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Meaningless Statements

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Abstract. Increasingly, discrete mathematics is influenced by connections with other fields. We give one example of a broad topic in the field of discrete mathematics that comes from the social sciences, and in particular from the theory of measurement that has been developed to put measurement, especially in the social sciences, on a firm mathematical foundation. The key concept is that of meaningful statement, a statement whose truth or falsity remains unchanged after admissible transformations of scales of measurement. We apply this concept to combinatorial optimization, graph coloring, scheduling, linear programming, 0-1 optimization, and multiperson games.

1. The Future of Discrete Mathematics

Increasingly, discrete mathematics is influenced by connections with other fields. In the mathematical sciences, it is influenced by connection with probability, geometry, algebra, analysis, topology, number theory, ... Outside the mathematical sciences, it is influenced by connection with biology, chemistry, physics, manufacturing, engineering, ... In this paper, I will give one example of a broad topic in the field of discrete mathematics that comes from the social sciences, but has been developed not only by economists and psychologists, but also by philosophers of science, physicists, logicians, and, of course, mathematicians. It is motivated by the theory of measurement that has been developed to put measurement, especially in the social sciences, on a firm mathematical foundation.

In this paper, I will give a brief overview of measurement theory and define a central concept of the theory, that of a meaningful statement. I will then apply the theory of meaningfulness to discrete mathematics in various ways. In particular, I will ask whether or not statements in combinatorial optimization, that a specific solution is optimal, are meaningful. I will talk about the meaningfulness of conclusions about the generalization of graph coloring called T-coloring. I will give a variety of examples of meaningful and meaningless conclusions about scheduling problems. This will lead to the question of when a greedy solution to an optimization problem is a meaningful optimal solution. I will talk in general about

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meaningfulness of conclusions in linear programming and in 0-1 optimization. Finally, I will mention meaningfulness of conclusions about solutions for multiperson games.

The reader will see that questions of meaningfulness give rise to interesting questions in discrete mathematics. The detailed questions and answers are, however, perhaps not as important as the new way of thinking, and new types of questions, that applications such as those from the social sciences are bringing into discrete mathematics.

2. The Theory of Measurement

The theory of measurement was developed to place issues of measurement on a firm mathematical foundation. A primary motivation for the development of this theory was the need to understand, using measurement in the physical sciences as a paradigm, what it means to measure things in the social sciences, things like preference, value, loudness, etc. The ideas of the modern theory of measurement go back to 19th century work of Helmholtz [**10**] and others. The modern theory of "representational measurement" goes back to the foundational papers of Scott and Suppes [**36**] and Suppes and Zinnes [**42**], and is described in such books as [**12, 15, 41, 25, 28**].

A scale of measurement arises when we assign numbers to objects. Thus, we can think of a *scale* as a function f from some set A of objects into the real numbers, $f : A \rightarrow \Re$. A key idea is that of admissible transformation of scale. An *admissible transformation* sends one acceptable scale into another, for example changing from centigrade into Fahrenheit or from kilograms into pounds. In most cases, one can think of an admissible transformation ϕ as a function defined on the range of $f; \phi : f(A) \rightarrow \Re$. For instance, the admissible transformation changing degrees centigrade into degrees Fahrenheit is given by $\phi(x) = (9/5)x + 32$ and that for changing kilograms into pounds is given by $\phi(x) = (2.2)x$.

A variety of interesting classes of scales can be described by studying the associated class of admissible transformations. This idea was introduced by the social scientist S.S. Stevens in a series of papers (e.g., [**37, 38, 39**]). For instance, if we are allowed to change the unit and nothing else, then the class of admissible transformations is given by the functions of the form $\phi(x) = \alpha x, \alpha > 0$. A scale with this as its class of admissible transformations is called a *ratio scale*. Ratio scales arise when we are measuring mass, temperature on the Kelvin scale when there is an absolute zero, and time intervals such as minutes, hours, seconds. If we are allowed to change both the unit and the zero point, then the class of admissible transformations is given by the functions of the form $\phi(x) = \alpha x + \beta, \alpha > 0$. A scale with this as its class of admissible transformations is called an *interval scale*. Interval scales arise in the measurement of temperature (as for example when we change from centigrade to Fahrenheit) and in measurement of time in the calendar sense (as in the statement that this is the year 1997). If only order matters, then the class of admissible transformations consists of all strictly increasing functions ϕ, i.e., functions ϕ so that $x \geq y \Leftrightarrow \phi(x) \geq \phi(y)$. In this case, we say we have an *ordinal scale*. Ordinal scales arise when we have gradations, as in grades of leather or lumber or wool, or in measures of hardness of minerals. They often arise when we measure preferences using utilities.

The key idea in this paper is that of meaningfulness. The definition I will use is due to Suppes [**40**] and Suppes and Zinnes [**42**]. We shall say that a statement involving numerical scales is *meaningful* if its truth or falsity is unchanged after any (or all) of the scales is transformed by an admissible transformation. (As an aside, we note that there are situations, called *irregular*, where one may not be able to obtain one acceptable scale from another by an admissible transformation. In this case, we have to modify the definition to say: A statement involving numerical scales is *meaningful* if its truth or falsity is unchanged after any (or all) of the scales is replaced by another acceptable scale. We shall disregard this complication. It is discussed, however, in the paper [**35**]. Alternative formulations of the definition are discussed in many places in the literature, for instance in [**7, 15**]. See [**33**] for other references.) To illustrate the definition, consider the statement "The piece of yellow chalk is three times as long as the piece of white chalk." Is this a meaningful statement? Length is measured on a ratio scale (we only change units, as for example from centimeters to inches). Hence, the admissible transformations are of the form $\phi(x) = \alpha x, \alpha > 0$. The statement in question is the statement

$$f(a) = 3f(b).$$

We want this to hold if and only if

$$\phi(f(a)) = 3\phi(f(b)).$$

But this latter statement becomes

$$\alpha f(a) = 3\alpha f(b),$$

which is equivalent to the original since $\alpha > 0$. Thus, the original statement is meaningful. In general, for ratio scales, it is meaningful to compare ratios: $f(a)/f(b) > f(c)/f(d)$. For interval scales, it is meaningful to compare intervals: $f(a) - f(b) > f(c) - f(d)$. For ordinal scales, it is meaningful to compare size: $f(a) > f(b)$.

Consider another example, the statement: "The mouse weighs 1000 times what the elephant weighs." This is meaningful as with the last example, since weight is measured on a ratio scale. However, it is presumably false. Meaningfulness is different from truth. It is concerned with the kinds of assertions it makes sense to make, in particular the kinds of assertions that are not accidents of the particular choice of scale, i.e., of units, zero points, etc. If the truth of a conclusion can depend upon some arbitrary choices such as about units or zero points, we would probably not want to put much weight behind that conclusion. The notion of meaningfulness has a long history in the theory of measurement, and is closely related to the concept of invariance that has played such a central role in the history of mathematics. It has found applications in the study of average performance, importance ratings, indices of consumer confidence, psychophysical scaling, blockmodeling of social structure, structural modeling in decisionmaking, analysis of matching experiments, statistical analysis, performance analysis, and elsewhere. Some of these applications are investigated in the book [**28**] and the papers [**33, 29**]. In this paper, I would like to discuss its relevance to problems of discrete mathematics, in particular problems having to do with optimization.

3. Meaningfulness of Conclusions from Combinatorial Optimization

A classical problem of combinatorial optimization is the shortest path problem. Consider a network in which there are three vertices, x, a, and y, and there are arcs from x to a with weight 2, a to y with weight 4, and x to y with weight 40. Then the shortest path from x to y is the path x, a, y. Is this conclusion meaningful? What if weights are measured on an interval scale? This could happen if the weight on an arc measures the temperature encountered while traveling along that arc and we are seeking a route that minimizes the sum of temperatures encountered. Consider the admissible transformation $\phi(x) = 2x + 70$. Now the path x, y has smaller weight (150) than the path x, a, y (weight 152). The original conclusion was meaningless. It is, however, meaningful if weights are measured on a ratio scale. This kind of observation was first noted in the paper [**30**]. The shortest path problem can be formulated as a linear programming problem, and so our example shows that the conclusion that z is an optimal solution in a linear programming problem can be meaningless if the cost parameters are measured on an interval scale.

Consider next the minimum spanning tree problem: Given a weighted graph, what is the spanning tree with a total sum of weights as small as possible? If weights are given on an interval scale, is the conclusion that a given spanning tree is minimum a meaningful conclusion? The answer turns out to be yes. This is a corollary of the result that Kruskal's algorithm (the greedy algorithm) (see [**13**] and [**19**]) gives a minimum spanning tree. In this algorithm, we order edges in increasing order of weight and examine edges in this order. We include an edge if it does not form a cycle with edges previously included. We stop when all vertices are included. Since this algorithm depends only on the order of the weights, the optimal solution remains unchanged under the scale transformations associated with interval scales, i.e., the transformations of the form $\phi(x) = \alpha x + \beta, \alpha > 0$. In fact, it remains unchanged under strictly increasing transformations, and so the conclusion of optimality is meaningful even for ordinal scales.

Note that the analysis of meaningfulness is something like sensitivity analysis. We are interested in the sensitivity (or invariance) of conclusions under certain kinds of changes. However, the classes of perturbations of data in the study of meaningfulness are quite different than those studied in sensitivity analysis. While sensitivity analysis deals mainly with small perturbations of data, meaningfulness deals with admissible transformations that preserve certain relationships among "similar" types of data, so whether the change in problem parameters is small or large is irrelevant as long as a relationship in question remains unchanged.

4. T-Coloring

The following problem arises from questions of channel assignment in communications. Let $G = (V, E)$ be a graph and let T be a set of nonnegative integers. Find an assignment f of a positive integer $f(x)$ to each vertex x in G so that

$$x, y \in E \Rightarrow |f(x) - f(y)| \notin T.$$

Such an assignment f is called a T-*coloring* of graph G. In channel assignments, V is a set of transmitters, there is an edge between two transmitters if they conflict, $f(x)$ is the channel assigned to transmitter x, and T is a set of disallowed separations between channels assigned to conflicting transmitters. If $T = \{0\}$, then the T-coloring is an ordinary graph coloring. We shall always assume that 0 belongs to

T, for otherwise the trivial assignment in which each transmitter gets the same channel is a T-coloring. If the elements of T are nonnegative integers and the values of $f(x)$ are positive integers, as we shall assume in what follows, then the case $T = \{0, 1\}$ means that conflicting transmitters get not only different channels, but non-adjacent channels. The notion of T-coloring was introduced by Hale [**9**] and by Cozzens and Roberts [**5**], and has been studied extensively. For instance, several Ph.D. theses have been written about these colorings; see [**3, 14, 26, 43, 45**] . See [**31, 32**] for surveys of the subject.

Consider a triangle with vertices a, b, c and a T-coloring with $T = \{0, 1, 4, 5\}$. T-coloring the vertices in a greedy manner using the vertex order a, b, c, always choosing for a given vertex the lowest channel that does not violate the T-coloring rules, we get the assignment $f(a) = 1, f(b) = 3, f(c) = 9$. However, there is a "better" solution: $f(a) = 1, f(b) = 4, f(c) = 7$. The second solution is better in the sense that $\max_{x,y \in V} |f(x) - f(y)|$ is smaller with the second assignment. The quantity $\max_{x,y \in V} |f(x) - f(y)|$ is called the *span* of f and the minimum span over all T-colorings f is called the *optimal span (optimal T-span)* of G. It is natural to ask the question: for what graphs and what sets T does the greedy algorithm give the optimal span? Unfortunately, this is a difficult question, and it is not yet completely solved even for complete graphs. There is a considerable literature devoted to this subject (see summaries in some of the theses and papers cited above). What if the numbers in T change by a scale transformation? Is the answer to the above question meaningful? Consider the case where elements of T are measured on a ratio scale. Then even for complete graphs, the conclusion that the greedy algorithm obtains the optimal span is meaningless. Take the previous example of the triangle, but consider the transformation $\phi(x) = 2x$. Then T becomes the set $\alpha T = \{0, 2, 8, 10\}$ and now the greedy algorithm gives the assignment $f(a) = 1, f(b) = 2, f(c) = 5$. This has an optimal span.

(As an aside, we note that it is not as easy to give an example of a complete graph K_n and a set T and an integer α so that the greedy algorithm obtains the optimal span for K_n with T, but not with αT. One interesting example due to Cozzens and Roberts [**6**] is K_6 with $T = \{0, 1, 2, 3, 4\} \cup \{8\} \cup \{10, 11, 12\} \cup \{31, 32\}$. Then the greedy T-coloring gives an optimal span for K_6, but does not do so for αT when $\alpha = 2$.)

THEOREM 4.1 (Cozzens and Roberts [**6**]). *For fixed T, the conclusion that the greedy algorithm obtains the optimal span for* all *complete graphs is meaningful if the elements of T are measured on a ratio scale.*

For example, the greedy algorithm obtains the optimal span for all complete graphs if $T = \{0, \alpha, 2\alpha, \ldots, r\alpha\}$ for any $\alpha = 1, 2, 3, \ldots$. This result and the related analysis allowed for the discovery of new examples of sets T for which the greedy algorithm gave optimal spans for all complete graphs. Thus, considerations of meaningfulness had a side benefit of leading to results of independent interest in graph theory.

Note that limiting the set T to integers could be problematical in the interpretation of this example. As Pierre Hansen has pointed out (personal communication), suppose $T = \{0, 1\}$ and frequencies are measured in kilohertz (kHz). Then frequencies at distance 1 kHZ are not allowed on conflicting transmitters. Neither, of course, are frequencies at distance 1/2 kHz However, if we take $\alpha = 1000$, then the new units are Hertz (Hz) and now frequencies at distance 500 Hz = 1/2 kHz are allowed on conflicting channels. This changes the problem.

Analyses of the type we have made in this section might be relevant to other types of generalizations of ordinary graph colorings such as set colorings (see survey in [**31**]).

5. Scheduling with Earliness and Tardiness Penalties and Priorities

Discrete mathematics plays a very important role in the analysis of scheduling problems. Here we shall ask the question: Is it meaningful to say that a given schedule is an optimal solution to a scheduling problem. In particular, we shall note that for certain kinds of scheduling problems that involve scales of measurement, this conclusion can be meaningless. The particular type of scheduling problem we shall consider arose from the Air Mobility Command of the United States Air Force, and it can be characterized as a scheduling problem with earliness/tardiness penalties and priorities. Suppose that n items (equipment, people) are to be transported from an origin to a destination. Item i has a *desired arrival time* d_i. For simplicity, we assume this is a positive integer. Transportation time will be assumed to be constant. For any given arrival time, there is available a certain *capacity* c for transportation (number of seats, cargo capacity, etc.) (Note that we are assuming constant capacity.) A *schedule* S is an assignment of a positive integer arrival time C_i to each item i subject to the constraint that the number of C_i that can equal a given integer is at most c. Penalties are applied for tardy arrivals and perhaps for early arrivals. These depend upon the desired and scheduled arrival times of the different items and also upon the priority or status of the different items. The priority of item i is denoted by w_i. The interest in scheduling problems where penalties are applied to early arrivals as well as late arrivals is closely tied to the concept of "just-in-time" production, the goal being to have the right amount of materials of the right quality at the right time in the right place to produce the right quantity of items demanded by the next step of the production.

In the transportation scheduling problem as we have described it, a *penalty function* depends upon the vectors $\mathbf{C}(S) = (C_1, C_2, \ldots, C_n) = \mathbf{C}$, $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. We denote the penalty function by an expression such as $F(\mathbf{C}, \mathbf{w}, \mathbf{d})$. It is natural to consider the case where F is *summable* in the sense that

$$F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = \sum_{i=1}^{n} g(C_i, w_i, d_i).$$

Summable penalty functions are called *separable* if g can be expressed as

$$g(C, w, d) = \begin{cases} h_t(w) f(C, d) & \text{if } C \geq d \\ h_e(w) f(C, d) & \text{otherwise} \end{cases}$$

for functions h_t and h_e defined on the positive reals and $f(C, d)$ defined on ordered pairs of natural numbers. A summable, separable penalty function is called *symmetric* if $h_t = h_e$. A simple example of a summable, separable, symmetric penalty function is the function

$$F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = \sum_{i=1}^{n} w_i |C_i - d_i|.$$

Here, we are using the weighted sum of the deviations between desired and scheduled arrival times, where the weighting factor is the priority. The scheduling problem we shall study is to find a schedule that minimizes the penalty. There is a large literature concerned with this problem. For example, if $h_e(w) = h_t(w)$ for all w and $f(C, d) = |C-d|$ and $c = 1$, Garey, Tarjan and Wilfong [**8**] showed that the problem is NP-complete if transportation times can vary, but solvable in polynomial time if they are constant as in our problem. Some survey papers that deal with penalty functions similar to the ones we have described are [**1, 2, 11**].

Suppose that priorities are measured on some scale. It then makes sense to ask whether or not the conclusion that schedule S is optimal is a meaningful conclusion. To give a simple example, suppose that priorities are measured on a ratio scale. We are interested in the assertion A:

$$F(\mathbf{C}, \mathbf{w}, \mathbf{d}) \leq F(\mathbf{D}, \mathbf{w}, \mathbf{d})$$

for all schedules $\mathbf{D}$. And we ask, if α is a positive number, does this assertion imply the assertion B:

$$F(\mathbf{C}, \alpha\mathbf{w}, \mathbf{d}) \leq \mathbf{F}(\mathbf{D}, \alpha\mathbf{w}, \mathbf{d})$$

for all schedules $\mathbf{D}$. Here, $\alpha\mathbf{w}$ means the vector $(\alpha w_1, \alpha w_2, \ldots, \alpha w_n)$. Suppose that F is summable and separable and that

$$h_e(\alpha w) = K(\alpha) h_e(w) \text{ for } \alpha > 0, K(\alpha) > 0. \tag{5.1}$$

Then assertion A implies assertion B, so, as noted in [**16**], it is meaningful to conclude that schedule $\mathbf{D}$ is optimal. Equation 5.1 arises in measurement theory in various contexts, and arose originally in psychophysics. See [**34**].

Consider a different example, also noted in [**16**]. Let F be a summable, separable, symmetric objective function defined by taking $c = 1$, $f(C, d) = |C - d|$, and $h_e(w) = h_t(w) = 2^{w-1}$ for all w. If $\mathbf{w} = (1, 2, 2, 2)$ and $\mathbf{d} = (1, 2, 2, 2)$, then consider $\mathbf{C} = (1, 2, 3, 4)$. We have $F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = 6$ and it is easy to see that $\mathbf{C}$ is optimal. However, after a change of scale from $\mathbf{w}$ to $\alpha\mathbf{w} = 2\mathbf{w}$, we find that $F(\mathbf{C}, 2\mathbf{w}, \mathbf{d}) = 24$ while $F(\mathbf{D}, 2\mathbf{w}, \mathbf{d}) = 22$ for the schedule defined by $\mathbf{D} = (4, 1, 2, 3)$. Thus, $\mathbf{C}$ is no longer optimal after a change of unit. As noted in [**16**], the priority functions h_e and h_t used in this example might be somewhat unrealistic. However, they illustrate the point. If we use the same penalty function except to change h_e and h_t to be linear, then the conclusion of optimality of a schedule is meaningful under ratio scales. However, this turns out not to be the case under interval scales, as is also noted in [**16**].

A special case of the scheduling problem we have considered is the case of constant desired arrival times (also known as the problem of common due dates). Suppose that the penalty function is summable, separable, and symmetric. Suppose it is also *w-increasing* in the sense that both h_e and h_t are strictly increasing in w and *$|C - d|$-increasing* in the sense that $f(C, d) = A(|C - d|)$ for A an increasing function. Mahadev, Pekeč, and Roberts [**16**] show that in this case, a schedule is optimal if and only if it is attainable by a greedy algorithm. Moreover, any strictly increasing transformation of scale values from w_i to $\phi(w_i)$ gives the same greedy solution. The following conclusion results:

THEOREM 5.1 (Mahadev, Pekeč, and Roberts [**16**]). *Suppose that the objective function is summable, separable, symmetric, w-increasing, and $|C - d|$-increasing. Then the statement that $\mathbf{C}$ is an optimal solution for the constant desired arrival* *time problem is meaningful if priorities are measured on an ordinal, interval, or ratio scale.*

Perhaps the simplest case of non-constant desired arrival times is that where $\mathbf{d} = (d, d, \ldots, d, k)$, where $k \neq d$. We shall call this the (d, k)-*problem.* This problem is very specialized, but it leads to some interesting mathematical questions. In the case of ratio scale priority measurement, if we use the objective function defined by

$$F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = \sum_{i=1}^{n} w_i |C_i - d_i|,$$

then the conclusion of optimality is meaningful for a (d, k) problem. This follows since Equation 5.1 holds. However, the conclusion of optimality fails if we have interval scale priority measurement. This is the case if $c = 1, \mathbf{w} = (\mathbf{9}, \mathbf{9}, \mathbf{9}, \mathbf{1}), \mathbf{d} = (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1})$. Then, as pointed out in [**16**], $\mathbf{C} = (2, 1, 3, 4)$ is an optimal schedule, but after the admissible transformation $\phi(w) = \alpha w + \beta$ with $\alpha = 1/8, \beta = 7/8$ is applied, the schedule $\mathbf{D} = (2, 3, 4, 1)$ has a lower penalty function than $\mathbf{C}$. This leads to the question of determining conditions under which the conclusion of optimality for a (d, k)-problem is meaningful under interval scale priority measurement.

THEOREM 5.2 (Mahadev, Pekeč, and Roberts [**16**]). *Suppose that the objective function is summable, separable, symmetric, and w-increasing, and $f(C, d) = K|C - d| + L$ for $K > 0$. Suppose that $k > d$ or $k < d$ and $d \geq \lceil (n+c)/2c \rceil$. Then the statement that C is an optimal solution for the (d, k)-problem is meaningful if priorities are measured on an ordinal, interval, or ratio scale.*

The results we have just described leave a variety of open problems. For example, little is known about meaningfulness even in the case of scheduling problems where $\mathbf{d} = (d, d, \ldots, d, k, k)$. We have also left open the meaningfulness for the (d, k)–problem when $k < d$ and $d < \lceil (n + c)/2c \rceil$ Little work has been done on the case of nonconstant transportation times. It remains an open question to find scheduling situations where the conclusion of optimality is meaningful for interval scales but not for ordinal scales.

6. Single Machine Scheduling

A problem analogous to the scheduling problem considered in the previous section arises when we study single machine scheduling. Consider the problem of scheduling n jobs on a single machine in which each job has a specified due date or completion time and a penalty is applied for a completion time different from the desired one. In many practical problems, a penalty is only applied for tardy completions; while more generally, a penalty is applied to both early and tardy completions, perhaps in a different way. Because penalties can be applied to early completions, we allow the machine to lie idle and we schedule without preemption, i.e., we do not allow a job to be interrupted once it is started. Among the penalties studied in the literature are those that involve weighting factors that weight the deviations from desired completion times.

Typically weights are measured using some scale of measurement, and we can ask about the effect on the solution to a scheduling problem if we make admissible changes of scale. Mahadev, Pekeč, and Roberts [**17**] show that in some cases such changes can transform an optimal solution into a non-optimal one, and systematically describe those situations when this anomaly occurs.

To formulate the problem precisely, let us suppose we have n jobs in which the i^{th} job has *processing time* $p_i > 0$ and *desired completion time* $d_i \geq 0$. At most one job can be carried out at a time and there is no preemption. A *schedule* S (*feasible schedule* S) assigns a *completion time* C_i to each job i, with time starting at 0 and each job i getting processing time p_i. As in the previous section we let $\mathbf{C}(S) = (C_1, C_2, \ldots, C_n)$ and we also use the vectors $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Since no job may start before time 0, we have $C_i \geq p_i$, for all i. We consider two special cases, those with *common due dates*, i.e., where $d_i = d$ for all i, and those with *common processing times*, i.e., where $p_i = p$ for all i. The first case is abbreviated **cdd**, with **ncdd** standing for the situation where we are not in case cdd. The second case is abbreviated **cpt**, with **ncpt** standing for the situation where we are not in case cpt.

Typically, we have an *objective function* $F(S) = F(d_1, d_2, \ldots, d_n, C_1, C_2, \ldots, C_n)$ and we seek a (feasible) schedule S that minimizes $F(S)$. Note that the objective function does not explicitly depend upon the processing times. These play a role in determining whether or not a schedule is feasible. We only consider objective functions that depend upon the deviations between desired and actual completion times, and specifically we define the *earliness* of the i-th job $E_i(S) = E_i = \max\{0, d_i - C_i\}$ and the *tardiness* of the i-th job $T_i(S) = T_i = \max\{0, C_i - d_i\}$. One objective function arises if we simply consider a weighted sum of earliness and tardiness, obtaining

$$F(S) = F_{sumE|T}(S) = \sum_{i=1}^{n} \alpha_i E_i + \sum_{i=1}^{n} \beta_i T_i,$$

where α_i and β_i are positive weights. If only tardiness is penalized, we consider the function

$$F(S) = F_{sumT}(S) = \sum_{i=1}^{n} \beta_i T_i.$$

In objective functions $F_{sumE|T}(S)$ and $F_{sumT}(S)$, we minimize the sum of the deviations. An alternative is to minimize the maximum deviation; we then have the objective functions

$$F(S) = F_{maxE|T} = \max\{\alpha_1 E_1, \alpha_2 E_2, \ldots, \alpha_n E_n, \beta_1 T_1, \beta_2 T_2, \ldots, \beta_n T_n\}$$

and

$$F(S) = F_{maxT} = \max\{\beta_1 T_1, \beta_2 T_2, \ldots, \beta_n T_n\}.$$

A variety of other objective functions have been studied in the literature.

In the case where $\alpha_i = \beta_i$ for all i, the objective function has symmetric weights and we say that we are in the case of *symmetry*; we abbreviate this case as **s**. If we are not in case s, we say we are in case **ns**. If $\beta_i = \beta$ for all i and if $\alpha_i = \alpha$ for all i when we are considering earliness penalties, then we say that we have *common weights* and abbreviate this case as **cw**. If we do not have common weights, we say we are in case **ncw**. In [**17**], Mahadev, Pekeč, and Roberts systematically study the four penalty functions $F_{sumE|T}$, F_{sumT}, $F_{maxE|T}$, and F_{maxT} under all combinations of the four assumptions cdd or ncdd, cpt or ncpt, s or ns, and cw or ncw. They show that if weights are measured on a ratio scale, then conclusions about optimality of a schedule are meaningful under the four objective functions we have defined. However, this is not always true for interval scales or ordinal scales. The results of [**17**] can be summarized both for interval and ordinal scales as follows: in the case of common weights, if weights are symmetric or if we only apply penalties for tardiness, then conclusions about optimality are meaningful for the four objective functions considered. In other cases, these conclusions can be meaningless (in the sense that there exist weights such that there is no optimal schedule for which the conclusion of optimality is meaningful), except in the cases ncw, cdd, cpt, s for objective function $F_{sumE|T}$ and ncw, cdd, cpt, ns for objective function F_{sumT}. The case of common due dates and non-common weights for objective function F_{maxT} is different than all other cases considered. In this case, there always exists an optimal schedule (given by the largest weight order) for which the conclusion of optimality is meaningful. However, there can be other optimal schedules for which the conclusion of optimality is meaningless.

The above results are mathematically very straightforward. They are more interesting for the points they make than for the mathematical issues involved. However, one surprising result is that the results about the meaningfulness of the conclusion of optimality are the same for interval and ordinal scales in all the cases that were considered in [**17**]. It is not hard to give examples of problems in combinatorial optimization where the conclusion of optimality is meaningful for interval scales but not for ordinal scales. (For instance, this is true for the traveling salesman problem where edge weights represent data measured on an interval scale.) It would be interesting to characterize scheduling problems for which the fact that the conclusion of optimality is meaningful for any choice of weights measured on an interval scale implies that the conclusion of optimality is meaningful for any choice of weights measured on an ordinal scale. This is a topic that will require further research.

The work of [**17**] leaves room for considerably more analysis. For one thing, it considers only single machines, rather than multiple machines. Second, the discussion is limited to four specific objective functions. A variety of other objective functions are studied in the literature (see for example the survey paper [**2**]) and it would be interesting to extend the analysis to some of these other objective functions as well.

7. When is a Greedy Solution a Meaningful Optimal Solution?

We have observed in Section 5 that in scheduling under ordinal scales, a "greedy" solution is optimal under certain conditions and this conclusion is meaningful. We would like to ask: Under a general setting, for ordinal scales, when is the greedy solution to an optimization problem a meaningful optimal solution? We give a general formulation due to Pekeč [**20, 21**]. In this section, we concentrate on the special case where weights are positive.

We use the notation $\Re^n_+$ for the set of vectors $\mathbf{x} \in \Re^n$ for which $x_i \geq 0$ for all i. Suppose $X = X_1 \times X_2 \times \ldots \times X_n$ and X_i is a metric space with a metric d_i. Consider an objective function $P : X \times \Re^n_+ \rightarrow \Re$. Let $F \subseteq X$ correspond to the set of feasible solutions. The problem is to minimize $P(\mathbf{x}, \mathbf{w})$ over $\mathbf{x} \in F$. Pekeč makes the following assumptions.

Axiom 1. For every i, there is a unique $a_i \in X_i$ so that for all $\mathbf{x} \in X, \mathbf{w} \in \Re^n_+$,

$$P(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n; \mathbf{w}) \leq \mathbf{P}(\mathbf{x}, \mathbf{w}).$$

In this axiom a_i is the "desirable value" for the i-th variable.

Axiom 2. If $x_j = y_j$ for $j \neq i$, then

$$P(\mathbf{x};\mathbf{w}) \leq P(\mathbf{y};\mathbf{w}) \Leftrightarrow d_i(x_i, a_i) \leq d_i(y_i, a_i)$$

This axiom says that P is increasing as the i-th variable moves away from the desirable value.

Axiom 3 (w-unboundedness). If $x_j = y_j$ for $j \neq i$, then

$$\lim_{w_i \to \infty} |P(\mathbf{x};\mathbf{w}) - P(\mathbf{y};\mathbf{w})| = \infty.$$

This axiom says that there can be an unbounded change in the value of P for any change of the value of the i-th variable.

Axiom 4 (Controllability). If $x_j = y_j$ for $j \neq i$, then there is $M_i(\mathbf{x},\mathbf{y})$ so that for all $\mathbf{w} \in \Re^n_+$,

$$\lim_{w_i \to 0} |P(\mathbf{x};\mathbf{w}) - P(\mathbf{y};\mathbf{w})| < M_i$$

This axiom says that moving a variable with a small weight cannot result in big changes in the value of the objective function.

These four axioms are satisfied by a linear objective function $P(\mathbf{x};\mathbf{w}) = \mathbf{w}^T\mathbf{x} = \sum w_i x_i$ and by a quadratic objective function $P(\mathbf{x};\mathbf{w}) = \sum w_i w_j x_i x_j$.

The w_i could be measured on scales. The decision as to whether or not $\mathbf{x} \in F$ could also depend upon scales. We shall only allow the former here. Moreover, we shall limit ourselves to the case where the w_i are measured on an ordinal scale.

THEOREM 7.1 (Pekeč [**20, 21**]). *Consider an optimization problem* $\min_{\mathbf{x}\in F} P(\mathbf{x};\mathbf{w})$, *where F is a closed set and P satisfies Axioms 1 to 4. If $w_1, w_2, \dots, w_n$ are measured on an ordinal scale, then every meaningful optimal solution is a "greedy" solution.*

Let us add two additional axioms.

Axiom 5. For $k = 1, 2, \dots, n$, $c \geq 0$, the equation $d_k(y_k, a_k) = c$ has at most one solution $y_k \in X_k$.

Axiom 6. For all $\mathbf{x} \in F$, the function $f_{\mathbf{x}} : \Re^n_+ \to \Re$ defined by $f_{\mathbf{x}}(\mathbf{w}) = P(\mathbf{x};\mathbf{w})$ is continuous.

THEOREM 7.2 (Pekeč [**20, 21**]). *Under Axioms 1 to 6, the following are equivalent:*

- **(a):** *For any choice of $w_1, w_2, \dots, w_n$ measured on an ordinal scale, any optimal solution to the problem* $\min_{\mathbf{x}\in F} P(\mathbf{x};\mathbf{w})$ *is meaningful.*
- **(b):** *For any such choice of $w_1, w_2, \dots, w_n$, any optimal solution to the problem* $\min_{\mathbf{x}\in F} P(\mathbf{x};\mathbf{w})$ *is greedy.*

These results have a negative implication for optimization under ordinal scales. Suppose there is a problem with a choice of weights so that the greedy algorithm doesn't give an optimal solution, for example the Traveling Salesman Problem. If w_i are measured on an ordinal scale, then there cannot be a meaningful optimal solution. In other words, here, it doesn't make sense to ask for an optimal solution!

It should be noted that Theorem 7.2 gives a characterization of problems for which only the greedy algorithm provides a solution, by characterizing the structure of the set of transformations of parameters of the objective function that leave the conclusion of optimality invariant. Problems solvable by a greedy algorithm are well understood in the case of optimizing a linear objective function over the set of 0-1 vectors–they are characterized by the matroid structure of the set of feasible solutions. (See for example [**19**]). However, the characterization in Theorem 7.2 applies in a much more general set-up. This indicates that it might be worth studying algorithms by way of understanding classes of transformations of parameters of the objective function under which the conclusion of optimality is invariant. For example, the least one should expect from an efficient optimization or approximation algorithm is that the same class of transformations that does not affect the conclusion of optimality should also not affect the solution obtained by the algorithm in question. Some observations along these lines are presented in [**20**].

It should be noted that the hypothesis that the function P is defined on $X \times \Re^n_+$ is needed for the proof of Theorem 7.2. It would be interesting to try to obtain a similar result if P is defined on $X \times \Re^n$.

8. Linear Programming

Theorem 7.2 suggests that for many problems, it will not make sense to ask for an optimal solution if we measure parameters of the objective function on ordinal scales. In this section, we consider interval and ratio scales. We concentrate on linear programming, and present just a few of the results of Pekeč [**20, 24**]. We have already seen in our discussion of the shortest path problem in Section 3 that if we have interval scales, the conclusion of optimality can be meaningless. Let us now consider a general linear programming problem:

$$\begin{array}{l} \max P(\mathbf{x};\mathbf{w}) = \mathbf{w}^T\mathbf{x} \\ \text{subject to:} \\ \mathbf{x} \in F := \{\mathbf{x} \in \Re^n : \mathbf{x} \geq 0, A\mathbf{x} = \mathbf{b}\} \end{array} \tag{8.1}$$

We limit ourselves to the situation where the w_i are measured on some scale, and assume that the a_{ij} and b_j are fixed. The situation when they are measured on scales also needs to be considered. A trivial observation is the following. Let $\mathbf{x}^*$ be an optimal solution to the linear programming problem 8.1. If $w_1, w_2, \ldots, w_n$ are measured on a ratio scale, then the conclusion that $\mathbf{x}^*$ is an optimal solution to 8.1 is meaningful. The following straightforward theorem handles the interval scale case.

THEOREM 8.1 (Pekeč [**20, 24**]). *Let $\mathbf{x}^*$ be an optimal solution to the linear programming problem 8.1. If $w_1, w_2, \ldots, w_n$ are measured on an interval scale, then the conclusion that $\mathbf{x}^*$ is an optimal solution to 8.1 is meaningful if and only if $x_1 + x_2 + \ldots + x_n$ is a constant on F.*

Analyzing meaningfulness of the conclusion of optimality when the set of feasible solutions depends on some problem parameters might not be a useful thing to do given that even a small change of, say, one coefficient in the constraint matrix A, can result in a change of all extremal points of the set of feasible solutions. However, there is a one to one correspondence between all extremal points and all basis matrices B, i.e., all possible sets of basic variables. (For definitions of basic variables and related concepts see any basic LP book, e.g., [**4**]). One can ask whether the set of basic variables of an optimal solution is invariant under certain transformations of problem parameters. In other words, one can ask whether the conclusion that the set S of basic variables defines an optimal solution is meaningful if, say, all coefficients in the i-th row of the constraint matrix A are measured on a ratio scale. In [**20, 24**] several problems of this type are analyzed. The following is a sample result.

THEOREM 8.2 (Pekeč [**20, 24**]). *Let B be an optimal basis for problem 8.1. If for every row i of the constraint matrix A, coefficients of the i-th row are measured on a common ratio scale, then the statement that the basic variables of B define an optimal basis for problem 8.1 is meaningful if and only if B^{-1} has no negative entries.*

The linear programming problems are the only problems where the meaningfulness of the conclusion of optimality, when the set of feasible solutions depends on the choice of the scale of measurement, has been studied. It would be interesting to see a similar analysis for problems where the structure of the set of feasible solutions is dependent on the problem parameters in a more complicated way (than simple linear inequalities). It is not even clear how this analysis would extend to semi-definite programming problems (given the absence of the notion of basic variable, basis, etc.).

9. 0-1 Optimization

Consider the linear 0-1 optimization problem

$$\max_{\mathbf{x}\in\mathcal{H}} \mathbf{w}^T\mathbf{x} \tag{9.1}$$

where $\mathcal{H} \subseteq \{0,1\}^n$. More generally, consider the problem

$$\max_{\mathbf{x}\in\mathcal{H}} P(\mathbf{x};\mathbf{w}) \tag{9.2}$$

where $\mathcal{H} \subseteq \{0,1\}^n, P : \{0,1\}^n \times \Re^n \to \Re$. A vector $\mathbf{x} \in \{0,1\}^n$ can be thought of as the incidence vector of a set $S \subseteq \{1,\ldots,n\}$. Hence, we can reformulate 9.2 as

$$\max_{S\in\mathcal{H}} f_S(\mathbf{w}) \tag{9.3}$$

where $\mathcal{H}$ is a collection of subsets of $\{1,\ldots,n\}$ and $f_S : \Re^n \to \Re$, in other words, $\mathcal{H}$ is a hypergraph. We ask: For what objective functions is the conclusion of optimality for a problem like 9.2 (or 9.3) meaningful? If w_i are measured on a ratio scale, the surprising result is that, under certain assumptions, the conclusion is meaningful only if the objective function is as in 9.1, i,.e., only if we have a linear 0-1 optimization problem.

Suppose that we have ratio scales. Then, meaningfulness means that for all $\alpha > 0$,

$$P(\mathbf{x}^*;\mathbf{w}) = max_{\mathbf{x}\in\mathcal{H}}P(\mathbf{x};\mathbf{w}) \Longleftrightarrow P(\mathbf{x}^*;\alpha\mathbf{w}) = max_{\mathbf{x}\in\mathcal{H}}P(\mathbf{x};\alpha\mathbf{w}).$$

We make the following assumptions.

Axiom L (Locality). For every $S \subset \{1,\ldots,n\}$, and for all $j \notin S$, $\frac{\partial f_S}{\partial w_j} = 0$.

This axiom says that changing the weight w_j corresponding to any element $j \notin S$ will not change the value of f_S.

Axiom N (Normality). For all i, $f_i(\mathbf{w}) = w_i$.

This axiom is not a serious restriction. One can always introduce new weights $w_i' = f_i(w_i)$.

Axiom C (Completeness). For all nonempty $S \subset \{1, \ldots, n\}$, $f_S(\Re^n) = \Re$.

This says that every number in $\Re$ is obtained for some vector in $\Re^n$.

Axiom S (Separability). For all f_S and all i, there is $g_i : \Re \to \Re$ such that $g_i \in C'(\Re)$ and $\frac{\partial f_s}{\partial w_i}(\mathbf{w}) = g_i(w_i)$.

This says that the rate of change of $f_S(\mathbf{w})$ with respect to changing w_i should depend only on w_i and not on values of $w_j, j \neq i$, and the dependence should be "smooth."

THEOREM 9.1 (Pekeč [**22**]). *Consider an optimization problem* $\max_{\mathbf{x}\in\mathcal{H}} P(\mathbf{x};\mathbf{w})$, *and suppose* $w_1, w_2, \ldots, w_n$ *are measured on a ratio scale. If Axioms L, N, C, and S hold, then every optimal solution is meaningful if and only if every* f_S *is linear, i.e., of the form*

$$f_S(\mathbf{w}) = \sum_{i\in S} c_{S,i} w_i.$$

Consider the example where

$$P(\mathbf{x};\mathbf{w}) = (x_1 {w_1}^p + \ldots + x_n {w_n}^p)^{1/p}.$$

This function satisfies Axioms L, N, and C, but not Axiom S. Note that it does satisfy meaningfulness under ratio scales.

As we noted in Section 6, it is not hard to give examples of problems in combinatorial optimization where the conclusion of optimality is meaningful for interval scales but not for ordinal scales. It would be interesting to characterize 0-1 optimization problems for which the fact that the conclusion of optimality is meaningful for any choice of w_i measured on an interval scale implies the same for any choice of w_i measured on an ordinal scale.

10. Multi-person Games

In game theory, the study of games with many players and presented in characteristic function form has many different solution concepts and many different interesting applications, going back to the classic book by von Neumann and Morgenstern [**44**]. We use the terminology of multi-person game theory from [**27**] and [**18**]. Let I be a set of players and consider *coalitions* or subsets of I. A game in *characteristic function form* is defined by a function $v : 2^I \backslash \{\emptyset\} \to \Re$. Sometimes, the function is defined on $\emptyset$ also, and then we assume $v(\emptyset) = 0$; however, we shall not take this approach here. Moreover, sometimes the condition

$$v(S \cup T) \geq v(S) + v(T) \text{ if } S \cap T = \emptyset,$$

known as *superadditivity*, is added to the definition of characteristic function, but we shall not make this assumption. The number $v(S)$ is thought of as the value that coalition S can guarantee itself. We shall study various solution concepts for games, and ask what happens to a solution if v is measured on some scale and we have a change of scale.

Solution concepts for games in characteristic function form are defined in terms of vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Re^n$, where $n = |I|$ and x_i represents the "payoff" to player i. The solution concepts we shall mention are called the core, the stable set, the Shapley value, the Banzhaf value, and the nucleolus, and all are widely used in game theory. We first consider the idea of normalization. Suppose that $D \subseteq \Re^n$. If there is $\alpha > 0$ so that for all $\mathbf{x} \in D, \sum_{i=1}^n \alpha x_i = 1$, we say that $\alpha D = \{\alpha\mathbf{x} : \mathbf{x} \in D\}$ is the *normalization* of D. We cannot expect a solution to a game to stay fixed after an admissible transformation of scale, but we can expect the normalization of the solution to stay fixed, i.e., the relative distribution of the payoffs to stay the same. We present some results of Pekeč [**20, 23**]. We first consider the case of ratio scales.

THEOREM 10.1 (Pekeč [**20, 23**]). *Let $D \subseteq \{\mathbf{x} \in \Re^n : \sum_{i=1}^n x_i = 1\}$. If v defines a ratio scale, then it is meaningful to say that D is the normalization of the core of v. The same conclusion holds for stable set, Shapley value, Banzhaf value, and nucleolus.*

The case of interval scales is more complicated. Consider the case where $I = \{1, 2\}, v(\{1\}) = 2, v(\{2\}) = 3, v(\{1, 2\}) = 6$. It is well known (see [**27**]) that the core can be computed as

$$C(v) = \{\mathbf{x} : \sum_{i=1}^{n} x_i = v(I), \text{ and for all } S \subseteq I, \sum_{i \in S} x_i \geq v(S)\}.$$

In our example,

$$C(v) = \{(x_1, x_2) : x_1 \geq 2, x_2 \geq 3, x_1 + x_2 = 6\}.$$

The normalization of $C(v)$ is

$$D = \{(x_1, x_2) : 1/3 \leq x_1 \leq 1/2, x_1 + x_2 = 1\}.$$

Let $v' = v - 1$, i.e., $v' = \alpha v + \beta, \alpha = 1, \beta = -1$. Then $v'(\{1\}) = 1, v'(\{2\}) = 2, v'(\{1, 2\}) = 5$ and

$$C(v') = \{(x_1, x_2) : x_1 \geq 1, x_2 \geq 2, x_1 + x_2 = 5\}.$$

The normalization of $C(v')$ is

$$D' = \{(x_1, x_2) : 1/5 \leq x_1 \leq 3/5, x_1 + x_2 = 1\}.$$

Thus, the statement that D is the normalization of the core is meaningless. So can be the statement that $\mathbf{x}$ is in the normalization of the core. The result is even stronger:

THEOREM 10.2 (Pekeč [**20, 23**]). *If a game v is defined on an interval scale and $D \subseteq \{\mathbf{x} \in \Re^n : \sum_{i=1}^n x_i = 1\}$, then the statement that D is the normalization of the core of v is meaningless. Moreover, if $\sum_{i=1}^n x_i = 1$, then the statement that $\mathbf{x}$ is in the normalization of the core of v is meaningless.*

There are similar results in [**20**] and [**23**] for stable sets.

Let $f(v)$ be the Shapley value, the Banzhaf value, or the nucleolus of game v. The normalization of $f(v)$ is the vector $[\sum_{i=1}^n f(v)_i]^{-1} f(v)$.

THEOREM 10.3 (Pekeč [**20, 23**]). *Let v be defined on an interval scale and let* $\mathbf{y} \neq (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, $\mathbf{y} \neq 0$. *Then if the statement that* $\mathbf{y}$ *is the normalization of the Shapley value is meaningful, the statement is false.*

In other words, except for 0 and symmetric vectors, if we can find an admissible transformation such that the vector is the normalization of the Shapley value, then we can find another admissible transformation such that it isn't. Similar results hold for the Banzhaf value and for the nucleolus.To illustrate the theorem, consider the example given prior to Theorem 10.2. Here, the Shapley value is given by (5/2, 7/2). Its normalization is (5/12, 7/12). If $v' = v - 1$, then the Shapley value becomes (2, 3) and its normalization is (2/5, 3/5). The statement that the normalization of the Shapley value is (5/12, 7/12) is meaningless.

In the case of ordinal scales, there is the following impossibility result:

THEOREM 10.4 (Pekeč [**23**]). *There is no solution concept* $f : \Re^{2^n-1} \to \Re^n$ *that attains infinitely many values in* $\Re^n$ *and such that for every ordinal scale n-person game there exists* $x \in \Re^n$ *so that the statement "*$x = f(v)$*" is meaningful.*

Put another way, if we exclude certain solution concepts that don't depend on game values, such as "player one takes everything," then for any solution concept f one can think of, we can construct an ordinal scale game so that "$f(v) = x$" is meaningless *for every* x.

It would be interesting to perform meaningfulness analyses for non-cooperative games, for instance to analyze the Nash equilibrium and its refinements from the point of view of measurement theory.

11. Closing

It is hard to predict what future trends and problems will stimulate the development of new mathematical methods. It is also hard to predict which methods of mathematics, in particular discrete mathematics, will turn out to be most useful, both theoretically and in applications to other areas. What is safe to say is that discrete mathematics, with its many connections to other fields, both mathematical and non-mathematical, is thriving.

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