    - Students relate the possible outcomes of tossing five coins with the binomial expansion of $(x + y)^5$ and the fifth row of Pascal's triangle, and generalize to values of n other than 5.
    - Students develop formulas for counting paths on grids or other simple street maps.
    - Students find the number of cuts needed in order to divide a giant pizza so that each student in the school gets at least one piece.
    - Students develop a precise description, using iteration, of the standard algorithm for adding two integers.

12. **Use basic principles to solve combinatorial and algorithmic problems.**

    - Students determine the number of ways of spelling "mathematics" in the array below by following a path from top to bottom in which each letter is directly below, and just to the right or left of the previous letter.

```
          M
         A A
        T T T
       H H H H
      E E E E E
     M M M M M M
      A A A A A
       T T T T
        I I I
         C C
          S
```

    - Students determine the number of ways a committee of three members could be selected from the class, and the number of ways three people with specified roles could be selected. They generalize this activity to finding a formula for the number of ways an n person committee can be selected from a class of m people, and the number of ways n people with specified roles can be selected from a class of m people.

- Students find the number of ways of lining up thirty students in a class, and compare that to other large numbers; for example, they might compare it to the number of raindrops (volume = .1 cc) it would take to fill a sphere the size of the earth (radius = 6507 KM).

- Students determine the number of ways of dividing 52 cards among four players, as in the game of bridge, and compare the number of ways of obtaining a flush (five cards of the same suit) and a full house (three cards of one denomination and two cards of another) in the game of poker.

- Students play Nim (and similar games) and discuss winning strategies using binary representations of numbers.

13. **Use discrete models to represent and solve problems.**

- Students study the four color theorem and its history. (*The Mathematicians' Coloring Book* provides a good background for coloring problems.)

- Students using graph coloring to determine the minimum number of guards (or cameras) needed for museums of various shapes (and similarly for placement of lawn sprinklers or motion-sensor burglar alarms).

- Students use directed graphs to represent tournaments (where an arrow drawn from A to B represents "A defeats B") and food webs (where an arrow drawn from A to B represents "A eats B"), and to construct one-way orientations of streets in a given town which involve the least inconvenience to drivers. (A directed graph is simply a graph where each edge is thought of as an arrow pointing from one endpoint to the other.)

- Students use tree diagrams to analyze the play of games such as tic-tac-toe or Nim, and to represent the solutions to weighing problems. Example: Given 12 coins one of which is "bad," find the bad one, and determine whether it is heavier or lighter than the others, using three weighings.

- Students use graph coloring to schedule the school's final examinations so that no student has a conflict, if at all possible, or to schedule traffic lights at an intersection.

- Students devise graphs for which there is a path that covers each edge of the graph exactly once, and other graphs which have no such paths, based on an understanding of necessary and sufficient conditions for the existence of such paths, called "Euler paths," in a graph. *Drawing Pictures With One Line* provides background and applications for Euler path problems.

- Students make models of polyhedra with straws and string, and explore the relationship between the numbers of edges, faces, and vertices, and generalize the conclusion to planar graphs.
- Students use graphs to solve problems like the "fire-station problem": *Given a city where the streets are laid out in a grid composed of many square blocks, how many fire stations are needed to provide adequate coverage of the city if each fire station services its square block and the four square blocks adjacent to that one?* The Maryland Science Center in Baltimore has a hands-on exhibit involving a fire-station problem for 35 square blocks arranged in a six-by-six grid with one corner designated a park.

14. **Analyze iterative processes with the aid of calculators and computers.**

- Students analyze the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ as a recurrence relation $A_{n+2} = A_n + A_{n+1}$ with connections to the golden ratio. *Fascinating Fibonaccis* illustrates a variety of connections between Fibonacci numbers and the golden ratio.
- Students solve problems involving compound interest using iteration on a calculator or on a spreadsheet.
- Students explore examples of linear growth, using the recursive model based on the formula $A_{n+1} = A_n + d$, where d is the common difference, and convert it to the explicit linear formula, $A_{n+1} = A_1 + n \cdot d$.
- Students explore examples of population growth, using the recursive model based on the formula $A_{n+1} = A_n \times r$, where r is the common multiple or growth rate, convert it to the explicit exponential formula $A_{n+1} = A_1 \times r^n$, and apply it to both economics (such as interest problems) and biology (such as concentration of medicine in blood supply).
- Students explore logistic growth models of population growth, using the recursive model based on the formula $A_{n+1} = A_n \times (1 - A_n) \times r$, where r is the growth rate and A_n is the fraction of the carrying capacity of the environment, and apply this to the population of fish in a pond. Using a spreadsheet, students experiment with various values of the initial value A_1 and of the growth rate, and describe the relationship between the values chosen and the long term behavior of the population.
- Students explore the pattern resulting from repeatedly multiplying $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ by itself.
- Students use a calculator or a computer to study simple Markov chains, such as weather prediction and population growth models. (See Chapter 7.3 of *Discrete Mathematics Through Applications.*)

- Students explore graphical iteration by choosing a function key on a calculator and pressing it repeatedly, after choosing an initial number, to get sequences of numbers like $2, 4, 8, 16, 32, \ldots$ or $2, \sqrt{2}, \sqrt{\sqrt{2}}, \sqrt{\sqrt{\sqrt{2}}}, \ldots$. They use the graphs of the functions to explain the behavior of the sequences obtained. They extend these explorations by iterating functions they program into the calculator, such as linear functions, where slope is the predictor of behavior, and quadratic functions $f(x) = ax(1-x)$, where $0 < x < 1$ and $1 < a < 4$, which exhibit chaotic behavior.

- Students explore iteration behavior using the function defined by the two cases
$$\begin{array}{ll} f(x) = x + \frac{1}{2} & \text{for } x \text{ between } 0 \text{ and } \frac{1}{2} \\ f(x) = 2 - 2x & \text{for } x \text{ between } \frac{1}{2} \text{ and } 1 \end{array}$$
They use the initial values 1/2, 2/3, 5/9, and 7/10, and then, with a calculator or computer, the initial values .501, .667, and .701 (which differ by a small amount from the first group of "nice" initial values). They compare the behavior of the sequences generated by these values to the sequences generated by the previous initial values.

- Students play the *Chaos Game.* Each pair of students is provided with an identical transparency on which have been drawn the three vertices L, T, and R of an equilateral triangle. Each team starts by selecting any point on the triangle. They roll a die and create a new point halfway to L if they roll 1 or 2, halfway to R if they roll 3 or 4, and halfway to T if they roll 5 or 6. They repeat 20 times, each time using the new point as the starting point for the next iteration. The teacher overlays all of the transparencies and out of this chaos comes ... the familiar Sierpinski triangle. (The Sierpinski triangle is discussed in detail in the sections for earlier grade levels. Also see Unit 2 in *Fractals for the Classroom. The Chaos Game* software allows students to try variations and explore the game further.)

15. **Apply discrete methods to storing, processing, and communicating information.**

- Students discuss various algorithms used for sorting large numbers of items alphabetically or numerically, and explain why some sorting algorithms are substantially faster than others. To introduce the topic of sorting, give each group of students 100 index cards each with one word on it, and let them devise strategies for efficiently putting the cards into alphabetical order.

- Students discuss how scanners of bar codes (zip codes, UPCs, and ISBNs) are able to detect errors in reading the codes, and evaluate and compare how error-detection is accomplished in different codes. (See the COMAP Module *Codes Galore* or Chapter 9 of *For All Practical Purposes.*)

- Students investigate methods of error correction used to transmit digitized pictures from space (Voyager or Mariner probes, or the Hubble space telescope) over noisy or unreliable channels, or to ensure the fidelity of a scratched CD recording. (See Chapter 10 of For All Practical Purposes.)
- Students read about coding and code-breaking machines and their role in World War II.
- Students research topics that are currently discussed in the press, such as public-key encryption, enabling messages to be transmitted securely, and data-compression, used to save space on a computer disk.

16. **Apply discrete methods to problems of voting, apportionment, and allocations, and use fundamental strategies of optimization to solve problems.**

- Students find the best route when a number of alternate routes are possible. For example: *In which order should you pick up the six friends you are driving to the school dance? In which order should you make the eight deliveries for the drug store where you work? In which order should you visit the seven "must-see" sites on your vacation trip?* In each case, you want to find the "best route," the one which involves the least total distance, or least total time, or least total expense. Students create their own problems, using actual locations and distances, and find the best route. For a larger project, students can try to improve the route taken by their school bus.
- Students study the role of apportionment in American history, focusing on the 1790 census (acting out the positions of the thirteen original states and discussing George Washington's first use of the presidential veto), and the disputed election of 1876, and discuss the relative merits of different systems of apportionment that have been proposed and used. (This activity provides an opportunity for mathematics and history teachers to work together.) They also devise a student government where the seats are fairly apportioned among all constituencies. (See the COMAP module *The Apportionment Problem* or Chapter 14 of *For All Practical Purposes.*)
- Students analyze mathematical methods for dividing an estate fairly among various heirs. (See Chapter 2 of *Discrete Mathematics Through Applications*, Chapter 3 of *Excursions in Modern Mathematics*, or Chapter 13 of *For All Practical Purposes.*)
- Students discuss various methods, such as preference schedules or approval voting, that can be used for determining the winner of an election involving three or more candidates (for example, the prom king or queen). With preference schedules, each voter ranks the candidates and the individual rankings are combined, using various techniques,

to obtain a group ranking; preference schedules are used, for example, in ranking sports teams or determining entertainment awards. In approval voting, each voter can vote once for each candidate which she finds acceptable; the candidate who receives the most votes then wins the election. (See the COMAP module *The Mathematical Theory of Elections* or Chapter 11 of *For All Practical Purposes.*)

- Students find an efficient way of doing a complex project (like preparing an airplane for its next trip) given which tasks precede which and how much time each task will take. (See Chapter 8 of *Excursions of Modern Mathematics* or Chapter 3 of *For All Practical Purposes.*)
- Students find an efficient way of assigning songs of various lengths to the two sides of an audio tape so that the total times on the two sides are as close together as possible. Similarly, they determine the minimal number of sheets of plywood needed to build a cabinet with pieces of specified dimensions.
- Students apply algorithms for matching in graphs to schedule when contestants play each other in the different rounds of a tournament.
- Students devise a strategy for finding a "secret number" from 1 to 1000 using questions of the form *Is your number bigger than 837?* and determine the least number of questions needed to find the secret number.

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