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Symmetric Class 0 subgraphs of complete graphs

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Abstract

In graph pebbling, a connected graph is called *Class 0* if it has a pebbling number equal to its number of vertices. This paper addresses the question of when it is possible to edge-partition a complete graph into k complementary Class 0 subgraphs. We define the notion of k-Class 0 graphs: a graph G is k-Class 0 if it contains k edge-disjoint subgraphs, where each subgraph is Class 0. We next present a family of k-Class 0 graphs for $k = 2$, showing that for $n \geq 9$, K_n is 2-Class 0. We finally provide a probabilistic argument to prove that $\forall k \in \mathbb{N}$, $\exists n \in \mathbb{N}$ such that K_n can be edge-partitioned into k cyclically symmetric subgraphs of diameter 2 and connectivity 3: that is K_n is k-Class 0.

1 Introduction

In the traditional pebbling framework for a graph G, we define a pebbling distribution $P : \mathcal{V}(G) \to \mathbb{N}$ which places a number of pebbles on each vertex of the graph[1]. A legal pebbling move consists of taking two pebbles from one vertex and removing them from the graph, while adding one pebble to an adjacent vertex. Many problems in graph pebbling relate to the pebbling number $\pi(G)$ of the graph (the minimum number of pebbles required such that any pebbling distribution placing that many pebbles on the graph will allow for any target vertex to be reached through a series of pebbling moves).

In this paper we will study the partitioning of complete graphs into complimentary cyclically symmetric Class 0 subgraphs. We define the notion of k-Class 0 graphs: a graph G is k-Class 0 if it contains k edge-disjoint subgraphs, where each subgraph is Class 0. To provide context, one can consider a multi-colored relative of the classic graph pebbling problem. The game in question is a generalization of graph pebbling to a version with k players, each representing one of k colors which are assigned to both pebbles and edges (see [2] for a similar game). When can the edges of a graph be colored with k colors, such that each implies a Class 0 subgraph? We show that for a given natural k, there exists an n such that the complete graph on n vertices can be edge-partitioned into k circularly symmetric Class 0 subgraphs, and hence is k-Class 0.

We'll begin by introducing definitions relevant to k-Class 0 graphs, as well as a known sufficient condition for a graph to be Class 0: diameter 2 and 3-connectivity ($2D3C$). We present a set of constructive operations on $2D3C$ graphs, and find a family of 2-Class 0 graphs. We then introduce a construction for cyclically symmetric subgraphs of complete graphs. We finally present a probabilistic argument to prove our main result.

2 Definitions and Framework

Recall the following important definitions from standard graph pebbling [3]:

Definition. A *pebbling move* consists of the removal of two pebbles from a single vertex and adding one to an adjacent vertex.

Definition. The *pebbling number* for a graph G, $\pi(G)$, is the minimum number such that any distribution of $\pi(G)$ pebbles on G will allow for any target vertex to be reached by a series of pebbling moves.

It is clear that $\pi(G) \geq |V(G)|$.

Definition. A graph G is *Class 0* if $\pi(G) = |V(G)|$.

We introduce the following pertinent definition concerning Class 0 subgraphs of a Class 0 graph.

Definition. A graph G on n vertices is considered *k-Class 0* if it contains k edge-disjoint Class 0 subgraphs spanning n vertices.

A graph which is k-Class 0 can hence be partitioned into k subgraphs, leaving room for a k-player graph pebbling game with each player moving on a Class 0 subgraph of the initial graph. It is clear that being k-Class 0 is a monotone property in graph edges; that is, adding edges to a graph which is already k-Class 0 will leave it k-Class 0. Note also that these subgraphs do not need to be a precise decomposition; unassigned edges won't affect k-Class 0 by monotonicity.

Two properties of graphs which are relevant in the discussion of Class 0 graphs are diameter and connectivity.

Definition. Let G be a connected graph. Then the *diameter* of G, $\text{diam}(G)$, is the greatest distance between any pair of vertices.

Definition. A graph G graph is said to be *k-connected* if there does not exist a set of $k-1$ vertices whose removal disconnects the graph. By convention, we require G to either have $k+2$ or greater vertices, or be the complete graph on $k+1$ vertices. For a given graph, the maximum k such that the graph is k-connected is written as $\kappa(G)$, the graph's *vertex connectivity*.

Intuititively, one can see that graphs of smaller diameter are more likely to be Class 0, and that low connectivity prevent a graph from being Class 0. There has been much work done characterizing Class 0 graphs, but one result which is used multiple times in our paper is given here.

Theorem (Clarke, et. al.). *If* $\text{diam}(G) = 2$, *and* $\kappa(G) \geq 3$, *then* G *is of Class 0 [4].*

It will be convenient to define this subset of Class 0 graphs, as results in this paper work directly with diameter and connectivity as a sufficient condition for Class 0.

Definition. We call a graph $2D3C$ if it has diameter 2 and is 3-connected.

From the previous theorem, we see that the $2D3C$ condition is stronger than that of Class 0. In particular, this condition implies that almost all graphs are Class 0, since almost all graphs are $2D3C$ [4]. While $2D3C$ is a sufficient condition for being Class 0, it is not necessary; a subsequent theorem [5] states that there exists a function $k(d)$, bounded in $\frac{2^d}{d} \leq k(d) \leq 2^{2d+3}$, such that if G is a graph of diameter d, and $\kappa(G) \geq k(d)$, then G is of Class 0. Below we present an example of a Class 0 graph with high connectivity but diameter greater than 2.

Example. Consider the icosahedron, a graph with 12 5-regular vertices, alternatively defined as having 20 equilateral triangle faces. One can show that the icosahedron has diameter 3 and is 5-connected, but is Class 0. Note that the connectivity does not lie within the bounds of the function $k(d)$ guaranteed by the theorem mentioned above [5].

3 Operations on 2D3C Graphs

We now present a pair of operations on graphs which preserve $2D3C$.

Definition (Strong Product)**.** Given graphs G and H, we'll define $G \boxtimes H$ as a graph with vertex set equal to the Cartesian Product of $V(G)$ and $V(H)$, and an edge relation such that $(u_i, v_i)(u_j, v_j) \in E(G \boxtimes H)$ if any one of the following holds:

(i) $u_i u_j \in E(G)$ and $v_i = v_j$

(ii) $u_i = u_j$ and $v_i v_j \in E(H)$

(iii) $u_i u_j \in E(G)$ and $v_i v_j \in E(H)$

We call this the *Strong Product* of the two graphs.

Definition (Vertex Splitting)**.** Let G be a graph on n vertices, and $v \in V(G)$. We *split* the vertex v to form graph G_v, duplicating the vertex by adding v_0, and connecting v_0 by an edge to all adjacencies of v. Note that G_v is now a graph on $n+1$ vertices.

In particular, we find that both strong products and vertex splitting preserve $2D3C$ across any number of subgraphs.

Theorem 1. *Let G and H be graphs which can be edge-partitioned into k $2D3C$ graphs. Then $G \boxtimes H$ can be edge-partitioned into k $2D3C$ graphs, and hence is k-Class 0.*

Theorem 2. *Let G be a graph which can be edge-partitioned into k $2D3C$ graphs. Then for any $v \in V(G)$, G_v can be edge-partitioned into k $2D3C$ graphs, and hence is k-Class 0.*

Since we can show that K_9 is 2-Class 0 by example (see Figure 1), a consequence of Theorem 2 is our first family of 2-Class 0 graphs.

Theorem 3. *For $n \geq 9$, complete graphs of the form K_n can be partitioned into two $2D3C$ subgraphs, and in particular are 2-Class 0.*

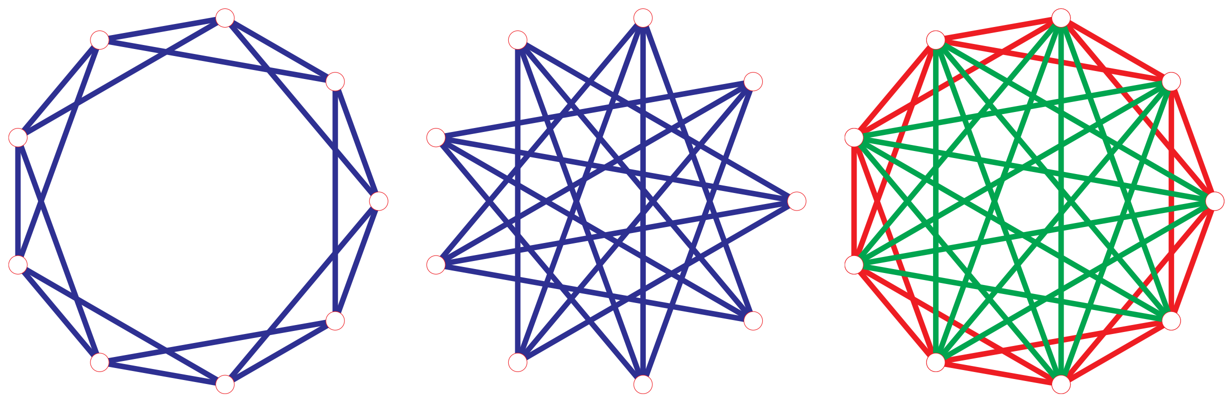
4 Cyclic Constructions

While the previous section provides a family of 2-Class 0 graphs and two constructive operations on $2D3C$ graphs, the next will introduce a concrete construction for this family when k is odd. This cyclically symmetric construction will lay the foundation for the existence result in the final section.

Definition (Cyclic Construction)**.** Let $n \in \mathbb{N}$, $S = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Let $A = \{a_1, a_2, \ldots\} \subseteq S$, and define $C_n(A) = C_n(a_1, a_2, \ldots)$ as the graph on vertices $v_1, v_2, \ldots, v_n$, where $v_i v_j \in E(C_n(A))$ if and only if $j - i = \pm m$ mod n for some $m \in A$.

Remark. Note that $C_n(A)$ is symmetric under cyclic permutations of $v_1, \ldots, v_n$, and is a subgraph of K_n. Note also that it suffices to consider S with cardinality $\lfloor \frac{n}{2} \rfloor$, as paths to the vertices of the opposite half will follow by symmetry.

Remark. Partitioning the edges of the graph into k cyclically symmetric subgraphs is equivalent to partitioning the set $S = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ into k subsets $A_1, \ldots, A_k$ such that the elements of a given subset will represent the edge lengths uniquely attributed to that subgraph according to our cyclic construction. Moreover, it is clear that $K_n = C_n(S) = C_n(A_1) \cup \cdots \cup C_n(A_k)$.



$G = C_9(1,2)$ $\quad$ $G^c = C_9(3,4)$ $\quad$ K_9

Figure 1: 2-Class 0 construction for $n = 9$

Example. We now focus on the particular subgraph construction that will be relevant in the next proof. Let $r \in \{2, 3, \ldots\}$, and consider $G = C_{4r+1}(1, \ldots, r)$ and $G^c = C_{4r+1}(r + 1, \ldots, 2r)$. We more clearly illustrate this construction when $r = 2, n = 9$ in Figure 1.

In Figure 1, $S = \{1, 2, 3, 4\}$, where $A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$ define $C_9(1, 2)$ and $C_9(3, 4)$. Note also that $K_9 = C_9(1, 2) \cup C_9(3, 4)$.

$C_n(A)$ in general appears to be highly connected. The following lemma gives an example of when 3-connectivity is guaranteed.

Lemma 4. *Let $n \in \mathbb{N}$, $A \subseteq S = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and consider $C_n(A)$. If A contains an element s such that $\gcd(s, n) = 1$, and at least one other element, then $C_n(A)$ is 3-connected.*

Proof. Let $s, t \in S$ such that s and n are coprime, and consider $C(s, t)$. Starting with a particular vertex v_0, we relabel the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ as $v_0, v_s, v_{2s}, \ldots, v_{(n-1)s}$, to show that $C_n(s, t) \simeq C_n(1, s^{-1}t)$. From an arbitrary 2-vertex cut, we are left with at most two connected components from the circular path of edge-length 1 alone (see Figure 2). One of the resulting arcs must have less than or equal to the number of vertices in the other, and we label the vertices circularly such that we removed v_0, v_k for $k \leq \lfloor \frac{n}{2} \rfloor$. Since $s^{-1}t \leq \lfloor \frac{n}{2} \rfloor$ and the arc containing our vertex is smaller, we have $k - 1 + s^{-1}t < n$. This implies that an edge connects v_{k-1} to a vertex (not v_0) in the opposite arc, from which we conclude that the two arcs are connected, and the graph itself is 3-connected. □

We surmise that the actual connectivity condition is stronger, but this lemma suffices. By restricting ourself to a prime number of vertices in a complete graph, any subpartition with two or more edge lengths will be 3-connected.

Theorem 5. *Complete graphs of the form K_{4r+1} for $r = 2, 3, \ldots$ can be edge-partitioned into two cyclically symmetric $2D3C$ subgraphs $C_{4r+1}(1, \ldots, r)$ and $C_{4r+1}(r + 1, \ldots, 2r)$, and hence are 2-Class 0.*

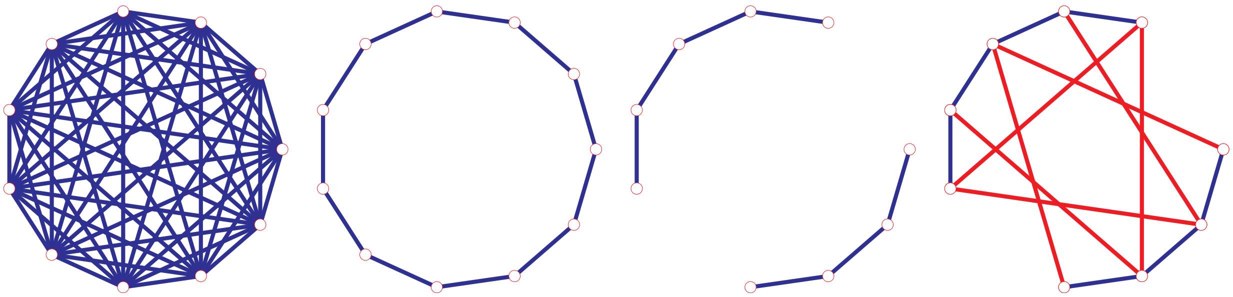


Figure 2: Illustration of proof method of Theorem 5 where $n = 11$, $s = 2, t = 3$. It is clear that as long as $s \neq t$, the two components must be connected by an element of the subgraph generated by t.

Proof. Let $r \in \{2, 3, \cdots\}$, and let $H = K_{4r+1}$. We now show that using the cyclic construction above, $G = C_{4r+1}(1, 2, \ldots, r)$ and its complement in H, $G^c = C_{4r+1}(r+1, \ldots, 2r)$, are Class 0 by showing that they are $2D3C$.
Claim: G, G^c are 3-connected. We note that $1 \in A_1$ is relatively prime to $4r+1$, and $2r \in A_2$ is as well. Since both A_1, A_2 contain multiple elements, by Lemma 4 the two subgraphs are each 3-connected.
Claim: $\mathrm{diam}(G) = \mathrm{diam}(G^c) = 2$. Let $v, w \in V(G)$ such that $vw \notin E(G)$. Since both v and w are connected to $2r$ vertices, but there are only $4r-1$ remaining distinct vertices in G, then v and w must have a common neighbor, implying that there is a two-path between them. Since the same logic follows for G^c, $\mathrm{diam}(G) = \mathrm{diam}(G^c) = 2$.

We have thus shown that when $n = 9, 13, \ldots$, K_n can be edge-partitioned into two cyclically symmetric $2D3C$ graphs, and hence is $2-$Class 0.

□

Through a similar method, we can extend a similar construction to graphs of the form K_{4r+3}, for $r = 2, 3, \ldots$, in which the number of possible neighbors for a given vertex is odd but not divisible by 4.

Theorem 6. *Complete graphs of the form K_{4r+3} for $r = 2, 3, \ldots$ can be edge-partitioned into two cyclically symmetric $2D3C$ subgraphs $C_{4r+3}(1, \ldots, r+2)$ and $C_{4r+1}(r+3, \ldots, 2r+1)$, and hence are 2-Class 0.*

Both of these cyclic constructions are for complete graphs with odd numbers of vertices. An open question we have is when it is possible to create cyclically symmetric partitions for $K_{10}, K_{12}, \ldots$? In Figure 4, we can use the cyclic construction for K_9 to demonstrate the effect of vertex splitting in creating a partition of K_{10}, showing that it indeed can be partitioned into two $2D3C$ subgraphs. However it is possible to show that K_{10} is not symmetric 2-Class 0, and we ask for what even-vertex complete graphs it is possible. Another question is to find families of constructions for partitioning complete graphs into $k > 2$ subgraphs which are $2D3C$; we tackle this problem through a different approach in the next section.

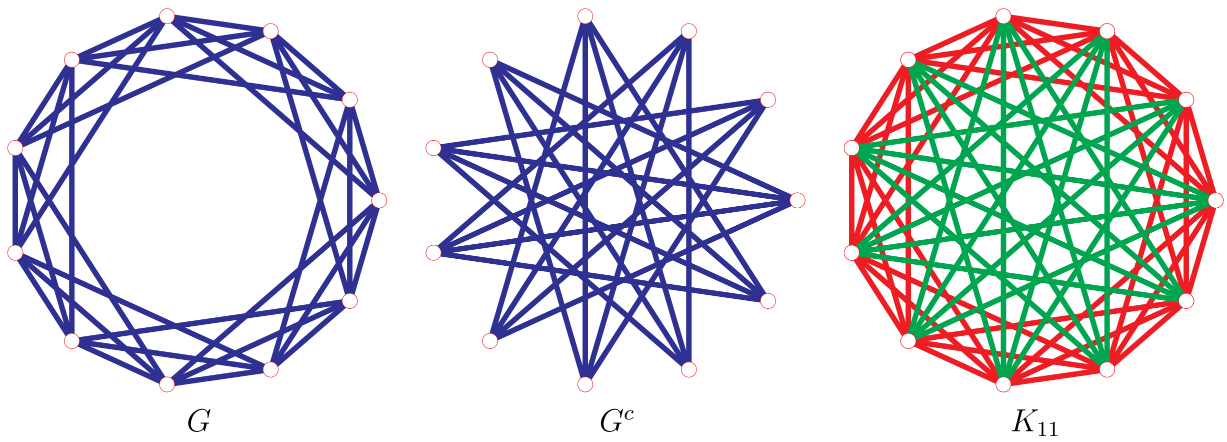


Figure 3: 2-Class 0 construction for K_{11}

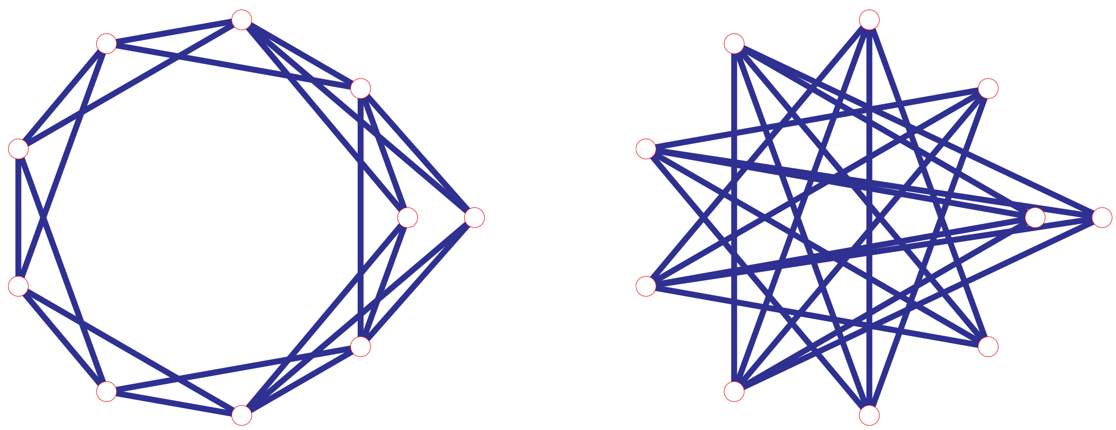


Figure 4: Splitting v_9 to form 2-Class 0 K_{10} from K_9

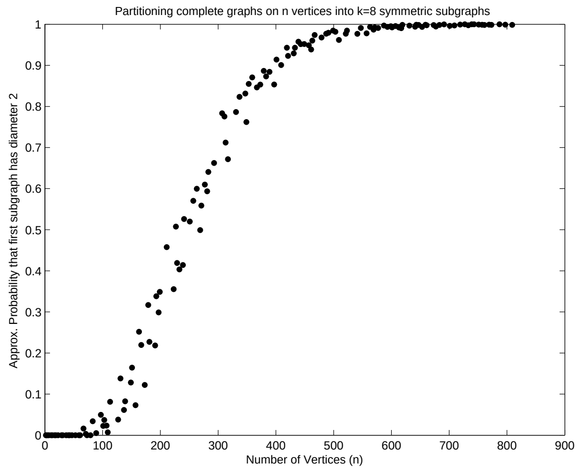
5 Existence of k-Class 0 graphs for all k

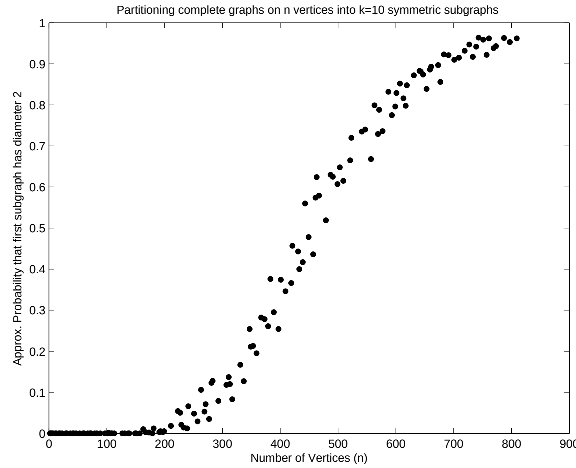
We will now prove the existence of a k-Class 0 decomposition for each $k \in \mathbb{N}$. In particular, we invoke the probabilistic method to show that there exists an $n \in \mathbb{N}$ such that K_n can be edge-partitioned into k cyclically symmetric Class 0 subgraphs.

We first show that for each $k \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that K_n can be edge-partitioned into k disjoint cyclically symmetric subgraphs, each of which has diameter 2. To do so, we will prove that for each $k > 2$, there is an n such that each subgraph in a random decomposition of K_n into k symmetric subgraphs has diameter 2. We can abstract this problem to one of basic additive number theory, in which the addition or subtraction of any two elements of a set $A \subseteq S$ represents a two-path to a vertex corresponding to the resulting element of S, and the size of the subsequent sum and difference set relates to the number of elements one can reach via a two-path.

To clarify this arithmetic, we define the *dihedral sumset* as follows.

Definition. Let $n \in \mathbb{N}$, $S = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and $A \subseteq S$. Then we define the *dihedral sumset*





Method: We let $n \in \mathbb{N}$ be a prime number, and fix some $k \in \mathbb{N}$ to be the *number* of partitions of the complete graph K_n. Next, we consider a random permutation of $1, \ldots, \lfloor \frac{n}{2} \rfloor$ and take (in order) $\left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{k} \right\rfloor$ elements out at a time, each of which corresponds to a random selection of an appropriately sized partition of K_n such that there will be a total of k partitions. We then attempt a simulation, in which we take random selections of appropriately sized subsets of S for various K_n, and evaluate the size of their dihedral sumset. We chose prime graph sizes n in order to deal with the 3-connected condition; the explanation of this lies in Lemma 4. In this figure we present the simulated probability that a given random symmetric subgraph of a complete graph with a prime number of vertices (x-axis) will be successfully of diameter 2, i.e. that the dihedral sumset generated by the corresponding $A \subseteq S$ is sufficiently large.

Figure 5: Simulated probability that a randomly selected subgraph of a complete graph on a prime number p of vertices will have diameter 2

of A to be the set

$$A \boxplus A = \{x \in S : x = a + b \text{ or } x = b - a \text{ or } x = n - (a + b), \text{ for } a, b \in A, b \geq a\}$$

Our definition of dihedral arithmetic exhausts the ways that one can use a pair of edge lengths to reach vertices around the graph. This is in comparison to the typical sumset, which under modulo n is defined as $A + A = \{x \in S : x = a + b \bmod n, \text{ for } a, b \in A\}$ [6].

Proposition 7. *$C_n(A)$ has diameter 2 if and only if $|A \boxplus A| = |S| = \lfloor \frac{n}{2} \rfloor$. In fact, $C_n(A)$ has diameter less than or equal to k if and only if $|\underbrace{A \boxplus A \boxplus \cdots \boxplus A}_{k \text{ times}}| = |S|$.*

This is intuitive; each additional dihedral sumset operation corresponds to length of a path between two elements. We also are interested in when an $x \in S$ can be reached via two-path using elements of a set A.

Definition. For a given $x \in S$, we consider pairs of elements $a, b \in S, a \leq b$ such that either $a + b = x$, $b - a = x$, or $n - (a + b) = x$. We call the event that both are included in a particular subset A a *pair event* (a, b) for x in S.

We can get some intuition on where this probabilistic argument comes from looking at Figure 5. In simulation, the larger the size of the complete graph (and hence the larger the size of a random subgraph, for fixed k), the more likely the subgraph will be of diameter 2. The proof begins below.

	Event class	Minimum number of disjoint events
1	$\{(1, x-1), \ldots, (\lfloor \frac{x}{2} \rfloor, \lceil \frac{x}{2} \rceil)\}$	$\lfloor \frac{x}{2} \rfloor$
2	$\{(\lfloor \frac{n-x}{2} \rfloor, \lceil \frac{n-x}{2} \rceil), (\lfloor \frac{n-x}{2} \rfloor - 1, \lceil \frac{n-x}{2} \rceil + 1), \ldots\}$	$\max\{\lfloor \lfloor \frac{n}{2} \rfloor - \lceil \frac{n-x}{2} \rceil \rfloor, \lfloor \lfloor \frac{n-x}{2} \rfloor - 1 \rfloor\}$
3	$\{(1, x+1), (2, x+2), \ldots, (\lfloor \frac{n}{2} - x \rfloor, \lfloor \frac{n}{2} \rfloor)\}$	$\lceil (\lfloor \frac{n}{2} \rfloor - x) \frac{1}{2} \rceil$

Figure 6: Distinct independent pair events for x in S.

Theorem 8. *Let $n \in \mathbb{N}$, $S = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. $\forall x \in S$, there are at least $f(n) = \lfloor \frac{n}{8} \rfloor + 1$ pair events in S, $\{(a_1, b_1), \ldots, (a_{f(n)}, b_{f(n)})\}$, such that $a_1, \ldots, a_{f(n)}, b_1, \ldots, b_{f(n)}$ are all distinct elements of S.*

Proof. Fix $v_0 \in K_n$ as 0, and consider some $x \in S$ as a target, corresponding to v_x. Figure 6 presents the number of distinct independent pair events for $x \in S$. We denote the classes of event pairs by the order of their rows in Figure 6 as 1, 2, and 3. There exists one more possibility: that x itself is an element of $A \subseteq S$. Grouping this single element event with class 1, we define $f_1(n, x) = \lfloor \frac{x}{2} \rfloor + 1$, $f_2(n, x) = \max\{\lfloor \lfloor \frac{n}{2} \rfloor - \lceil \frac{n-x}{2} \rceil \rfloor, \lfloor \lfloor \frac{n-x}{2} \rfloor - 1 \rfloor\}$, and $f_3(n, x) = \lceil (\lfloor \frac{n}{2} \rfloor - x)\frac{1}{2} \rceil$.

We note that f_3 is weakly decreasing in x and f_1 is weakly increasing in x, and hence the intersection of the two functions will yield a lower bound for the number of independent pair events for x. We define a function $f(n, x)$ which selects the greatest number of independent pairs depending on x, as below:

$$f_0(n, x) = \begin{cases} f_1(n, x) & \text{if } x \geq \lfloor \frac{n}{4} \rfloor \\ f_3(n, x) & \text{if } x < \lfloor \frac{n}{4} \rfloor \end{cases}$$

This will bound the smallest number of independent pair events. The minimum of this function on S is achieved when $\lfloor \frac{x}{2} \rfloor + 1 = \lceil (\lfloor \frac{n}{2} \rfloor - x)\frac{1}{2} \rceil$, which occurs when $x = \lfloor \frac{n}{4} \rfloor$. The minimum for this function will be equivalent to $\lfloor \frac{n}{8} \rfloor + 1$, and thus there exist at least $f(n) = \lfloor \frac{n}{8} \rfloor + 1$ distinct pair events in S for each $x \in S$. $\square$

Definition (Independent sorting process)**.** We utilize the following process to sort elements of S into k distinct partitions: we randomly assign each element of S to a partition of A_i with probability $\frac{1}{k}$ that a particular partition will end up containing a particular element.

Hence, from the perspective of the partition itself each element will be placed independently with probability $\frac{1}{k}$.

Theorem 9. *For $k \in \mathbb{N}$ and $n \in \mathbb{N}$, let $A_1, \ldots, A_k$ be defined according to the independent sorting process. Then*

$$\lim_{n \to \infty} P(\operatorname{diam}(C_n(A_i)) = 2 \textit{ for all } i \in \{1, \ldots, k\}) = 1.$$

In particular, the probability that at least one of $A_1, \ldots, A_k$ is not of diameter 2 is bounded above by

$$k \left\lfloor \frac{n}{2} \right\rfloor \left[1 - \left(\frac{1}{k} \right)^2 \right]^{\lfloor \frac{n}{8} \rfloor + 1}$$

Proof. We approach this using a probabilistic argument. Consider the probability of being able to reach some $x \in S$, given $A_i \subseteq S$. Since we are assuming independent sorting of elements of S into partitions with probability $\frac{1}{k}$, the probability that a single fixed pair event fails will be bounded above by $1 - \left(\frac{1}{k}\right)^2$. Since we have at least $f(n) = \lfloor \frac{n}{8} \rfloor + 1$ independent events, we know that

$$P(\text{all the events fail for } x) \leq \left[1 - \left(\frac{1}{k}\right)^2\right]^{\lfloor \frac{n}{8} \rfloor + 1}$$

By this fact and noting that there are exactly $\lfloor \frac{n}{2} \rfloor$ elements in S,

$$P(\text{at least one } x \in S \text{ is inaccessible in } A_i) \leq \left\lfloor \frac{n}{2} \right\rfloor \left[1 - \left(\frac{1}{k}\right)^2\right]^{\lfloor \frac{n}{8} \rfloor + 1}$$

and subsequently

$$P\left(\begin{array}{c}\text{at least one } x \in S \text{ will be inaccessible} \\ \text{in some } A_1, \ldots, A_k\end{array}\right) \leq k \left\lfloor \frac{n}{2} \right\rfloor \left[1 - \left(\frac{1}{k}\right)^2\right]^{\lfloor \frac{n}{8} \rfloor + 1}.$$

Hence,

$$\lim_{n \to \infty} P(\operatorname{diam}(C_n(A_i)) = 2 \text{ for all } i \in \{1, \ldots, k\}) = 1$$

□

In particular, we have just shown that for each $k \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that K_n can be edge-partitioned into k disjoint cyclically symmetric subgraphs, each of which has diameter 2. We can now conclude with the following theorem.

Theorem 10. *For each $k \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that K_n can be edge-partitioned into k cyclically symmetric $2D3C$ subgraphs. That is, there exists some n large enough such that K_n is $k-$Class 0.*

Proof. Theorem 9 gives a probabilistic bound on the diameter 2 condition. To bound the 3-connectivity condition, consider the case of n prime, $n \geq 11$, $S = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and $A_1, \ldots, A_k$ defined according to the random sorting process.

It follows from Lemma 4 that restricting ourselves to prime n, for a partition of S into $A_1, \ldots, A_k$ according to our independent sorting process, as long as for each $i \in \mathbb{N}$, A_i has more than one element, then K_n is partitionable into k 3-connected subgraphs. Note that according to our sorting process,

$$P(|A_i| \leq 1) = \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{k} \left(\frac{k-1}{k}\right)^{\lfloor \frac{n}{2} \rfloor - 1} + \left(\frac{k-1}{k}\right)^{\lfloor \frac{n}{2} \rfloor}$$

Moreover, by the same logic as in the previous theorem,

$$P(|A_i| \leq 1 \text{ for at least one } i) = \left\lfloor \frac{n}{2} \right\rfloor \left(\frac{k-1}{k}\right)^{\lfloor \frac{n}{2} \rfloor - 1} + k \left(\frac{k-1}{k}\right)^{\lfloor \frac{n}{2} \rfloor}$$

Combining the result from Theorem 9,

$$P\left(\ |A_i| \leq 1 \text{ for at least one } i \quad OR \quad \begin{array}{c} \text{at least one } x \in S \text{ will be inaccessible} \\ \text{in at least one of the } k \text{ subsets} \end{array} \right)$$

$$\leq \left\lfloor \frac{n}{2} \right\rfloor \left(\frac{k-1}{k} \right)^{\lfloor \frac{n}{2} \rfloor - 1} + k \left(\frac{k-1}{k} \right)^{\lfloor \frac{n}{2} \rfloor} + k \left\lfloor \frac{n}{2} \right\rfloor \left[1 - \left(\frac{1}{k} \right)^2 \right]^{\lfloor \frac{n}{8} \rfloor + 1}$$

There exists a prime n_0 large enough such that

$$\left\lfloor \frac{n}{2} \right\rfloor \left(\frac{k-1}{k} \right)^{\lfloor \frac{n}{2} \rfloor - 1} + k \left(\frac{k-1}{k} \right)^{\lfloor \frac{n}{2} \rfloor} + k \left\lfloor \frac{n}{2} \right\rfloor \left[1 - \left(\frac{1}{k} \right)^2 \right]^{\lfloor \frac{n}{8} \rfloor + 1} \leq 1$$

and hence there exists by probabilistic argument a partition of K_{n_0} into k symmetric $2D3C$ subgraphs. □

6 Future steps

In Figure 7, for selected k we note the smallest k-Class 0 complete graph guaranteed by Theorem 10, along with the smallest we have found by inspection.

Number of partitions	Best by asymptotic bound	Best known
2	137	9
3	467	19
4	929	43
10	8501	?

Figure 7: For fixed number of partitions k, existence result implied by theorem

It is evident that the existence result is not by any means a tight bound. Among possible future steps, we hope to:

(i) Reconcile the gap between smallest observed k-Class 0 graphs and the ones guaranteed by probabilistic argument. A first step would be to more cleanly address the 3-connectivity result.

(ii) Shift focus from complete graphs to large random graphs, and ask the same questions.

(iii) Investigate how strong products and vertex splitting affect general Class 0 graphs, and specifically assess the conjecture that vertex splitting preserves Class 0 on graphs with two vertices or more.

(iv) Begin to analyze strategies and characteristics of the multicolor pebbling game.

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