Linking and Caging

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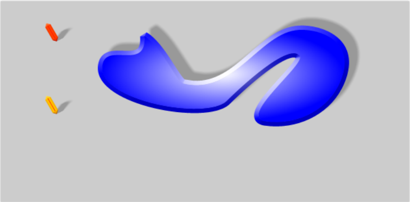

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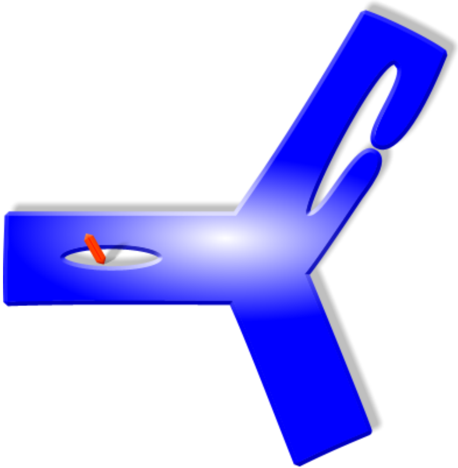
The bounding bodies here are mere points; more general cases can be handled without much overhead (in the simplest situation when the bounding bodies are round disks of the same radius, one can replace $\mathcal{D}$ by its parallel body).

caging

Example of caging with one finger



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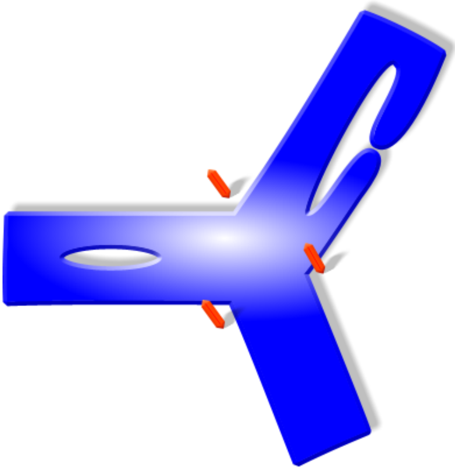
Example of caging with two fingers

caging

Example of caging with three fingers



caging as a topological problem

This work deals with the topological aspects of caging, essentially with the topology of certain configuration spaces, defined by geometric constraints.

Most important type of questions asked about such spaces question is about the structure of their connected components: there is a component corresponding to configuration far away from the body, and all other components correspond to caging configurations.

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The caging seems to be a purely geometric phenomenon, and is best dealt with using tools and methods of computational geometry. We argue however, that topological techniques allow deeper insight and, potentially, routes to new algorithms.

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Topological aspects of caging were addressed, implicitly, in the beautiful '89 paper by Goodman, Pach and Yep on *Mountain Climbing, Ladder Moving, and the Ring-Width of a Polygon*; more recently by Rimon and Blake ('96) and Mason and Rodriguez ('08).

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We start with some warm-up stories.

on cars and wagons

Consider the following (well-known) problem from *V.I. Arnold's* book on ordinary differential equations:

Problem 1. Two nonintersecting roads lead from City A to City B (Fig. 1). Suppose it is known that two cars connected by a rope of length less than $2l$ manage to go from A to B along different roads without breaking the rope. Can two circular wagons of radius l whose centers move along the roads in opposite directions pass each other without colliding?

Solution. Consider the square

$$M = \{(x_1, x_2) : 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$$

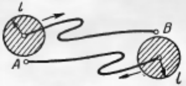

Fig. 1 Initial position of the wagons.

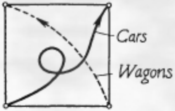

Fig. 2 Phase space of a pair of vehicles.

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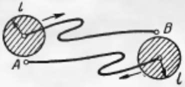

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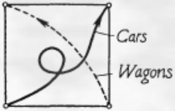

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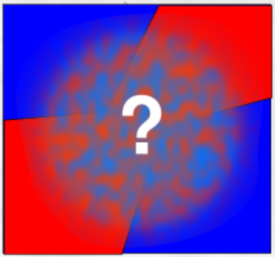


on cars and wagons, cont'd

We see that there is a certain dichotomy: either the cars, or the wagons can perform the task.

The problem is essentially topological (despite its geometric appearance): the obstacle to two red corners belonging to the same connected component is the fact that two blue corners are in the same component, and vice versa.

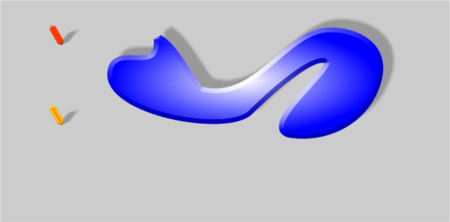
> *There is an obvious resemblance to the dualities of the optimization problems:* max *of the primary functional is equal to* min *of the functional of the dual problem.*

squeezing a camel through the eye of a needle

Let us consider the following problem related to caging.

We start with the planar domain $\mathcal{D}$ (with piece-wise smooth or semialgebraic boundary) homeomorphic to a open disk, and two point configuration $\mathbf{C} = \{p_1, p_2\}$ on plane.

Problem 1 *Can one* squeeze $\mathcal{D}$ *between the point configuration* $\mathbf{C}$*?*



pulling the needle around the camel

Equivalently, can one pull the points p_1, p_2 (preserving the distance between them) around $\mathcal{D}$?

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Definition 1 Pulling **C** *around* $\mathcal{D}$ *is a loop* π *in* $\mathcal{E}$ *(that is a continuous mapping* $S^1 \to \mathcal{E}$*) such that the loops* πp_i *in* $\mathbb{R}^2$ *do not meet* $\mathcal{D}$ *and the following "winding" condition holds: index of the loop* πp_1 *with respect to* $\mathcal{D}$ *is* 1*, and the index of* πp_2 *is* 0*.*

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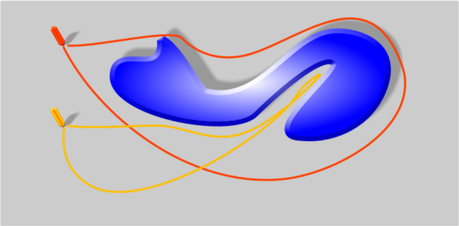

dichotomy

Now, what about somewhat more complicated shapes?

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Proposition 1 *Either one can pull two point configuration* $\mathbf{C}$ *around* $\mathcal{D}$*, or there exists a full rotation of* $\mathbf{C}$ entirely within $\mathcal{D}$*, that is a loop* $\pi' : S^1 \to \mathcal{E}$ *such that the vector* $\pi'_\theta p_1 - \pi'_\theta p_2$ *turns around the origin (perhaps, several times), and both loops* $\pi'_\theta p_1, \pi'_\theta p_2$ *stay within* $\mathcal{D}$*.*

Euclidean motions of the plane

The topology of $\mathcal{E}$ is important: $\mathcal{E}$ is diffeomorphic to the (open) solid torus,

$$\mathcal{E} \cong \mathbb{R}^2 \times S^1 :$$

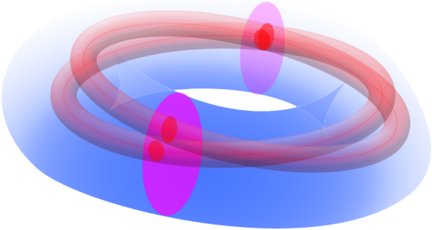
indeed, an orientation-preserving Euclidean motion of the plane can be uniquely represented as a composition of a parallel translation (i.e. $\mathbb{R}^2$), and a rotation (i.e. S^1).

Remark 1 *In fact, one can work with somewhat more convenient 3-dimensional sphere, $S^3 \supset \mathcal{E}$. Indeed, one could without restricting generality first pass to the closed solid torus (by restricting admissible motions to those which do not take p_1 outside of some pre-specified disk $B(R)$ of large enough radius R, and then by allowing the configurations with $p_1' \in \partial B(R); |p_2' - p_1'| \leq |p_1 - p_2|$. The resulting configuration space is homeomorphic to S^3 and the notion of "pulling through" remains unaffected.*

useful subsets of $\mathcal{E}$

Definition 2 *We will use the following subsets:*

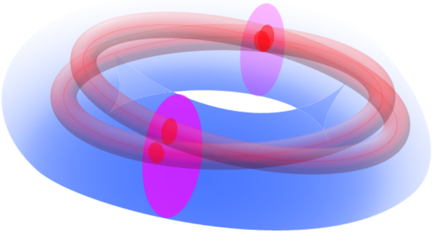
- $O_i \subset \mathcal{E}, i = 1, 2$ *is the set of motions taking* p_i *to* $\mathcal{D}$;
- $F_i := \mathcal{E} - O_i$;
- $O_{12} := O_1 \cap O_2$; $F_{12} := \mathcal{E} - O_{12}$;
- $O := O_1 \cup O_2$; $F := \mathcal{E} - O$.



some properties or F and O

Some trivial properties of $O_{\cdot}, F_{\cdot}$:

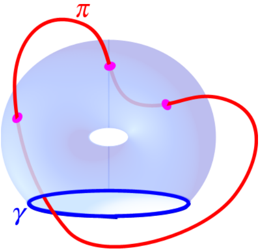
- Obstacles O_i are diffeomorphic to the solid tori (like $\mathcal{E}$);
- Free spaces F_i are homotopy equivalent to $\mathbb{T}^2$ ("thick torus");
- a loop $\pi : S^1 \to \mathcal{E}$ avoids F *iff* the loops πp_i in $\mathbb{R}^2$ avoid $\mathcal{D}$.



winding and linking numbers

The winding number of the loop πp_i around $\mathcal{D}$ can be represented as the *linking* of the loop π in $\mathcal{E}$ with the equator in O_i, that is the set of motions in $\mathcal{E}$ taking p_i into a given point in $\mathcal{D}$.

Definition 3 *Let $\pi, \gamma : S^1 \to \mathcal{E}$ be two loops in $\mathcal{E}$, such that γ can be patched by an oriented surface S. Then the intersection number of S and π is called the linking of π and γ.*



Linking number of π and γ is 1.

rotating **C** inside $\mathcal{D}$

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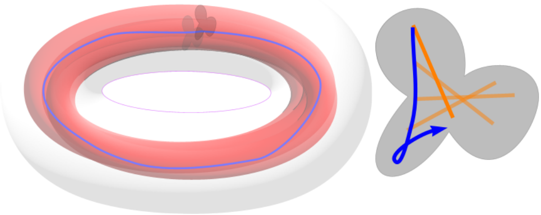
- The fact that a loop π in $\mathcal{E}$ is such that both πp_i remain within $\mathcal{D}$ is equivalent to π being within O_{12}.
- The fact that the loop "turns **C** around" means that π goes around the solid torus $\mathcal{E}$ (perhaps, several times).



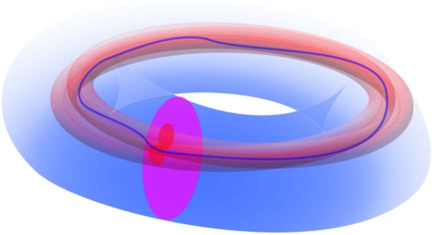
(almost) elementary proof

Now, to proof of Proposition **??**.

One direction is easy:

If there is a rotation of **C** within $\mathcal{D}$, that is there exists a loop π in O_{12} running around the torus $\mathcal{E}$, then it also runs around $O_i, i = 1, 2$.

Hence, if a loop $\gamma \in F$ has nonzero linking number with O_1, it has nonzero linking number with π, and therefore also nonzero linking number with O_2. Hence, pulling $\mathcal{D}$ through **C** is impossible.



proof, cont'd

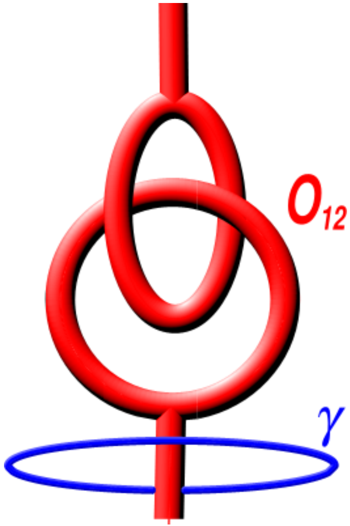
To get the reverse direction, we need a form of *Alexander duality* which in our situation says:

> *The only obstacle to patching a meridian-like loop γ in $F_{12} = \mathcal{E} - O_{12}$ by a chain avoiding O_{12} is a loop in O_{12} running around $\mathcal{E}$.*

Now, if there is *no* such loop in O_{12}, that is if the configuration **C** cannot be rotated within the $\mathcal{D}$, then the large loop γ in $\mathcal{E}$ having linking 1 with both $O_i, i = 1, 2$ can be patched by a 2-dimensional surface S (with boundary γ).

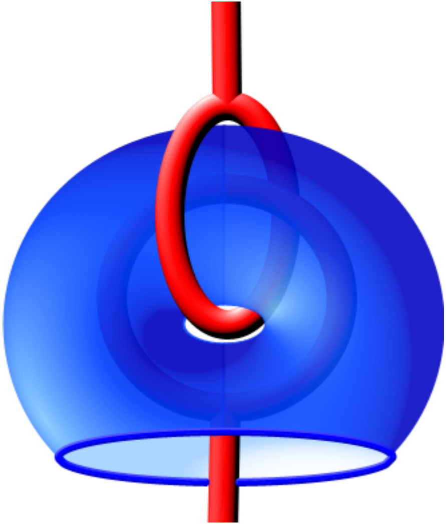
proof, cont'd

- "Large meridian loop" is an encircling of $\mathcal{D}$ by **C** at distance.



proof, cont'd

- "Large meridian loop" is an encircling of $\mathcal{D}$ by **C** at distance.
- Impossibility to rotate the two-point configuration **C** within $\mathcal{D}$ implies that this loop is homologically trivial.



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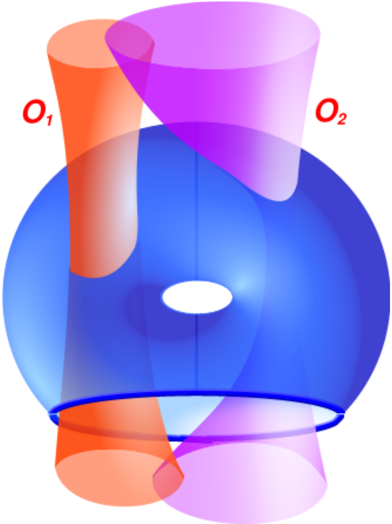

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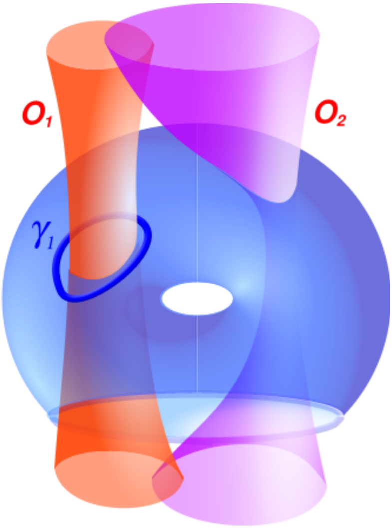
- "Large meridian loop" is an encircling of $\mathcal{D}$ by $\mathbf{C}$ at distance.
- Impossibility to rotate the two-point configuration $\mathbf{C}$ within $\mathcal{D}$ implies that this loop is homologically trivial.
- The solid tori O_i intersect S, but not together.
- Hence, one can choose a loop in S encircling O_1 but not O_2. This loop gives the desired pulling.



"scientific proof"

An alternative proof uses the so-called homological Meyer-Vietoris exact sequence:

$$\ldots \xrightarrow{\partial} H_1(\tilde{F}) \xrightarrow{i_1 \oplus i_2} H_1(\tilde{F}_1) \oplus H_1(\tilde{F}_2) \xrightarrow{p} H_1(\tilde{F}_{12}) \xrightarrow{\partial} \ldots$$

(here $\tilde{F}.$'s are the complements to the obstacles not in the solid torus $\mathcal{E}$, but in the 3-sphere $S^3 \supset \mathcal{E}$).

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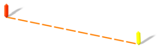
(here $\tilde{F}_\cdot$'s are the complements to the obstacles not in the solid torus $\mathcal{E}$, but in the 3-sphere $S^3 \supset \mathcal{E}$).

The abelian groups $H_1(\tilde{F}_1), H(\tilde{F}_2)$ and $H_1(\tilde{F}_1 \cap \tilde{F}_2)$ have subgroups generated by the classes λ of large “meridian loops” (in fact, $H_1(\tilde{F}_i)$ are freely generated by these classes). The image of $i_1 \oplus i_2$ contains the diagonal in $H_1(\tilde{F}_1 \cap \tilde{F}_2)$, as $i_1 \oplus i_2 : \lambda \mapsto \lambda \oplus \lambda$. Pulling **C** around $\mathcal{D}$ is possible if and only if the rank of $Im(i_1 \oplus i_2) = 2$, that is if and only if the class of the large meridian loops in $H_1(\tilde{F}_{12})$ vanishes, that is, by Alexander duality, if and only if there is no loop in O_{12} sent to the generator of $H_1(\mathcal{E})$ by the inclusion.

caging

Now we go to our main problem, two-finger caging. We keep assumption that a planar domain $\mathcal{D}$ is homeomorphic to an open disk. We will assume that a reference point configuration $g_*\mathbf{C}, g \in \mathcal{E}$ is far away from $\mathcal{D}$.



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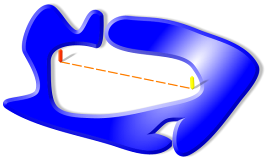

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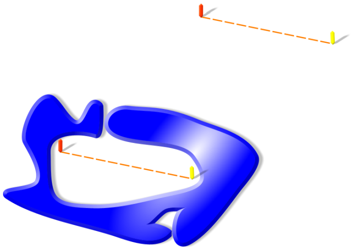

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The configuration is *not caging* if there exists a path in $\mathcal{E}$ connecting e and g and avoiding O.

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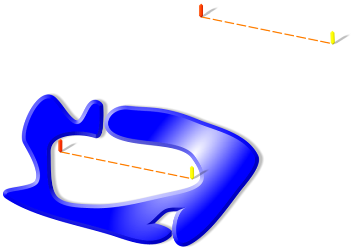

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The configuration is *not caging* if there exists a path in $\mathcal{E}$ connecting e and g and avoiding O.
In other words, there is no caging, if e and g_* are in the same connected component of $F = \mathcal{E} - O$.

Meyer-Vietoris exact sequence, once again

This means we want to investigate $H_0(F)$.

Consider again the Mayer-Vietoris sequence, but at a different place:

$$\dots \xrightarrow{i_1 \oplus i_2} H_1(F_1) \oplus H_1(F_2) \xrightarrow{p} H_1(F_{12}) \xrightarrow{\partial} H_0(F) \to H_0(F_1) \oplus H_0(F_2) \xrightarrow{\cdots}$$

Proposition 2 *The number of connected components of F not containing g_* equals the rank of the subgroup of $H_1(O_{12})$ having zero linking number with the class of large meridian.*

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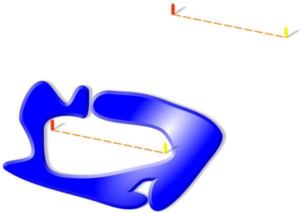
From this viewpoint, the *witnesses* of separation of the components in $\mathcal{E} - O$ are not 2-dimensional surfaces (as one would expect in a 3-dimensional manifold), but 1-dimensional loops.

alternative view

One can associate to two-point configuration **C** in a canonical way a class $\gamma_{\mathbf{C}}$ in $H_1(F_{12})$ (that is, a loop defined up to deformations avoiding O_{12}). The construction is simple:



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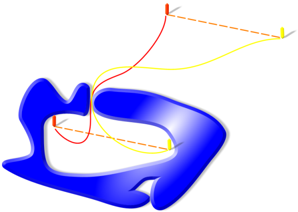

- There exist paths $\gamma_i : I \to \mathcal{E}, i = 1, 2$ connecting e and g_* such that $\gamma_i p_i$ avoids $\mathcal{D}$ at all times (equivalently, the range of γ_i is in F_i.

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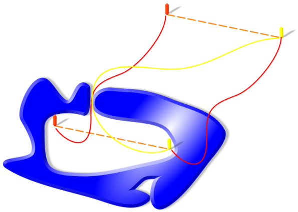

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- Joining these two paths we obtain a loop avoiding O_{12}. This loop (non-unique) defines a class in $H_1(F_{12})$.

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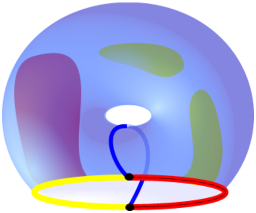
One can associate to two-point configuration $\mathbf{C}$ in a canonical way a class $\gamma_{\mathbf{C}}$ in $H_1(F_{12})$ (that is, a loop defined up to deformations avoiding O_{12}). The construction is simple:



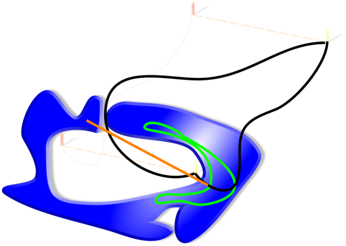
- There exist paths $\gamma_i : I \to \mathcal{E}, i = 1, 2$ connecting e and g_* such that $\gamma_i p_i$ avoids $\mathcal{D}$ at all times (equivalently, the range of γ_i is in F_i.
- Joining these two paths we obtain a loop avoiding O_{12}. This loop (non-unique) defines a class in $H_1(F_{12})$.

Proposition **??** implies that if class $\gamma_{\mathbf{C}}$ is trivial, then e and g_* are in the same connected component of F.

Coupling between $H_1(O_{12})$ and $H_1(F)$

The Proposition **??** identifies the obstacles to extracting the 2-finger configuration with loops in O_{12}, that is the loops in $\mathcal{E}$ which keep both points of the configuration within $\mathcal{D}$: a configuration $\mathbf{C}$ is caging if the loop $\gamma_{\mathbf{C}}$ has nontrivial linking with a loop in O_{12}.

To apply this criterion, one needs an efficient algorithm computing the linking number.



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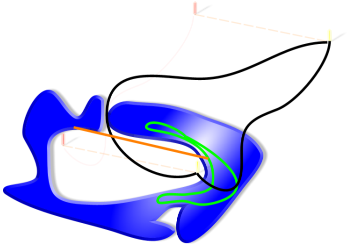

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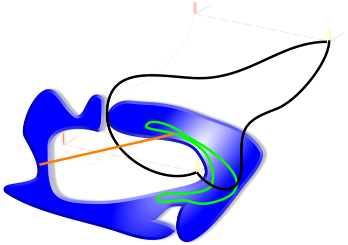

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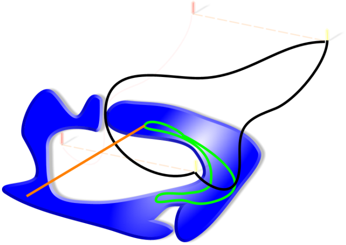

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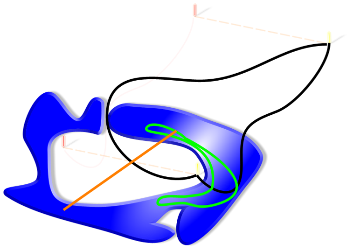

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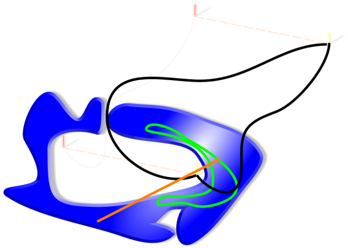

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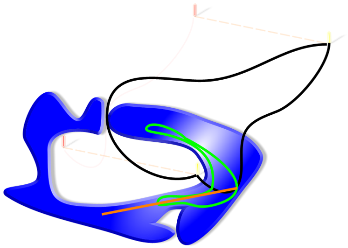

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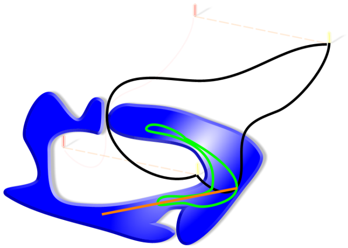

It is more convenient to consider *framed plane curves* in lieu of curves in $\mathcal{E}$.

Computing linking numbers

A framed planar curve is an oriented curve in $\mathbb{R}^2$ with a (continuously varying) unit tangent vector at each point.

A path π in $\mathcal{E}$ and a 2-point configuration $\mathbf{C}$ define a planar framed curve πp_1 framed by the vectors $(\pi p_2 - \pi p_1)/|\pi p_2 - \pi p_1|$.

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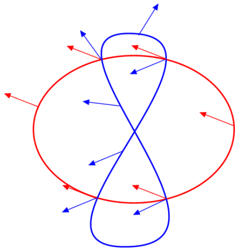

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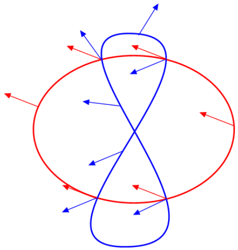

Proposition 3 *The linking number of two plane framed curves is equal to*

$$\sum w_i d_i,$$

where the summation is over the segments of C_2 into which C_1 partitions it, w_i is the winding of the segment, and σ_i is the index of C_1 with respect to this segment.

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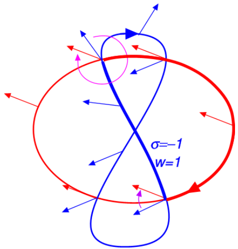

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What are the take-aways and future directions:

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- The fact that the witnesses for caging are 1-dimensional cycles leads to a novel algorithms of identifying all caging two-finger configurations *(work in progress!)*
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THANK YOU!