Beck's three permutations conjecture: A counterexample and some consequences

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Abstract

Given three permutations on the integers 1 through n, consider the set system consisting of each interval in each of the three permutations. In 1982, Beck conjectured that the discrepancy of this set system is $O(1)$. In other words, the conjecture says that each integer from 1 through n can be colored either red or blue so that the number of red and blue integers in each interval of each permutations differs only by a constant. (The discrepancy of a set system based on two permutations is at most two.)

Our main result is a counterexample to this conjecture: for any positive integer $n = 3^k$, we construct three permutations whose corresponding set system has discrepancy $\Omega(\log n)$. Our counterexample is based on a simple recursive construction, and our proof of the discrepancy lower bound is by induction. This construction also disproves a generalization of Beck's conjecture due to Spencer, Srinivasan and Tetali, who conjectured that a set system corresponding to ℓ permutations has discrepancy $O(\sqrt{\ell})$.

Our work was inspired by an intriguing paper from SODA 2011 by Eisenbrand, Pálvölgyi and Rothvoß, who show a surprising connection between the discrepancy of three permutations and the bin packing problem: They show that Beck's conjecture implies a constant worst-case bound on the additive integrality gap for the Gilmore-Gomory LP relaxation for bin packing in the special case when all items have sizes strictly between 1/4 and 1/2, also known as the three partition problem. Our counterexample shows that this approach to bounding the additive integrality gap for bin packing will not work. We can, however, prove an interesting implication of our construction in the reverse direction: there are instances of bin packing and corresponding optimal basic feasible solutions for the Gilmore-Gomory LP relaxation such that any packing that contains only patterns from the support of these solutions requires at least $\mathrm{OPT} + \Omega(\log m)$ bins, where m is the number of items.

Finally, we discuss some implications that our construction has for other areas of discrepancy theory.

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1 Introduction

Consider a set system $\mathcal{S}$ of M sets on n elements and consider an assignment to the elements, $\chi : [n] \rightarrow \{-1, +1\}$. The value of a set $S_j \in \mathcal{S}$ with respect to a fixed assignment χ is $|\sum_{i \in S_j} \chi(i)|$. The *discrepancy* of a set system $\mathcal{S}$ is the maximum value over all sets, minimized over all assignments. This quantity is sometimes referred to as the *red-blue discrepancy* of a set system. A famous result of Spencer shows that when $M = O(n)$, the discrepancy of a set system is at most $O(\sqrt{n})$ [Spe85]. Recently, Bansal gave an efficient algorithm for finding such a low discrepancy assignment or coloring [Ban10], and even more recently Lovett and Meka gave an another efficient algorithm yielding an independent proof of Spencer's result [LM12].

Spencer's result holds for general set systems. A well-studied research topic in combinatorial discrepancy theory is to determine the discrepancy of set systems with certain restrictions, such as bounded VC-dimension or bounded degree. In this paper, we consider set systems based on permutations. Given a permutation on the integers 1 through n, consider the set system consisting of each interval of this permutation. Without loss of generality, we can assume that this is the identity permutation. Thus, this set system contains the set $\{1, 2, 3\}$ and $\{3, 4, 5, 6\}$, etc., but it does not, for example, contain the set $\{2, 4\}$. It is easy to see that the discrepancy of this set system is one. When the set system consists of all intervals from two permutations, each on the integers from 1 through n, then the discrepancy is at most two [Spe87].

What is the discrepancy of a set system based on three permutations? Beck conjectured that the discrepancy of this set system is $O(1)$. In other words, he conjectured there is always an assignment $\chi : [n] \rightarrow \{-1, +1\}$ such that the value of any set in this set system is $O(1)$. Another way to view the problem is that each integer from 1 through n can be colored either red or blue so that the number of red and blue integers in each interval of each permutations differs only by a constant.

Our main result is a counterexample to this conjecture. In particular, for each integer $k > 0$, we give an instance of three permutations on the ground set 1 through 3^k such that the discrepancy is at least $\lceil k/3 + 1 \rceil$. Setting $n = 3^k$, this yields a set of three permutations with discrepancy at least $\lceil (\log_3 n)/3 + 1 \rceil = \Omega(\log n)$.

1.1 Background on Beck's conjecture

Beck first stated this conjecture in 1982 [Bec11]. The earliest written reference to this conjecture that we have found is on page 42 of the 1987 edition of Spencer's "Ten Lectures on the Probabilistic Method" [Spe87]. Spencer describes a clever proof that the discrepancy of two permutations is at most two, states the conjecture for three permutations, and offers \$100 for its resolution. In the 1994 edition, Spencer attributes this conjecture to Beck. In a more recent book, Matoušek says (on page 126) that resolving Beck's conjecture "remains one of the most tantalizing questions in combinatorial discrepancy" [Mat10].

Although it was not resolved until now, the conjecture did receive some attention. Fishburn and Gehrlein give an example of three permutations based on a geometric configuration of rectangles, for which if the coloring is restricted so that no set corresponding to one of the permutations has absolute value more than one, then the set system has unbounded discrepancy [FG90]. Without this restriction on the colorings, however, their construction has discrepancy two.

Citing Beck's conjecture as motivation, Bohus shows that a set system based on ℓ permutations always has discrepancy $O(\ell \log n)$ [Boh90]. This was later improved by Spencer, Srinivasan and Tetali who show that such a set system actually has a coloring with discrepancy $O(\sqrt{\ell} \log n)$ [SST01]. While Bohus gives an efficient algorithm to find a coloring matching his upper bound, Spencer et al. leave open the question of whether a coloring matching their bound can be found efficiently. Since these latter results are via the entropy method, it is possible that a constructive algorithm can be obtained via the recent methods of Bansal, who gives constructive algorithms for finding low discrepancy colorings for general set systems [Ban10]. Our results show that the bounds of Bohus and of Spencer et al. are tight up to the factor containing the number of permutations, ℓ, i.e. these upper bounds are tight for set systems based on a fixed number of permutations. Spencer et al. also generalize Beck's conjecture positing that any set system based on ℓ permutations has discrepancy $O(\sqrt{\ell})$ [SST01]. We note that our construction disproves this stronger conjecture as well.

We note that one possible reason that the conjecture may have been believed to be true is because, prior to our work, it appears that there were no constructions of three permutations known to have discrepancy greater than two.

1.2 Consequences of Beck's conjecture

Recently, Eisenbrand, Pálvölgyi and Rothvoß made a surprising connection between Beck's conjecture and the additive integrality gap of a well-studied LP relaxation for bin packing [EPR11]. In the bin packing problem we are given an instance $\mathcal{I}$ of m items where each item $i \in \mathcal{I}$ has a size $s(i) \in (0, 1]$. The objective is to pack the items into the minimum number of capacity one bins. For a bin packing instance $\mathcal{I}$ denote by $\mathrm{OPT}(\mathcal{I})$ the optimal solution, i.e. the minimum number of bins necessary to pack all of the items. A pattern $p \subseteq \mathcal{I}$ is legal if the items it contains fit into one bin, that is, if $\sum_{i \in p} s(i) \leq 1$. Let $\mathcal{P}$ be the set of all legal patterns. The following is known as the Gilmore-Gomory LP relaxation for bin packing [Eis57, GG61]:

$$\begin{aligned} \min \sum_{p \in \mathcal{P}} x_p & \\ \sum_{p \in \mathcal{P} : i \in p} x_p \geq 1, & \quad \forall i \in \mathcal{I} \\ x_p \geq 0. & \end{aligned} \tag{LP}$$

Rounding this LP relaxation is a basic component of the famous Karmarkar-Karp algorithm for bin packing, which results in a packing with at most $\mathrm{OPT}_{LP}(\mathcal{I}) + O(\log^2 m)$ bins [KK82], where $\mathrm{OPT}_{LP}(\mathcal{I})$ is the optimal value of the LP for a given instance $\mathcal{I}$. For the special case of bin packing, called three partition, in which each item has a size $s(i) \in (1/4, 1/2)$, the Karmarkar-Karp algorithm results in a packing with $\mathrm{OPT}_{LP}(\mathcal{I}) + O(\log m)$ bins.

Let $\mathcal{I}$ be an instance of three partition. Eisenbrand et al. show that if Beck's conjecture were true, then $\mathrm{OPT}(\mathcal{I}) \leq \mathrm{OPT}_{LP}(\mathcal{I}) + O(1)$ [EPR11]. In other words, they would be able to bound the additive integrality gap by a constant in this special case of the bin packing problem! They leave open the question of whether a reduction in the other direction can be established: Does an upper bound of $\mathrm{OPT}_{LP}(\mathcal{I}) + O(1)$ on the size of an optimal integral solution for three partition imply an $O(1)$ upper bound on the discrepancy of three permutations? In light of our results, such a reduction would disprove the long-standing conjecture that $\mathrm{OPT}(\mathcal{I})$ is upper bounded by $\mathrm{OPT}_{LP}(\mathcal{I}) + O(1)$.

1.3 Consequences of a disproof of Beck's conjecture

Despite the fact that the approach of Eisenbrand et al. for proving a constant additive integrality gap on the bin packing LP will not work, we show that our construction disproving Beck's conjecture nevertheless has some interesting implications for the bin packing problem: there are instances of bin packing (instances of three partition, in fact) and corresponding optimal *basic* feasible solutions for the Gilmore-Gomory LP relaxation, such that any packing that contains only patterns from the support of its corresponding solution requires at least $\mathrm{OPT}(\mathcal{I}) + \Omega(\log m)$ bins. This implication is a lower bound on all algorithms that use patterns from the support of a basic feasible solution to the LP.

The additive factor is tight for the three partition problem due to the Karmarkar-Karp algorithm, which can be slightly modified to use only patterns from the LP support. Specifically, the algorithm proceeds in iterations. At each iteration, it "discards" some items, uses the support to pack some items, and recurses on the leftover items. In the case of three partition, the total number of discarded items is $O(\log m)$. Thus, we can easily pack these items using an additional $O(\log m)$ patterns, yielding a packing that uses a total of $\mathrm{OPT}_{LP}(\mathcal{I}) + O(\log m)$ bins, each of which corresponds to a pattern from the support.

We note that, independently, in an extended journal version of their paper, Eisenbrand et al. use our construction to make a similar observation: Roughly speaking, they show that there are instances of bin packing and corresponding optimal solutions (not necessarily basic) to the LP, for which any set of at most $\mathrm{OPT}(\mathcal{I})$ patterns from the support of the LP solution leaves $\Omega(\log^2 m)$ items uncovered [EPR, EPR10]. This matches the worst-case guarantee of the Karmarkar-Karp algorithm for the general case of bin packing [KK82].

1.4 Consequences for discrepancy theory

Another consequence of our lower bound is that it provides an alternate solution to a question posed by Sós, which asks how large is the discrepancy of a union of set systems when each set system has constant hereditary discrepancy [LSV86, Spe87]. In Section 7, we discuss this and questions related to other notions of discrepancy.

1.5 Organization

In Section 2, we give some basic definitions and notation. In Section 3, we define the construction of three permutations that we will use to prove our main result. In Section 4 and Section 5, we state and prove our main theorem: the set system associated with prefixes of the three permutations from Section 3 has discrepancy $\Omega(\log n)$. In Section 6, we present the implications that our construction has for the bin packing problem. Finally, in Section 7, we discuss applications of our construction to other problems in discrepancy.

2 Basic definitions and notation

Recall that for a set system $\mathcal{S} = \{S_1, S_2, S_3, \dots S_M\}$, the *discrepancy* of the set system is:

$$\mathrm{disc}(\mathcal{S}) \quad = \quad \min_{\chi} \max_{j \in [M]} |\sum_{i \in S_j} \chi(i)|. \tag{1}$$

Let $[n]$ denote the set of integers from 1 through n, and let $[x, y]$ (where $x < y$) denote all integers from x through y. For a coloring $\chi : [n] \to \{-1, +1\}$, if $S \subseteq [n]$, let $\chi(S) = \sum_{j \in S} \chi(j)$. We will use n to denote the length of the permutations, i.e. $n = 3^k$ for some specified integer $k > 0$.

For some fixed k, the corresponding three permutations described in Section 3 will be denoted by π_1^k, π_2^k and π_3^k. Let $\alpha_i^k(x)$ denote the elements in positions 1 through x in the permutation π_i^k, where $x \in [0, n]$. In other words, $\alpha_i^k(x)$ is a prefix of π_i^k of length x. Note that $\alpha_i^k(0)$ represents the empty set. Given the three permutations π_1^k, π_2^k and π_3^k, the set system $\mathcal{S}_k$ consists of all sets $\alpha_i^k(x)$ for $x \in [3^k]$. In other words, $\mathcal{S}_k$ is the set system of all prefixes of the three permutations, π_1^k, π_2^k and π_3^k. Note that if we prove a lower bound on the set system $\mathcal{S}_k$, the same lower bound holds on the set system containing all intervals of each of the permutations.

We will also use the notion of sets corresponding to suffixes of the permutations, even though these sets do not appear in our set systems. Let $\omega_i^k(x)$ denote the elements in positions x through 3^k in the permutation π_i^k, where $x \in [3^k + 1]$. In other words, $\omega_i^k(x)$ is a suffix of π_i^k of length $3^k - x + 1$. We define $\omega_i^k(3^k + 1)$ to be the empty suffix.

3 Recursive construction

We give a construction for three permutations on the integers 1 through n, where $n = 3^k$ for some integer $k > 0$. Consider the following recursive construction of three lists:

$$\begin{array}{ccc} A & B & C \\ C & A & B \\ B & C & A, \end{array}$$

where A represents the interval $[1, n/3]$, B the interval $[n/3+1, 2n/3]$, and C the interval$[2n/3+1, n]$. Each of the three copies of A (and B and C, respectively) is divided further into three equal sized blocks of consecutive elements, and these three blocks are permuted as in the above construction. This process of dividing the blocks into three equal sized blocks and permuting them according to the above construction is iterated k times. To illustrate these actions, when $n = 9$, this construction results in the following three permutations:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \\ 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1. \end{array}$$

When $n = 27$, the three permutations are:

$$\begin{array}{ccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\
27 & 25 & 26 & 21 & 19 & 20 & 24 & 22 & 23 & 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 & 18 & 16 & 17 & 12 & 10 & 11 & 15 & 13 & 14 \\
14 & 15 & 13 & 17 & 18 & 16 & 11 & 12 & 10 & 23 & 24 & 22 & 26 & 27 & 25 & 20 & 21 & 19 & 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1.
\end{array}$$

One useful observation about this construction pertains to the symmetry of these three permutations. If we consider the set of permutations π_1^k, π_2^k and π_3^k, then the three permutations induced by $\{\pi_i^k\}$ on the set of integers $[1, 3^{k-1}]$ are isomorphic to the permutations $\{\pi_i^{k-1}\}$. This also holds for the permutations induced by $\{\pi_i^k\}$ on $[3^{k-1}+1, 2\cdot 3^{k-1}]$ and to the permutations induced by $\{\pi_i^k\}$ on $[2\cdot 3^{k-1}+1, 3^k]$.

Fact 1. *Given permutations $\{\pi_i^k\}$, the three permutations induced on $[1, 3^{k-1}]$ (and on $[3^{k-1}+1, 2\cdot 3^{k-1}]$, $[2\cdot 3^{k-1}+1, 3^k]$, respectively) are isomorphic to the permutations $\{\pi_i^{k-1}\}$.*

4 Main Theorem

Let $\mathcal{S}_k$ refer to the set system consisting of all prefixes of the three permutations, π_1^k, π_2^k and π_3^k, on $n = 3^k$ elements described in Section 3. Note that the set of all prefixes of the permutations is a subset of all intervals of the permutations. Since we are proving a lower bound, it suffices to consider the set system consisting only of prefixes. Our main theorem is:

Theorem 1. $\mathrm{disc}(\mathcal{S}_k) \geq \lceil \frac{k}{3} + 1 \rceil = \lceil \frac{\log_3 n}{3} + 1 \rceil$.

5 Proof of Main Theorem

In our construction, as k increases by 1, it is not necessarily the case that the discrepancy increases by 1. If this were true, then we could prove a lower bound of $\log_3 n$ rather than $\log_3 n/3$. However, one of our key ideas—roughly speaking—is that the *sum* of the discrepancies of the set systems, each corresponding to one of the permutations, increases by 1 as k increases by 1. We will use the following definitions, which denote the maximum/minimum sum of the prefixes of the set systems corresponding to each permutation for a fixed coloring χ:

$$\begin{aligned}
\mathrm{disc}^k_{\mathrm{L}^+}(\chi) &:= \max_{x,y,z\in[0,3^k]} \Big(\chi(\alpha_1^k(x)) + \chi(\alpha_2^k(y)) + \chi(\alpha_3^k(z)) \Big), \\
\mathrm{disc}^k_{\mathrm{L}^-}(\chi) &:= \min_{x,y,z\in[0,3^k]} \Big(\chi(\alpha_1^k(x)) + \chi(\alpha_2^k(y)) + \chi(\alpha_3^k(z)) \Big). \qquad (2)
\end{aligned}$$

Although our set systems do not contain suffixes, we will also use the following definitions:

$$\begin{aligned}
\mathrm{disc}^k_{\mathrm{R}^+}(\chi) &:= \max_{x,y,z\in[1,3^k+1]} \Big(\chi(\omega_1^k(x)) + \chi(\omega_2^k(y)) + \chi(\omega_3^k(z)) \Big), \\
\mathrm{disc}^k_{\mathrm{R}^-}(\chi) &:= \min_{x,y,z\in[1,3^k+1]} \Big(\chi(\omega_1^k(x)) + \chi(\omega_2^k(y)) + \chi(\omega_3^k(z)) \Big). \qquad (3)
\end{aligned}$$

For a coloring $\chi : [3^k] \to \{-1, +1\}$, let $\Sigma = \chi([3^k])$. If $\Sigma \geq 1$, then our goal is to show the following:

$$\mathrm{disc}^k_{\mathrm{L}^+}(\chi) \quad \geq \quad k + 3. \tag{4}$$

Alternatively, if $\Sigma \leq -1$, then we want to show:

$$\mathrm{disc}^k_{\mathrm{L}^-}(\chi) \quad \leq \quad -k - 3. \tag{5}$$

If we can show the appropriate inequality for every coloring χ, then this would imply our main theorem, as one of the three set systems must then have discrepancy at least $\lceil |(k+3)/3| \rceil$. However, we do not see how to directly use (4) and (5) as an inductive hypothesis. Thus, we need a stronger inductive hypothesis, which is stated in the following lemma and corollary.

Lemma 1. *Let $\Sigma = \chi([3^k])$. If $\Sigma \geq 1$, then:*

$$\mathrm{disc}^k_{\mathrm{L}^+}(\chi),\ \mathrm{disc}^k_{\mathrm{R}^+}(\chi) \ \geq \ k + \Sigma + 2.$$

If $\Sigma \leq -1$, then:

$$\mathrm{disc}^k_{\mathrm{L}^-}(\chi),\ \mathrm{disc}^k_{\mathrm{R}^-}(\chi) \ \leq \ -k + \Sigma - 2.$$

Note that Lemma 1 implies our stated goal in (4) and (5) and, therefore, our Main Theorem. Indeed, since 3^k is odd, it must be the case for any coloring $\chi : [3^k] \to \{-1, +1\}$ that $|\Sigma| \geq 1$ and the theorem follows. Before we prove Lemma 1, we show that Lemma 1 implies the following corollary, which will be useful in our inductive proof.

Corollary 2. *Let $\Sigma = \chi([3^k])$. If $\Sigma \leq -1$, then:*

$$\mathrm{disc}^k_{\mathrm{L}^+}(\chi),\ \mathrm{disc}^k_{\mathrm{R}^+}(\chi) \geq k + 2\Sigma + 2.$$

If $\Sigma \geq 1$,

$$\mathrm{disc}^k_{\mathrm{L}^-}(\chi),\ \mathrm{disc}^k_{\mathrm{R}^-}(\chi) \leq -k + 2\Sigma - 2.$$

Proof: Let us first consider the case in which $\Sigma \leq -1$. Note that for each π^k_i, it is the case that for each $x \in [0, 3^k]$, $\chi(\alpha^k_i(x)) + \chi(\omega^k_i(x+1)) = \Sigma$. Therefore, for some coloring χ, consider an $x \in [0, 3^k]$ that maximizes $\chi(\alpha^k_i(x))$. Then $y = x + 1$ is a value of $y \in [1, 3^k + 1]$ that minimizes $\chi(\omega^k_i(y))$. Thus, we have:

$$\begin{aligned} \mathrm{disc}^k_{\mathrm{R}^-}(\chi) + \mathrm{disc}^k_{\mathrm{L}^+}(\chi) &= 3\Sigma \quad \Rightarrow \\ \mathrm{disc}^k_{\mathrm{L}^+}(\chi) &= 3\Sigma - \mathrm{disc}^k_{\mathrm{R}^-}(\chi). \end{aligned}$$

By Lemma 1, we have:

$$\begin{aligned} \mathrm{disc}^k_{\mathrm{L}^+}(\chi) &\geq 3\Sigma + k - \Sigma + 2 \\ &= k + 2\Sigma + 2. \end{aligned}$$

An analogous argument works to give the same lower bound on $\mathrm{disc}^k_{\mathrm{R}^+}$ when $\Sigma \leq -1$. Now consider the case in which $\Sigma \geq 1$. We have:

$$\begin{aligned} \mathrm{disc}^k_{\mathrm{R}^+}(\chi) + \mathrm{disc}^k_{\mathrm{L}^-}(\chi) &= 3\Sigma \quad \Rightarrow \\ \mathrm{disc}^k_{\mathrm{L}^-}(\chi) &= 3\Sigma - \mathrm{disc}^k_{\mathrm{R}^+}(\chi). \end{aligned}$$

By Lemma 1, we have:

$$\begin{aligned}\mathrm{disc}^k_{\mathrm{L}^-}(\chi) &\leq 3\Sigma - k - \Sigma - 2 \\ &= -k + 2\Sigma - 2.\end{aligned}$$

The argument for the upper bound on $\mathrm{disc}^k_{\mathrm{R}^-}$ when $\Sigma \geq 1$ is symmetric. □

5.1 Proof of Lemma 1

Now we will prove Lemma 1 using induction. Note that in our inductive hypothesis, we will assume Lemma 1 for $k-1$. This will allow us to also assume the bounds stated in Corollary 2, since we have shown that, for a given value of k, Lemma 1 implies Corollary 2.

Base Case: $k = 1$

Suppose that $\Sigma = \chi([3]) \geq 1$. Let $\pi^1_1 = (a, b, c), \pi^1_2 = (c, a, b)$ and $\pi^1_3 = (b, c, a)$. Without loss of generality, there are only two possibilities for such colorings:

$$\begin{pmatrix}\chi(1) & \chi(2) & \chi(3) \\ \chi(3) & \chi(1) & \chi(2) \\ \chi(2) & \chi(3) & \chi(1)\end{pmatrix} = \begin{pmatrix}1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1\end{pmatrix}.$$

Suppose $\Sigma = \chi([3]) = 1$. The only way to achieve such a coloring is to have two of the elements be colored '+1' and one element be colored '−1'. Without loss of generality, in the above case, we have assigned -1 to element 2. Then one of the permutations has a prefix (suffix) with value two, while each of the other two permutations have prefixes (suffixes) with value one. Specifically, in this case, π^1_2 has a prefix of value two and both other permutations have prefixes with value one. Thus, we have: $\mathrm{disc}^1_{\mathrm{L}^+}(\chi), \mathrm{disc}^1_{\mathrm{R}^+}(\chi) = 4 \geq k+\Sigma+2 = 4$. Now suppose that $\Sigma = \chi([3]) = 3$. In this case, each permutation has a prefix (suffix) with value three. Thus, $\mathrm{disc}^1_{\mathrm{L}^+}(\chi), \mathrm{disc}^1_{\mathrm{R}^+}(\chi) = 9 \geq k+\Sigma+2 = 6$. Thus, Lemma 1 holds for $\Sigma \geq 1$ when $k = 1$.

When $\Sigma = \chi([3]) = -1$, the same arguments can be used to show that $\mathrm{disc}^1_{\mathrm{L}^-}(\chi), \mathrm{disc}^1_{\mathrm{R}^-}(\chi) = -4 \leq -k - \Sigma - 2 = -4$. Similarly, when $\Sigma = \chi([3]) = -3$, $\mathrm{disc}^1_{\mathrm{L}^-}(\chi), \mathrm{disc}^1_{\mathrm{R}^-}(\chi) = -9 \leq -6$. This concludes the proof of the base case.

Inductive step

Now we assume that Lemma 1 and thus its Corollary 2 are true for $k-1$ and prove the Lemma (and thus, the Corollary) true for k.

For some fixed $\chi : [3^k] \to \{-1, +1\}$, let $\Sigma = \chi([3^k])$. Let a, b and c denote the values of the three blocks of 3^{k-1} consecutive integers in the recursive construction, i.e. $\chi([1, 3^{k-1}]), \chi([3^{k-1} + 1, 2 \cdot 3^{k-1}])$ and $\chi([2 \cdot 3^{k-1} + 1, 3^k])$, although not necessarily in this order. We always assume that $a \geq b \geq c$, i.e. the value of the block with the largest value is denoted by a, etc. Note that a, b and c are each odd numbers, because they always represent the values of intervals with odd length. Each permutation in $\{\pi^k_i\}$ corresponds to some permutation of a, b and c and the elements within.

Without changing the discrepancy, we can re-order or re-label the three permutations to form one of the following two configurations, in which each row corresponds to one of the three permutations in $\{\pi_i^k\}$.

$$(\mathrm{I})\ \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = \begin{pmatrix} b & c & a \\ a & b & c \\ c & a & b \end{pmatrix}, \qquad (\mathrm{II})\ \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}.$$

First we consider the case in which $\Sigma = \chi([3^k]) \geq 1$. This implies that $a+b+c \geq 1$. There are two subcases:

(i) $a \geq b \geq 1$ (and $c \geq 1$ or $c \leq -1$),

(ii) $a \geq 1$ and $c \leq b \leq -1$.

First, we consider case (i) and configuration (I). If we look at a permutation of the rows so that the blocks with value b are on the diagonal (as shown), then in configuration (I), the value of the blocks below the diagonal are positive (which is desirable). Thus, we can consider the three prefixes corresponding to the permutations of the block with value b. Suppose, without loss of generality (and for ease of notation) that the block with value b is $[1, 3^{k-1}]$. In this case, the permutations on the diagonal are π_1^{k-1}, π_2^{k-1} and π_3^{k-1}. By the inductive assumption, for any $\chi : [3^{k-1}] \rightarrow \{-1, +1\}$, there are three corresponding prefixes $\alpha_1^{k-1}(x_1), \alpha_2^{k-1}(x_2)$ and $\alpha_3^{k-1}(x_3)$, for some integers $x_1, x_2, x_3 \in [0, 3^{k-1}]$, such that:

$$\begin{aligned} \chi(\alpha_1^{k-1}(x_1)) + \chi(\alpha_2^{k-1}(x_2)) + \chi(\alpha_3^{k-1}(x_3)) &= \operatorname{disc}_{\mathrm{L}^+}^{k-1}(\chi) \\ &\geq (k-1) + b + 2. \end{aligned} \tag{6}$$

Note that if either the block $[3^{k-1}+1, 2\cdot 3^{k-1}]$ or the block $[2\cdot 3^{k-1}+1, 3^k]$ had value b, and therefore appeared on the diagonal of configuration (I), then by Fact 1, we see that these permutations are isomorphic to $\{\pi_i^{k-1}\}$. This allows us to use the inductive hypothesis in these cases as well, and to draw the same conclusion as we drew in (6).

Now we consider some $\chi : [3^k] \rightarrow \{-1, +1\}$. This coloring induces a coloring on $[3^{k-1}]$ for which the above assumption in (6) holds. Suppose that π_h^{k-1}, π_j^{k-1} and π_ℓ^{k-1}, for $h, j, \ell \in \{1, 2, 3\}$, correspond to the permutations of block $[3^{k-1}]$ that appear in the first, second and third rows of the configuration, respectively. For the fixed coloring χ on $[3^k]$, our goal is to show that there are three prefixes of the three permutations $\{\pi_i^k\}$ such that we can lower bound the value of the sum of these prefixes with respect to the fixed coloring χ. The prefix of the permutation corresponding to the first row of the configuration is $\alpha_h^{k-1}(x_h)$. For the permutation corresponding to the second row of the configuration, we add the block with value a to the front of $\alpha_j^{k-1}(x_j)$. For the permutation corresponding to the third row of the configuration, we add the block with value a to the front of

$\alpha_\ell^{k-1}(x_\ell)$ preceded by the block with value c. Thus, by the inductive hypothesis, we have that:

$$\begin{aligned}
\mathrm{disc}^k_{\mathrm{L}^+}(\chi) &\geq \chi(\alpha_h^{k-1}(x_h)) + \Big(a + \chi(\alpha_j^{k-1}(x_j))\Big) + \Big(c + a + \chi(\alpha_\ell^{k-1}(x_\ell))\Big) \\
&= \mathrm{disc}^{k-1}_{\mathrm{L}^+}(\chi) + 2a + c \\
&\geq (k-1) + b + 2 + 2a + c \\
&\geq k + \Sigma + 1 + a \\
&\geq k + \Sigma + 2.
\end{aligned}$$

The last inequality follows from the fact that in case (i), $a \geq 1$. Thus, the inductive step holds for case (i), configuration (I).

Now let us consider configuration (II). In this case, we consider a permutation of the rows so that the blocks with value a occupy the diagonal. By the same reasoning as discussed previously and by induction, we have:

$$\begin{aligned}
\mathrm{disc}^k_{\mathrm{L}^+}(\chi) &\geq \mathrm{disc}^{k-1}_{\mathrm{L}^+}(\chi) + 2b + c \\
&\geq (k-1) + a + 2 + 2b + c \\
&\geq k + \Sigma + b + 1 \\
&\geq k + \Sigma + 2.
\end{aligned}$$

Since in case (i), $b \geq 1$, the inductive step holds for case (i), configuration (II).

Now we consider case (ii), when $a \geq 1$ and $c \leq b \leq -1$. In this case, we again have the above two configurations:

$$\text{(I)}\ \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = \begin{pmatrix} b & c & a \\ a & b & c \\ c & a & b \end{pmatrix}, \qquad \text{(II)}\ \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix} = \begin{pmatrix} c & b & a \\ a & c & b \\ b & a & c \end{pmatrix}.$$

Note that in case (ii), for both configurations (I) and (II), we use Corollary 2. We consider configuration (I) first.

$$\begin{aligned}
\mathrm{disc}^k_{\mathrm{L}^+}(\chi) &\geq \mathrm{disc}^{k-1}_{\mathrm{L}^+}(\chi) + 2a + c \\
&\geq (k-1) + 2b + 2 + 2a + c \\
&\geq k + \Sigma + a + b + 1 \\
&\geq k + \Sigma + 2.
\end{aligned}$$

Since we have $a + b + c \geq 1$, it follows that $a + b \geq 1 - c \geq 2$. Thus, case (ii) holds for configuration (I). Now let us consider configuration (II). We have:

$$\begin{aligned}
\mathrm{disc}^k_{\mathrm{L}^+}(\chi) &\geq \mathrm{disc}^{k-1}_{\mathrm{L}^+} + 2a + b \\
&\geq (k-1) + 2c + 2 + 2a + b \\
&\geq k + \Sigma + a + c + 1 \\
&\geq k + \Sigma + 2.
\end{aligned}$$

Since we have $a+b+c \geq 1$, it follows that $a+c \geq 1-b \geq 2$. Thus, case (ii) holds for configuration (II).

The proof of the lower bound on $\mathrm{disc}^k_{\mathrm{R}+}(\chi)$ is symmetric to the one we have just given for $\mathrm{disc}^k_{\mathrm{L}+}(\chi)$. Instead of adding the blocks whose values lie in the lower left hand triangle to form the new prefixes, we use the blocks whose values lie in the upper right hand triangle.

Finally, we need to show that if $\Sigma = \chi([3^k]) \leq -1$, then:

$$\mathrm{disc}^k_{\mathrm{L}-}(\chi),\ \mathrm{disc}^k_{\mathrm{R}-}(\chi) \ \leq \ -k+\Sigma-2. \tag{7}$$

Note that this follows from our proof of the first part of Lemma 1, namely that when $\Sigma = \chi([3^k]) \geq 1$, then:

$$\mathrm{disc}^k_{\mathrm{L}+}(\chi),\ \mathrm{disc}^k_{\mathrm{R}+}(\chi) \ \geq \ k+\Sigma+2. \tag{8}$$

If we consider a coloring $\chi : [3^k] \to \{-1,+1\}$ such that $\chi([3^k]) \leq -1$, and it is the case that (7) does *not* hold, then consider $\chi^- = -\chi$, i.e. the negation of χ. It follows that $\chi^-([3^k]) \geq 1$, but (8) does not hold for coloring χ^-, which is a contradiction. □

6 Consequences for bin packing LP

The main result of this section is the following:

Theorem 3. *For infinitely many integers m, there exists a bin packing instance $\mathcal{I}$ on m items, and an optimal* basic *feasible solution x to* (LP)*, such that any integral solution y to* (LP) *satisfying $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$ has value at least $\mathrm{OPT}(\mathcal{I}) + \Omega(\log m)$.*

We actually prove a slightly stronger statement in Remark 1. The *replacement property* allows an algorithm to replace item of size s by any item of size $s' \leq s$ in any legal pattern. We show that our lower bound on the size of any integral solution y to (LP) with $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$, holds when allowing replacements. Using an observation made by [EPR], we can even allow the integral solution to use the same pattern multiple times.

6.1 The Construction

The instance is very simple, in a sense that all the sizes are numbers in the interval $[1/4, 1/3]$, so that any pattern containing at most three items is legal. The more challenging part is to construct a basic feasible solution with the required property.

Fix any integer $k > 0$, and recall that the three permutations $\pi^k_1, \pi^k_2, \pi^k_3$ are permutations on the first $n = 3^k$ positive integers (we omit the superscript when k is clear by the context). We shall append the number 0 at the end of each permutation (that is, define $\pi_i(0) = n+1$ for $i = 1,2,3$). Let $m = 3(n+1)/2$ and define an instance $\mathcal{I} = \mathcal{I}(k)$ of m items. Each item corresponds to a pair of consecutive numbers in one of the permutations, that is, for each $i \in [3]$ and $j \in [(n+1)/2]$ we have an item (i,j), this item corresponds to the pair $(\pi_i(2j-1), \pi_i(2j))$ and has size $1/3 - (i + 2j/(n+1))/48$. See Figure 1 for the definition of items in the case $k = 2$. Define

π_1	**1 2**	3 4	5 6	7 8	9 0
π_2	9 7	8 3	**1 2**	6 4	5 0
π_3	5 6	4 8	9 7	2 3	**1 0**

Figure 1: The construction of the bin packing instance for $k = 2$, each underlined pair corresponds to an item. The items in p_1 are in bold font.

$$\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

Figure 2: The incidence matrix A for the case $k = 2$. The rows correspond to items, in each block of $(n + 1)/2 = 5$ rows are the items defined by one of the permutations. The $n + 1 = 10$ columns correspond to the patterns.

a set $P \subseteq \mathcal{P}$ of $n + 1$ patterns $P = \{p_0, p_1, \ldots, p_n\}$, where the pattern p_r contains the three items that correspond to the number r. More formally, the pattern p_r contains the items $(1, j_1)$, $(2, j_2)$, $(3, j_3)$, where j_i is such that $\pi_i^{-1}(r) \in \{2j_i - 1, 2j_i\}$ for $i \in [3]$. See Figure 2 for the incidence matrix of items and patterns for $k = 2$.

Observation 1. *Each item is contained in exactly two patterns, and each pattern contains exactly three items.*

Observation 2. *The item sizes are all in $[1/4, 1/3]$ and are strictly decreasing (ordered lexicographically by (i, j)).*

6.2 The Proof

The proof of Theorem 3 will use the discrepancy of permutations established in the previous section. A coloring $\chi : [n] \to \{-1, 1\}$ naturally corresponds to an integral solution of the bin packing instance we defined, where pattern p_r is taken if and only if $\chi(r) = 1$ (pattern p_0 does not correspond to a coloring, but it changes the cost of the solution by at most additive 1, so we may assume it is always taken).

Lemma 2. *Let $k \geq 8$ be an integer. Any feasible integral solution $y \in \{0,1\}^{n+1}$ to the instance $\mathcal{I}(k)$ such that $\text{supp}(y) \subseteq P$, satisfies $|\text{supp}(y)| \geq n/2 + k/16$.*

Proof. Seeking contradiction, assume that $|\text{supp}(y)| < n/2 + k/16$, and we will show that y cannot be feasible. Consider the coloring $\chi : [n] \to \{-1, 1\}$ defined by $\chi(j) = 2y_j - 1$. Observe that $\Sigma = \chi([n]) = |\{j \in [n] : y_j = 1\}| - |\{j \in [n] : y_j = -1\}| \leq (n/2+k/16) - (n/2-k/16-1) = k/8+1$. Clearly we may assume $|\text{supp}(y)| \geq n/2 + 1$ (since adding more patterns will make it more likely that y is feasible), so that $\chi([n]) > 0$. By Corollary 2,

$$\text{disc}^k_{L-}(\chi) \leq -k + 2\Sigma - 2 \leq -3k/4 \ .$$

Recall the definition of $\text{disc}^k_{L-}(\chi)$ in (2), which suggests that there exists an $i \in [3]$ and a prefix $w \in [n]$ for which

$$\chi(\alpha^k_i(w)) \leq -k/4 \leq -2 \ . \tag{9}$$

We conclude that there must be an item (i, j) for some integer $0 < j \leq w/2$ and $i \in [3]$, such that $\chi(\pi_i(2j - 1)) = \chi(\pi_i(2j)) = -1$, meaning the item is not covered by any pattern of y. This contradicts the fact that y is a feasible solution. ☐

Remark 1. *The proof of Lemma 2 holds even when allowing replacements and using patterns multiple times.*

Proof. Observe that in (9) we have $i \in [3]$ and a prefix $w \in [n]$ for which the discrepancy is at most $-k/4$, in other words $|\{j \in [w] : y_{\pi_i(j)} = 1\}| \leq \lfloor w/2 - k/8 \rfloor$. There are $\lfloor w/2 \rfloor$ items corresponding to this prefix, so in fact there are at least $k/8$ items that are not covered by any pattern chosen by y. In a setting that allows replacements, we may place such an uncovered item in available spots of other patterns, provided that the spot was intended for an item of greater size. By Observation 2 we have that these items cannot take the place of items corresponding to pairs (i', j') with $(i', j') > (i, j)$ (lexicographically), and as these uncovered items correspond to a prefix of π_i, the only option is to take the place of items corresponding to pairs (i', j') with $i' < i$. However, by our assumption on y, that $|\text{supp}(y)| < n/2 + k/16$, there are less than $k/16$ available spots in each permutation i' (obviously taking out larger items to create space is not helpful). As there are at most two values of $i' < i$, we have that there is no sufficient place for all the $k/8$ items.

When allowing replacements, it is also natural to allow multiple uses of the same pattern in the integral solution, that is, $y \in \mathbb{N}^{n+1}$. Since in Lemma 2 we defined $y \in \{0,1\}^{n+1}$, this is inherently not allowed. However, an observation made in [EPR], that the proof of Theorem 1 holds even for coloring with odd integers, in fact enables multiple uses of the same pattern. This is because the coloring is defined as $\chi(j) = 2y_j - 1$, which is always odd. Replacing $|\text{supp}(y)|$ by $\sum_j y_j$ in the proof of Lemma 2, we will still have a prefix with large negative discrepancy, so the same argument holds. ☐

Lemma 3. *The solution $x_p = 1/2$ for all $p \in P$, and $x_p = 0$ otherwise, is an optimal* basic *feasible solution of cost $(n + 1)/2$.*

Proof. By Observation 1 x is a feasible solution, and its cost is $(n + 1)/2$ because there are $n + 1$ patterns in P. This is clearly an optimal solution, since any bin can contain at most 3 items, so the number of required bins is at least $m/3 = (n + 1)/2$.

In order to prove that x is a basic solution, we need to exhibit $|\mathcal{P}|$ linearly independent tight constraints. As $|\text{supp}(x)| = n+1$, there are $|\mathcal{P}|-(n+1)$ tight constraints of the form $x_p = 0$, and it remains to show that there are $n + 1$ linearly independent constraints of the form $\sum_{p\in\mathcal{P}:i\in p} x_p = 1$. Since every item appears in exactly two patterns, all these constraints are in fact tight. The proof will follow once we establish the fact that there are $n+1$ linearly independent such constraints. Let A be the incidence matrix of the items in the patterns of P, which has $m = 3(n+1)/2$ rows and $n+1$ columns (see Figure 2). In what follows we prove that A has full rank. Abusing notation slightly, let p_r be the r-th column vector of A. Seeking contradiction, assume that there are coefficients $\alpha_0, \alpha_1 \dots, \alpha_n$, not all equal to 0, such that

$$\sum_{r=0}^{n} \alpha_r p_r = 0. \tag{10}$$

We will prove the following claims.

Claim 1. *If (10) holds, then there is some $\alpha > 0$ such that $|\alpha_r| = \alpha$ for every $0 \le r \le n$.*

Claim 2. *There are three numbers $1 \le i_1 < i_2 < i_3 \le 9$, such that there are three items corresponding to the pairs (i_1, i_2), (i_2, i_3), (i_1, i_3).*

Let us now conclude by showing a contradiction. By Claim 1 the absolute value of all coefficients is $\alpha > 0$. Let $1 \le i_1 < i_2 < i_3 \le 9$ be as in Claim 2. Note that at least two of the coefficients of i_1, i_2, i_3 must have the same sign, say w.l.o.g i_1, i_2. Since there is an item corresponding to (i_1, i_2), the absolute value of the appropriate entry in the vector $\sum_{r=0}^{n} \alpha_r p_r$ will be $2\alpha \neq 0$, a contradiction. ◻

It remains to prove the claims stated above.

Proof of Claim 1. The basic idea is the following: By Observation 1 every item is contained in exactly two patterns, so that if (10) is to hold, the two patterns containing the item must have coefficients whose sum is 0. Roughly speaking, if we have a nonzero coefficient α for some pattern, any item it contains appears in one more pattern, which must have coefficient $-\alpha$. This pattern contain more items, which will force coefficient α for yet other patterns, and so on. Finally, we will have that all patterns have $\pm\alpha$ coefficient. Next we prove this formally.

For the sake of analysis, we ignore the number 0 appended at each permutation end (if the nonzero coefficient is α_0, then the items containing 0 will also induce nonzero coefficients for some pattern $h > 0$). As k grows, the position of blocks of numbers in some of the permutations may shift, and thus affect the pairing of numbers into items. Since we aim at an inductive proof, we need to consider all possible locations of a block, and in what follows we define the possible ways of pairing the numbers into items. Observe that there are two ways to create pairs from consecutive numbers in each of the permutations: starting at the first or second position. For example when $k = 2$, π_2 can be paired into items as

$$\underline{9\ \ 7}\ \ \underline{8\ \ 3}\ \ \underline{1\ \ 2}\ \ \underline{6\ \ 4}\ \ 5$$

or as

$$9\ \ \underline{7\ \ 8}\ \ \underline{3\ \ 1}\ \ \underline{2\ \ 6}\ \ \underline{4\ \ 5}$$

Consider the 8 possible pairing schemes of the three permutations. A pairing scheme may be defined by the set $C = \{1, 2\}^3$, where each symbol indicates from which position to start pairing in each permutation. For example (122) means the following pairing scheme (for $k = 2$)

$$\begin{array}{l} \underline{\mathbf{1}\ \ \mathbf{2}}\ \ \underline{3\ \ 4}\ \ \underline{5\ \ 6}\ \ \underline{7\ \ 8}\ \ 9 \\ 9\ \ \underline{7\ \ 8}\ \ \underline{\mathbf{3}\ \ \mathbf{1}}\ \ \underline{2\ \ 6}\ \ \underline{4\ \ 5} \\ 5\ \ \underline{6\ \ 4}\ \ \underline{8\ \ 9}\ \ \underline{7\ \ 2}\ \ \underline{\mathbf{3}\ \ \mathbf{1}} \end{array} \tag{11}$$

For every pairing scheme $c \in C$ let $G_k^{(c)} = (V, E)$ be a graph where $V = [3^k]$ and $(u, v) \in E$ iff there is an item corresponding to the pair of integers (u, v) in one of the π_i with the pairing scheme c. An edge (u, v) indicates that the patterns p_u, p_v must have opposite coefficients, so the connectivity of the graph suggests that all coefficients are $\pm\alpha$, as required. We will prove by induction on k that for any possible pairing scheme $c \in C$, $G_k^{(c)}$ is connected. The base case when $k = 1$ holds,

$$\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}$$

because every permutation has one pair, and note that it cannot be that all the three pairs are the same, thus we have one number which is connected to the other two. For the induction step, assume the assertion holds for k, and prove for $k+1$. By Fact 1 we have that the permutations induced on the three blocks $[1, 3^k]$, $[3^k+1, 2\cdot 3^k]$ and $[2\cdot 3^k+1, 3^{k+1}]$ are isomorphic to the permutations of $[3^k]$. Fix any pairing scheme of the three permutations, which induces a pairing scheme on each of the three blocks. Let G_1, G_2, G_3 be the three graphs induced by the pairing scheme of the permutations of each block. By the induction hypothesis, all three graphs are connected. It remains to see that these graphs are connected to each other. To see this, note that any block will be in the middle section in one of the three permutations. Now, since it contains an odd number of elements, one of its numbers will be paired with a number from another block, establishing the connection to the graph of another block. Thus $G_k^{(c)}$ is connected, which concludes the proof.

□

Proof of Claim 2. By Fact 1, the block [9] appears consecutively in each of the permutations, ordered exactly as depicted in Figure 1. As in the proof of Claim 1, we need to consider all the possible pairing schemes of items in each permutation, and there are 8 cases to inspect[1]. By a simple case analysis one can check that there are three numbers satisfying the assertion of the claim,

[1]In fact there are less cases, because in π_1 the block [9] never shifts.

no matter which pairing scheme of C is chosen. For instance, in the pairing scheme (122) depicted in (11), the numbers $4, 5, 6$ satisfy the claim, in the pairing scheme (111) depicted in Figure 1, the numbers $3, 4, 8$ satisfy it. The following is a matching between the 8 possible pairings and the numbers i_1, i_2, i_3.

$$\begin{aligned}
(111) &\Longleftrightarrow 3, 4, 8\\
(112) &\Longleftrightarrow 7, 8, 9\\
(121) &\Longleftrightarrow 1, 2, 3\\
(122) &\Longleftrightarrow 4, 5, 6\\
(211) &\Longleftrightarrow 4, 5, 6\\
(212) &\Longleftrightarrow 1, 2, 3\\
(221) &\Longleftrightarrow 7, 8, 9\\
(222) &\Longleftrightarrow 2, 6, 7
\end{aligned}$$

□

Proof of Theorem 3. Combining Lemmata 2, 3, and noting that $k = \log_3 n = \Omega(\log m)$, yields the proof of the theorem. □

7 Consequences for discrepancy theory

A topic of interest in the field of combinatorial discrepancy theory is the worst-case relationships between various definitions of discrepancy. In this section, we discuss how our lower bound may be useful in studying the gaps between these different quantities.

7.1 Union of set systems with low hereditary discrepancy

Let $\mathcal{S} = \{S_1, \ldots, S_M\}$ be a set system on $[n]$. Given a set $W \subseteq [n]$, we define the trace of $\mathcal{S}$ on W as the set system $\mathcal{S}|_W = \{S_1 \cap W, \ldots, S_M \cap W\}$. The *hereditary discrepancy* of $\mathcal{S}$ is defined as

$$\operatorname{herdisc}(\mathcal{S}) = \max_{W \subseteq [n]} \operatorname{disc}(\mathcal{S}|_W). \tag{12}$$

Hereditary discrepancy is in some sense a more robust measure of the complexity of a set system than discrepancy. An intriguing question, raised by Sós [LSV86, Spe87], concerns quantifying the robustness of hereditary discrepancy: given set systems $\mathcal{T}$ and $\mathcal{U}$, each with hereditary discrepancy one, how large can the hereditary discrepancy of $\mathcal{T} \cup \mathcal{U}$ be as a function of n? A clever example due to Hoffmann shows that the hereditary discrepancy of $\mathcal{T} \cup \mathcal{U}$ is $\Omega(\log n / \log\log n)$ [Spe87]. More recently, Matoušek showed an upper bound of $O(\log(M \cdot n)\sqrt{\log n})$ [Mat11]. Furthermore, Matoušek presented a construction due to Pálvögyi, which improves on Hoffmann's example: there exist $\mathcal{T}$ and $\mathcal{U}$ each with hereditary discrepancy one, where $\operatorname{herdisc} \mathcal{T} \cup \mathcal{U}$ is $\Omega(\log n)$. Pálvögyi's construction was originally presented geometrically in the context of cover-decomposable planar sets [Pál10].

Here, we provide another example of two set systems with hereditary discrepancy one whose union has discrepancy $\Omega(\log n)$. Let $\mathcal{S}'_k$ be the set system consisting of the prefix intervals $\{\alpha_1^k(x)\}_{x=1}^n$ of π_1^k, and let $\mathcal{S}''_k$ be the set system consisting of the prefix intervals $\{\alpha_2^k(x)\}_{x=1}^n \cup \{\alpha_3^k(x)\}_{x=1}^n$ of π_2^k, π_3^k. We have,

$$\operatorname{herdisc}(\mathcal{S}'_k) = 1, \tag{13}$$

$$\operatorname{herdisc}(\mathcal{S}''_k) = 1, \tag{14}$$

$$\operatorname{herdisc}(\mathcal{S}'_k \cup \mathcal{S}''_k) = \operatorname{herdisc}(\mathcal{S}_k) = \Omega(\log n). \tag{15}$$

Equation (13) follows from the fact that the discrepancy of one permutation is one. Equation (14) follows by the fact that any set system consisting of the prefix intervals of two permutations has discrepancy at most one. Note that the discrepancy of the set system based on *all* intervals of two permutations is at most two [Spe87]. Equation (15) follows from Theorem 1.

7.2 Vector discrepancy of permutations

Another notion that has proven useful in studying discrepancy, especially from an algorithmic perspective is a relaxation of discrepancy called *vector discrepancy*. The vector discrepancy of a set system $\mathcal{S}$ is defined as

$$\operatorname{vecdisc}(\mathcal{S}) = \min_{\mathbf{u_1},\ldots,\mathbf{u_n}} \max_{S_j \in \mathcal{S}} \left\| \sum_{i \in S_j} \mathbf{u_i} \right\|_2, \tag{16}$$

where $\mathbf{u_1}, \ldots, \mathbf{u_n}$ ranges over unit length vectors in $\mathbb{R}^n$. We define *hereditary vector discrepancy* analogously to hereditary discrepancy:

$$\operatorname{hervecdisc}(\mathcal{S}) = \max_{W \subseteq [n]} \operatorname{vecdisc}(\mathcal{S}|_W). \tag{17}$$

The best currently known upper bound on the gap between the discrepancy and the hereditary vector discrepancy is $O(\log M \cdot n)$, due to Bansal [Ban10].

Via the determinant lower bound of Lovasz, Spencer, and Vesztergombi [LSV86], Matoušek recently proved that if $\mathcal{T}$ and $\mathcal{U}$ are set systems with constant hereditary discrepancy, then $\operatorname{hervecdisc}(\mathcal{T} \cup \mathcal{U}) \leq O(\sqrt{\log n})$ [Mat11]. Combining this theorem with our observations in Section 7.1, we see that the hereditary vector discrepancy of the set system consisting of the intervals of any three permutations is $O(\sqrt{\log n})$. In light of our lower bound on the discrepancy of three permutations, an improved upper bound on the hereditary vector discrepancy of three permutations would disprove a conjecture of Matoušek that hereditary discrepancy is at most a factor of $O(\sqrt{\log n})$ higher than hereditary vector discrepancy [Mat11].

Thus, we conclude with an open problem. Let ω_0, ω_1 and ω_2 be the third roots of unity. In our construction, we can assign the vector ω_i to each element equivalent to $i \pmod 3$. This coloring suffices to show that $\operatorname{vecdisc}(\mathcal{S}_k) \leq 1$. What is the value of $\operatorname{hervecdisc}(\mathcal{S}_k)$? More generally, what is the worst-case vector discrepancy of three permutations?

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