Problems to Explore
Rutgers Young Scholars Program

PROBLEM 1: DECREASING AGE RATIOS

Two years ago John was 3 times his sister's age. Four years ago he was 5 times her age. In how many years will John be twice his sister's age?

PROBLEM 2: FAST AND SLOW CLOCKS

Electric clock A gains 2 seconds every hour, i.e., when 1 hour has actually passed, the clock shows that 1 hour and 2 seconds has passed.

Electric clock B loses 3 seconds every hour

Suppose that both clocks are started at the same correct time.

(a) How long will it be in days before the clocks agree again?

(b) How long will it be in days before the clocks simultaneously show the correct time again?

PROBLEM 3: THE LARGEST n DIGIT nth POWER

The integer $125 = 5^3$ is a 3 digit number that is a cube and the integer $1296 = 6^4$ is a 4 digit number that is a fourth power.

What is the largest number that for some positive integer n is both an nth power and has n digits?

PROBLEM 4: FAST AND THE SLOW TRAINS

An express, or fast, and a local, or slow, train run on parallel tracks between cities A and B. Assume that the speed of each train is constant.

As both trains are going from A to B, the express train takes 24 seconds to pass the local train completely. Turning around at B and returning, the express meets the local, still heading to B, and the two trains pass completely in 4 seconds. When the two trains meet, the express has completed 40% of its trip from B back to A, and it arrives at A one hour later, while the local continues on to B.

Does the local arrive at B before or after the express arrives at A, and by how many minutes?

PROBLEM 5: SOME SUMS OF SIX DIGIT INTEGERS

Suppose that a, b, c are distinct non-zero digits.

(a) Find a formula, depending on a, b, c, for the sum of all six digit integers whose only digits are a, b, c.

(b) What is the sum of all six digit integers having all non-zero digits AND EXACTLY THREE DIFFERENT DIGITS?

PROBLEM 6: A SELF-DESCRIPTIVE CRYPTARITHM

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      A
P A I N
    I N
  T H E
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N E C K
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In the addition at the left each letter stands for one of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9. Assume that different letters represent different digits. In addition, to avoid confusion assume that the letter I corresponds to the digit 1. Under these assumptions, find all possible additions having this pattern.

PROBLEM 7: ATTACKING QUEENS

Recall that in chess a queen attacks any square that is on a straight line — horizontally, vertically, or diagonally — from the square on which the queen stands. Regard the queen as also attacking the square which she occupies. What is the smallest number of queens required to attack all of the squares of a 6×6 chessboard?

PROBLEM 8: ADDITION BY GEOMETRIC DISSECTION

Verify that $9^2 + 2^2 = 7^2 + 6^2$ by cutting a 9×9 square into four pieces that together with a 2×2 square can be rearranged to give a 7×7 square and a 6×6 square.

PROBLEM 9: ORDERED TRIPLES OF 1, 2, 3, 4

Let S be the set of distinct ordered triples comprised of the numbers 1, 2, 3, 4. To say that the triple is distinct means that no number occurs twice in the triple. To say that the triple is ordered means that two triples in which the same numbers appear in a different order are considered to be different triples.

Some of the elements of S are: (1,2,3), (1,2,4), (3,2,1), (3,2,4), (4,2,1), (4,3,2).

We wish to list all of the elements of S by a certain system. In this system the triple listed following (i, j, k) is one of the triples that begins with (j, k). For example, assuming that we start with $(1, 2, 3)$, then the next triple listed must be either $(2, 3, 1)$ or $(2, 3, 4)$.

Starting with $(1, 2, 3)$, there are four possible ways to list the first three triples:

$(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$;

$(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 4)$;

$(1, 2, 3)$, $(2, 3, 4)$, $(3, 4, 1)$;

$(1, 2, 3)$, $(2, 3, 4)$, $(3, 4, 2)$.

The question is: can we list all 24 triples of S according to this system, starting with $(1, 2, 3)$ and listing no triple twice, so that the last triple listed is a predecessor of $(1, 2, 3)$, or, in other words, is either $(3, 1, 2)$ or $(4, 1, 2)$?

PROBLEM 10: WHICH EQUILATERAL TRIANGLE?

A point within an equilateral triangle is at distance 5, 7, 8 from each of the vertices. What is the length of a side of the triangle?

PROBLEM 11: MAXIMUM SETS WITH MINIMUM SUMS

Let S be a set of positive integers, each of which is less than 20.
(a) What is the maximum size m of S if S has the property that no sum of distinct elements of S is equal to 20.
(b) Of all sets S of positive integers less than 20 having maximal size m, and having the property that no sum of distinct elements of S is equal to 20, which are the sets having the smallest possible sum?

PROBLEM 12: A THIRD AS A FRACTION USING ALL NON-ZERO DIGITS

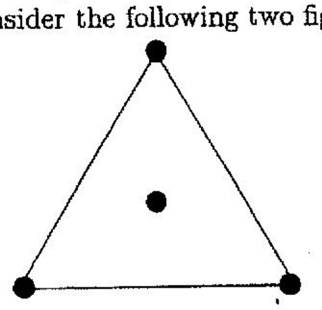
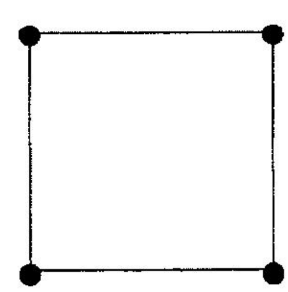
Using each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once form two numbers x and y so that $x/y = 1/3$.
(*Suggestion:* In the beginning it may be useful to recall that an integer is divisible by 3 or 9 if and only if the sum of its digits is divisible by 3 or 9, respectively.)

PROBLEM 13: THE PHOTOGRAPHER'S PROBLEM

A photographer wished to arrange twelve people of different heights in two rows of six each. Each person in the first row must be shorter than the person directly behind and, going from left to right, the people must get taller.
How many such arrangements are there?

PROBLEM 14: FOUR POINTS IN THE PLANE

Four points in the plane have the property that just two distances occur among the four points. For example, consider the following two figures:

In the first figure, the four points consist of three vertices of an equilateral triangle together with its center. The distance between any two vertices of the triangle is the same, and each of the vertices is at the same distance from the center. Thus, only two different distances occur among the four points.
In the second figure, the four points are the four corners of a square. The distance between any two adjacent corners of the square is the length of the side of the square. The distance between opposite corners of the square is equal to the length of the diagonal of the square. Again, just two different distances occur among the four points.
(a) Which other sets of four points in the plane have the property that just two different distances occur among the four points?
(b) Is there any set of five points in the plane having the property that just two different distances occur among the five points?

PROBLEM 15: CROSSWORD PUZZLE PATTERNS

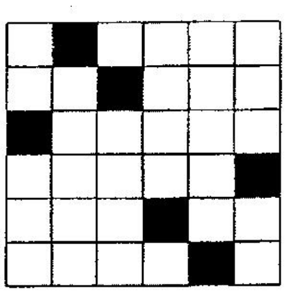
A crossword puzzle pattern is produced by darkening certain squares of $n \times n$ grid so that:

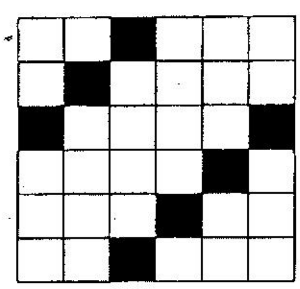
(1) All words have length at most four.

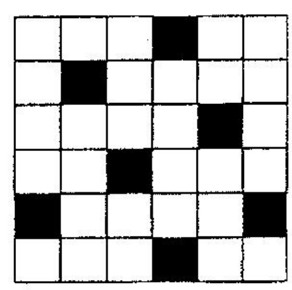
(2) The undarkened squares are connected in the sense that one can go from any undarkened square to any other by proceeding in a series of steps, each step horizontal or vertical, through other undarkened squares.

The problem is to determine the smallest number of squares one must darken to achieve such a pattern.

As an example consider the following three patterns on a 6×6 grid. The first pattern fails to satisfy condition (1), since there is a word of length 5. In the second example, since all words have length at most 4, condition (1) holds, but condition (2) fails. In the third example, both conditions (1) and (2) hold. Note that in the third example, seven squares have been darkened. It is possible to show, on a 6×6 grid, that one cannot obtain a crossword puzzle pattern that satisfies (1) and (2) by darkening only six squares. Thus, the minimum number of squares that must be colored to achieve a pattern satisfying conditions (1) and (2) is seven.







What is the minimum number of squares that must be darkened to achieve a pattern satisfying conditions (1) and (2):

(a) On a 7×7 grid? (b) On an 8×8 grid?

(c) On a 9×9 grid? (d) On a 10×10 grid?

Explain why your answer is a minimum.