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SELF-ORGANIZING SOCIAL HIERARCHIES ON SCALE-FREE NETWORKS

LAZAROS K. GALLOS

Department of Physics, University of Thessaloniki
54124 Thessaloniki, Greece
gallos@physics.auth.gr

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In this work, we extend the model of Bonabeau *et al.* in the case of scale-free networks. A sharp transition is observed from an egalitarian to an hierarchical society, with a very low population density threshold. The exact threshold value also depends on the network size. We find that in an hierarchical society the number of individuals with strong winning attitude is much lower than the number of the community members that have a low winning probability.

Keywords: Sociophysics; hierarchies; phase transition; scale-free networks.

1. Introduction

Self-organization of society structures and the formation of hierarchies has always been an important issue in sociological studies.[1,2] Recently, a fresh point of view in the same problem was introduced through application of statistical physics concepts and methods. A simple, yet powerful enough, model was introduced by Bonabeau *et al.*[3] in order to explain the uneven distribution of fame, wealth, etc. The model was further modified later by Stauffer,[4,5] who introduced a feedback mechanism for determining the probability of one's social rise or fall in the hierarchy.

The above model places the interacting individuals on a lattice, so that the space, as experienced by a participant, is homogenous. Recently, though, a huge number of observations on social (among many others) systems has revealed a strongly inhomogeneous character in the number of connections between individuals.[6,7] In the present study, we extend the model of Bonabeau *et al.* for the case where the substrate of the agents motion and interaction is such a scale-free network.

2. The Model

In the original version of the model proposed by Bonabeau *et al.*,[3] a number of agents are distributed randomly on a $L \times L$ lattice, occupying a concentration p

of the total number of lattice sites. Each site can host only one individual. These individuals perform isotropic random walks on the lattice. A random agent is chosen and moves equiprobably to one of its four neighboring sites, while the system time advances by $1/pN$ (when all individuals have moved on average once, time is considered to have advanced by one Monte Carlo step). Each person i is characterized by a history parameter h_i which is a measure of an individual's "fitness" and can represent wealth, power, or any property that is judged to be important in a society. Initially, all participating agents are of equal status ($h_i = 1$) and there is no hierarchy in the population. When in the process of the random walk, though, an individual i tries to visit a site already occupied by another person j, there is a fight between the two. If the "attacking" person wins then i and j exchange their positions. Otherwise, they remain in their original sites. The outcome of the fight depends on the "strength" h of the two opponents, with a probability q that i wins over j:

$$q = \frac{1}{1 + \exp[\eta(h_j - h_i)]}\,, \tag{1}$$

where η is a free parameter, with a constant value within each realization. After a fight the fitness h of a person participating in a fight is updated: the fitness of the winner h increases by 1, while the fitness of the loser decreases by 1. Thus, the variable h_i measures the number of wins minus the number of losses, but it is also modified by an effect of fading memory. After one Monte Carlo step the fitness of all individuals decreases to 90% of its current value. In other words, in order to keep a large enough strength, it is not enough to have won a lot of fights in the past and remain inactive, but one must always retain one's strength by participating (and winning) in fights. When the density of participants is low, this memory loss is the prevailing mechanism that drives the system towards the egalitarian status, since fights in that case are rare.

The level of separation in a society is measured via an order parameter, which is taken to be the dispersion in the probability of winning a fight

$$\sigma = (\langle q^2 \rangle - \langle q \rangle^2)^{1/2}\,. \tag{2}$$

The average is considered over all fights occuring within one Monte Carlo time step. A large value of σ reveals an hierarchical society where the probability of winning differs considerably among the population. On the contrary, values of σ close to zero imply that on average all society members "fight" each other in terms of equivalent strengths.

In the original paper, a phase transition was observed upon increasing the density p, from $\sigma = 0$ to a finite σ value. Sousa and Stauffer,[4] though, pointed out that the transition was an artifact of the simulations and this transition was in fact absent. Later, Stauffer proposed a different mechanism for calculating the winning probability,[5] where feedback from the current system state was introduced in the following form:

$$q = \frac{1}{1 + \exp[\sigma(h_j - h_i)]} \,. \tag{3}$$

In this case, an hierarchically organized population (large σ value) enhances the probability of the strongest member to win, and thus introduces a preference towards already strong individuals. This mechanism restored the sharp transition of σ with increasing p, yielding a critical value close to $p_c = 0.32$.

In this work, we apply the modified model of Eq. (3) on scale-free networks. A scale-free network is a graph where the probability that a node has k links to other nodes follows a power law

$$P(k) \sim k^{-\gamma} \,, \tag{4}$$

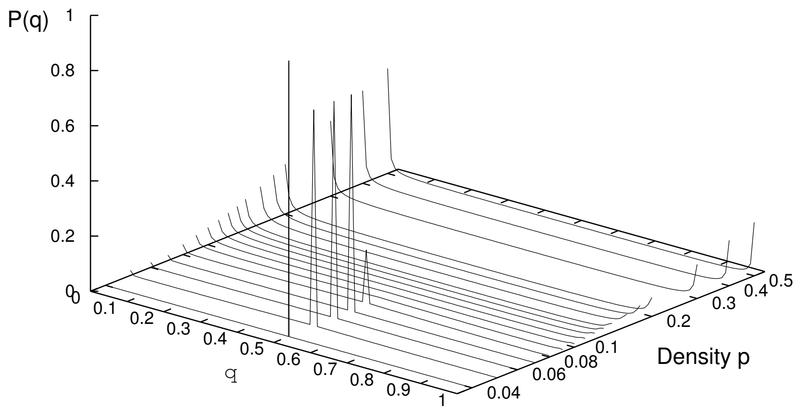
where usually $2 < \gamma < 4$. We prepare a number of different networks (typically of the order of 100) with a given γ value using the configuration model: First, the number of links for each node is determined by drawing random numbers from the distribution (Eq. (4)) and then links are established between randomly chosen pairs of nodes. Care is taken to avoid self-links or double links between two nodes. This process may create isolated clusters of nodes, so in our simulations we only keep the largest cluster in the system which (depending on γ) comprises 35–100% of the number of system nodes N.

Individuals are randomly placed on the system nodes and move along the links. A person on a node with k connections choses randomly one of the connected nodes with probability $1/k$ and tries to jump there. If the node is occupied a fight takes place under the same rules as in the case of the lattice.

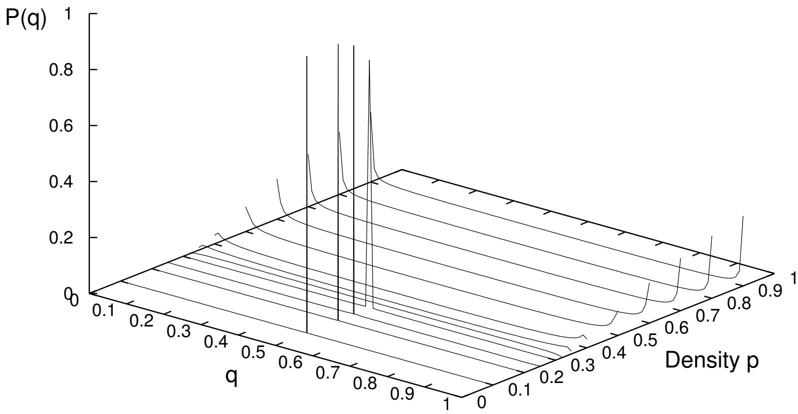
3. Results

In Fig. 1(a), we present the distribution of q for different population densities p, for networks with $\gamma = 3.0$, and $N = 10^5$ nodes. We have verified that the observed behavior is in general valid for other values of γ as well. When p is very small, there are only rare encounters between the individuals and all winning probabilities are equal to 0.5, which yields a delta function distribution up to $p = 0.04$. When p becomes greater than $p = 0.05$ the form of the distribution changes drastically. The peak is getting lower, until it completely dissapears. Now, in the region around $p = 0.1$ all winning probabilities are almost equiprobable and evenly distributed among the population. Upon further increasing p a strong polarization arises in the population with most people having a vanishing winning probability. Very few individuals have intermediate values of q, and another peak appears in the distribution close to the area of complete dominance $q = 1$. The intensity of this peak is lower than the peak at $q = 0$, indicating that the clique of “strong” individuals has fewer members than the community of low-“strength” members.

Comparison with the case of a lattice (shown in Fig. 1(b)) reveals some interesting features. The general behavior is similar (going from a delta function to



(a)



(b)

Fig. 1. Evolution of the distribution $P(q)$ with increasing density p of the population, (a) on the largest cluster of a network with $\gamma = 3.0$ and $N = 10^5$ nodes, and (b) on regular two-dimensional lattice.

uniform distribution to increasing peaks at the edges of the distribution). However, the range over where these transitions take place is very different, with networks leaving the egalitarian state in much lower densities (notice the logarithmic axis of p in Fig. 1(a)). More important is the observation that on a lattice the two peaks of the winning probability distribution at high population densities are completely symmetric. This symmetry is due to the homogeneity of the lattice, contrary to the result for the scale-free networks. On a network, an individual with high winning probability placed on a hub, will fight against many opponents who have lower q. These low-q individuals at the branches of the hub try to pass through the hub, where they will probably lose the fight. In this way, they will become weaker while they will further strengthen the already strong person. In practice, one strong individual can keep a quite large number of weaker people into a losing state, which is the mechanism underlying the observed asymmetry in the two peaks. In general, roughly 60–65% of the individuals belong to the $q \sim 0$ community and 20–25% belong to the $q \sim 1$ clique.

The observed change in the distribution shape with increasing p already hints the existence of a sharp phase transition. This transition is indeed verified by our simulations, when using our order parameter σ. The results are presented in Fig. 2.

The critical threshold for all γ values is significantly lower than in the case of lattices (where $p_c = 0.32$ is confirmed). In fact, when $\gamma = 2.25$ or $\gamma = 2.5$ there is almost no threshold and an hierarchical society emerges as soon as there is a nonzero population on the lattice, due to the frequent encounters. For $\gamma \geq 3$, the threshold has a finite value, which is still in low densities, of the order of 0.05. It is also noteworthy that for networks with low γ, the asymptotic value of the order parameter is smaller than the one for networks with large γ values. This shows that the most-connected networks initially establish an hierarchical society at lower densities than less connected networks, but retain a lower level of hierarchy at larger densities.

The network heterogeneity also introduces another effect, apart from moving the critical threshold closer to $p = 0$. For concentrations close to criticality from

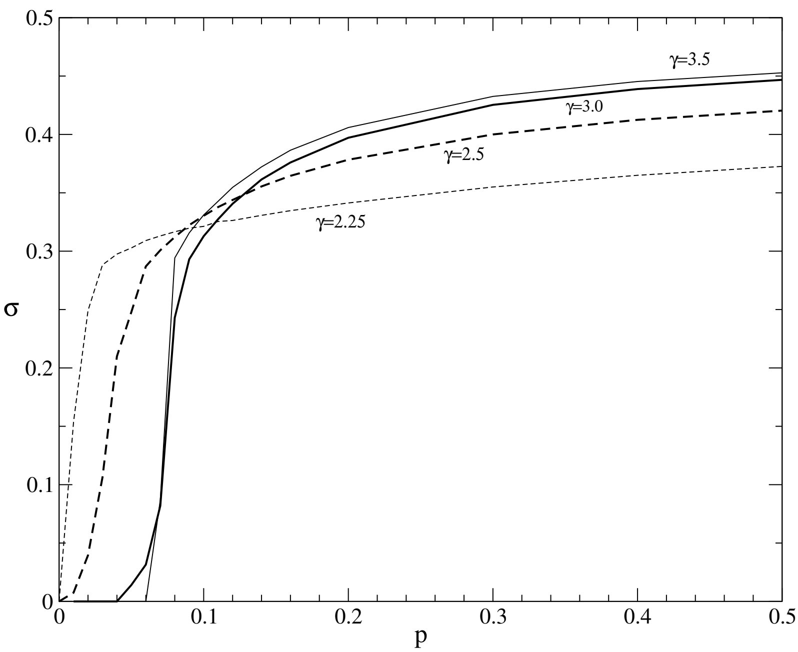


Fig. 2. Order parameter σ as a function of the population density p for scale-free networks with exponent $\gamma = 2.25$, 2.5, 3.0, and 3.5 (shown on the plot). Results were averaged over 100 different network realizations of $N = 10^5$ nodes, using typically 10^5 steps per run.

below, the behavior of σ in networks of the same γ may be very different. Thus, for e.g., $\gamma = 3.0$ and $p = 0.05$, in most realizations the value σ vanishes in a few hundrend steps. In a significant percentage (roughly 10–15%) of the realizations, though, we have observed that σ would retain a large value and fluctuate around $\sigma = 0.3$ even after 10^4 steps. Inspection of these realizations revealed that almost all of them finally converge to $\sigma = 0$, but the time for equilibration may be of the order of 10^6 steps or even more, while at the same time the fluctuations in the value of σ with time are large (σ can assume values very close to 0 or rise up to 0.45, before settling to $\sigma = 0$). These long relaxation times and the wide dispersion of σ show that a society with a density close to the criticality on a scale-free network may remain in turbulence for a long time, and even a small number of individuals may separate into different hierarchies for a significant duration, before finally settling to an egalitarian society.

Finally, we studied the effect of the network size on our presented results (Fig. 3). The curves seem to converge for networks of $N = 10^5$–10^6 nodes. For a given γ value, all network sizes used follow roughly a common curve at large population densities. The transition threshold, on the other hand, varies with N. Increasing the network size leads to a lower transition value p_c. The value of p_c for $\gamma = 2.5$ tends to 0, for large enough networks, while for $\gamma = 3.0$ it tends to a small value of around $p_c = 0.04$. Inspection of other γ values indicates that in the range $2 < \gamma < 3$ the value of p_c tends to zero with increasing network size, while when $3 < \gamma < 4$ the transition point is around $p_c = 0.05$.

In the inset of Fig. 3(b), the threshold density value p_c is presented as a function of the network size N, for $\gamma = 3.0$.[a] It seems that this threshold value converges towards its asymptotic value by following a power law $p_c \sim k^{-a}$, with an approximate exponent of $a = 0.3$, before settling to a constant value that remains unchanged with increasing N (at least for networks up to $N = 10^6$ which are the largest studied in this work).

4. Conclusions

In this work, we studied the model of Bonabeau *et al.* for the case where the population moves on the nodes of a scale-free network. A number of important differences were observed, as compared to the case of lattice diffusion. The heterogeneity of the scale-free structure and the different behavior of the diffusion process strongly affect the results of the model. For example, it is known that diffusion is not a very efficient process on networks, in the sense that a random walker can never really get away from the origin on finite-size networks.[8,9] This factor causes the individuals to remain close to each other and a large number of encounters take place, even if there are only few individuals. This results in an extremely low value of the density threshold that separates egalitarian from hierarchical societies. In fact, for $\gamma < 3$

[a] In Fig. 3(b), it is obvious that for small networks the transition is no longer sharp, so we chose to identify p_c as the approximate value where σ first assumes a nonzero value.

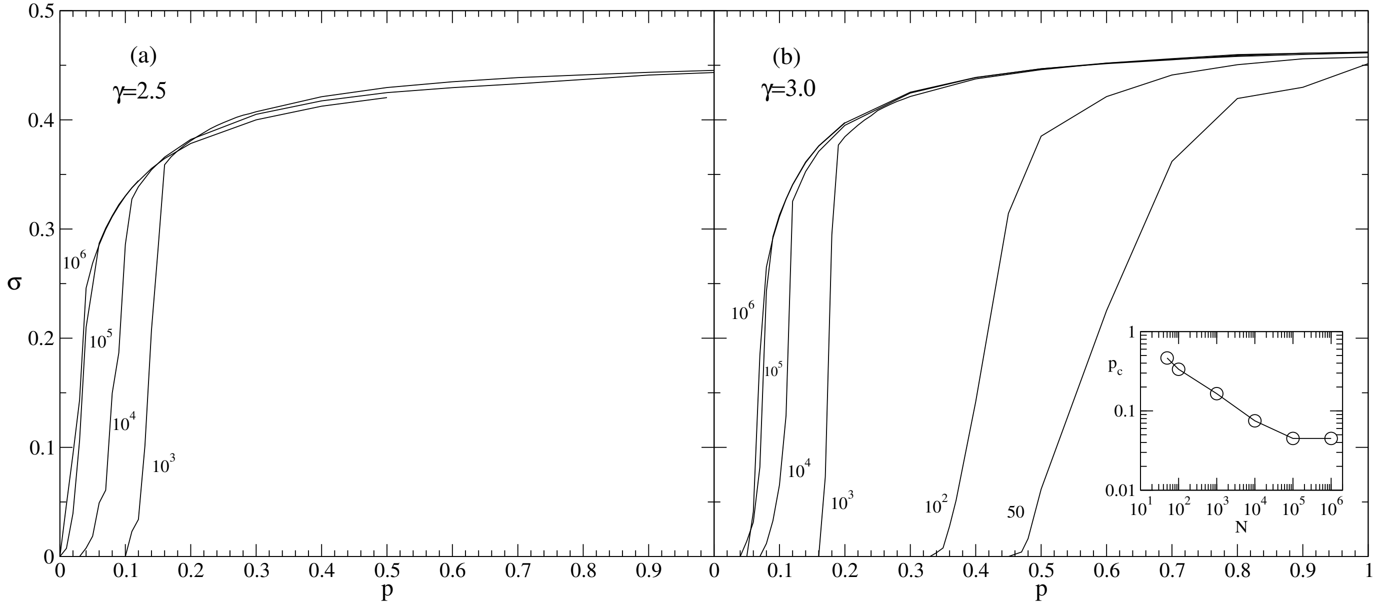


Fig. 3. Order parameter σ as a function of the population density p for scale-free networks of varying sizes from $N = 50$ to $N = 10^6$. Results for (a) $\gamma = 2.5$ and (b) $\gamma = 3.0$. Inset of (b): Threshold density p_c as a function of the network size for $\gamma = 3.0$.

there is a strong indication from the simulations that $p_c = 0$, at least for large network sizes N.

The number of individuals with strong probability of winning is also significantly lower than the number of people that cannot easily win a fight and thus climb in the hierarchy. This asymmetry is not observed in lattices, where the isotropic environment of motion equally favors the development of the two separated communities, but with equal number of members.

After completion of this work, we were informed about two papers on similar problems.[10,11] In Ref. 10, an interesting extension was proposed, where there is no spatial component in the problem, and each agent may interact with any other agent. The system was studied under different memory factors f, where the rate of memory fading is varied, leading to a phase transition at a critical value f_c. In Ref. 11, an asymmetry was introduced in the lattice version of the model, where a fight loss results in a penalty for the history parameter h greater than 1, while the gain remains 1 (such an asymmetry emerges naturally in a scale-free network without this rule, as we have already discussed). In Fig. 3 of the same paper, there are also results for σ as a function of p for the case of a Barabasi–Albert network (which is known to yield an exponent $\gamma \simeq 3$). A direct comparison between our model and this BA network is not possible, mainly because asymmetry was introduced in that case and because the size of the BA network was not mentioned in the paper. The location of the crossover point at $p_c = 0.19$ seems close to our results for a scale-free network of $\gamma = 3.0$ and $N \sim 10^3$. The transition in our case, though, is much sharper.

Acknowledgments

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References

1. H. G. Landau, *Bull. Math. Biophys.* **13**, 1 (1951).
2. I. Chase, *Behav. Sci.* **19**, 374 (1974).
3. E. Bonabeau, G. Theraulaz and J.-L. Deneubourg, *Physica A* **217**, 373 (1995).
4. A. O. Sousa and D. Stauffer, *Int. J. Mod. Phys. C* **11**, 1063 (2000).
5. D. Stauffer, *Int. J. Mod. Phys. C* **14**, 237 (2000).
6. R. Albert and A.-L. Barabasi, *Rev. Mod. Phys.* **74**, 47 (2002).
7. S. N. Dorogovtsev and J. F. F. Mendes, *Adv. Phys.* **51**, 1079 (2002).
8. L. K. Gallos, *Phys. Rev. E* **70**, 046116 (2004).
9. E. M. Bollt and D. ben-Avraham, *New J. Phys.* **7**, 26 (2005).
10. K. Malarz, D. Stauffer and K. Kulakowski, e-print physics/0502118 (2005).
11. D. Stauffer and J. S. Sá Martins, *Adv. Complex Systems* **6**, 559 (2003).