DIMACS Technical Report 2005-08
February 2005

On Cyclically Orientable Graphs

by

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ABSTRACT

Graph G is called cyclically orientable (CO) if it admits an orientation in which every simple chordless cycle is cyclically oriented. This family of graphs was introduced by Barot, Geiss, and Zelevinsky in their paper "Cluster algebras of finite type and positive symmetrizable matrices", *J. London Math. Soc.* **73** Part 3 (2006), 545-564. The authors obtained several nice characterizations of CO-graphs, being motivated primarily by their applications in cluster algebras. Here we obtain several new characterizations that provide algorithms for recognizing CO-graphs and obtaining their cyclic orientations in linear time. We show that CO-graphs are edge maximal 2-trees; that is, $G = (V, E)$ is a 2-tree if and only if G is CO and $G' = (V, E')$ is not CO whenever E is a proper subset of E'.

Keywords: cluster algebra, graph, chromatic number, planar graph, series-parallel graph, cycle, chord, chordless cycle, orientation, cyclic orientation, 2-tree.

1 Introduction

In this paper we consider only the graphs without loops and parallel edges. Given a graph G, a cycle C in G is called *chordless* if every edge of G connecting two vertices of C belongs to C. Similarly, we define chordless paths in G.

A graph G is called *cyclically orientable* (CO) if it admits an orientation in which every simple chordless cycle is cyclically oriented. Such an orientation we will also call *cyclic*. CO-graphs were introduced in [1], where it is shown that the following claims (1-3) are equivalent:

(1) Graph G is CO.

(2) The edges of G can be linearly ordered so that different chordless cycles of G have different maximal edges.

Equivalently, in terms of [2] condition (2) means that, the hypergraph H whose vertices are the edges of G and hyperedges are the chordless cycles of G is 1-degenerate.

(3) $|Cyc| = |Edg| - |Ver| + |Con|$

where $Cyc = Cyc(G), Edg = Edg(G), Ver = Ver(G)$, and $Con = Con(G)$ are respectively the sets of chordless cycles, edges, vertices, and connected components of G.

It is also shown in [1] that the corresponding inequalities $(2', 3')$ hold for an arbitrary graph G.

$(2')$ The edges of G can be linearly ordered so that an edge is maximal in a chordless cycle if and only if it is one of the last $|Edg| - |Ver| + |Con|$ edges of the list;

$(3')$ $|Cyc| \geq |Edg| - |Ver| + |Con|$.

The above characterizations of CO-graphs have important applications in cluster algebras. However, these characterizations (including the definition) are not constructive; that is, they provide neither efficient recognition algorithms nor cyclic orientations for CO-graphs. To get such algorithms and orientations we will derive several new characterizations.

The same results were obtained independently by David E. Speyer; cf. [6] and [3]. Also, one of the referees noticed that these results are closely related to [5], Proposition 5.10 of section 5.

2 Characterization, recognition, and cyclic orientation of CO-graphs

2.1 Cyclically orientable graphs and 2-trees

It appears that CO-graphs naturally generalize so-called 2-trees.

Obviously, all trees can be generated by the following recursion. Start with one vertex and, in general, given a tree T, choose a vertex a in T, a new vertex b, and add the new edge (a, b) to T.

By definition, all 2-trees are recursively generated as follows. Start with one edge and, in general, given a 2-tree $\mathcal{T}$, choose an edge (a, b) in $\mathcal{T}$, a new vertex c, and add two new edges (a, c) and (b, c) to $\mathcal{T}$.

CO-graphs are characterized similarly by the following, a little bit more general, recursion suggested in [6]. Given a graph G, choose an edge (a, b) in G and add to G a simple path p between a and b that contains at least 3 vertices, all new, except a and b.

Let us denote the graph obtained from G by $G' = R(G, p, a, b)$.

Clearly, the path p *must* be of length at least 2. In other words, p must contain at least 3 vertices and 2 edges, since otherwise an edge parallel to (a, b) would appear in G'.

We start with a single edge and apply operation R recursively. If in each step of the recursion p consists of 2 edges, (a, c) and (c, b), then we obtain 2-trees. Two examples are given in the Figure. In general, we get 2-connected CO-graphs. More precisely, the following claim holds.

Theorem 1 *([6]) (i) A graph G is CO if and only if each its 2-connected component is CO.*

(ii) Beginning with a single edge and applying the operation R recursively, one obtains a 2-connected CO-graph.

(iii) Each 2-connected CO-graph can be obtained in this way.

Proof of (i) and (ii). Claim (i) is obvious, since, given a graph G and a simple cycle C (chordless or not) in G, all vertices and edges of C belong to one 2-connected component of G.

Lemma 1 *The graph $G' = R(G, p, a, b)$ is CO (respectively, 2-connected) if and only if G is CO (respectively, 2-connected).*

Yet, to be precise, we should mention an obvious exception. If G consists of a single edge (a, b) then G' is a simple cycle. Hence, in this case G' is 2-connected, while G is not.

Proof of the Lemma. Each edge e of p belongs to a unique chordless cycle. Indeed, this cycle is formed by p and (a, b), while every other cycle through e has a chord, namely, (a, b).

Both claims of the Lemma follow from this observation. □

Clearly, this Lemma implies (ii). Again, to be precise, we should mention that, beginning with a single edge, one should apply the operation R at least once to get a 2-connected CO-graph. □

Claim (iii) is more difficult. We will prove it in Section 5. Now let us derive some corollaries.

2.2 Recognizing CO-graphs and getting their cyclic orientation

In this sections we will invert the recursive procedure of Theorem 1 to get a recognition algorithm for the CO-graphs. By Theorem 1 (i), graph G is CO if and only if all its 2-connected components are CO. Hence, without loss of generality, we can assume that G is 2-connected.

Suppose that $G = R(G', p, a, b)$, where G' is a reduced graph. Then, by Lemma 1, G' is 2-connected (unless it consists of a single edge), because G is 2-connected, and, moreover, G is CO if and only if G' is CO. Hence, the recognition problem for G is reduced to the same problem for a smaller graph G'. On the other hand, by Theorem 1 (iii), every CO-graph G can be represented as $G = R(G', p, a, b)$. In other words, if there is no such a representation then G is not CO. We will either find a representation $G = R(G', p, a, b)$ or prove that it does not exist. Denote by A the set of all vertices of degree 2 in G. These vertices may form a cycle C. Clearly, C is chordless, since all its vertices are of degree 2. Furthermore, since G is 2-connected, each of its vertices is in C; that is, $G = C$. Hence, in this case G is a CO-graph.

If the vertices of A do not form a cycle then they must form several disjoint paths in G. Every such path p_i has exactly two adjacent vertices, a_i and b_i, where $i \in I$ are some indices. If for an $i \in I$ the pair a_i, b_i is an edge of G, then $G = R(G', p_i, a_i, b_i)$, and we get a desired representation. Indeed, in this case, we get G' by deleting all vertices of p_i from G. If no pair a_i, b_i is an edge of G then, by Theorem 1 (iii), G is not CO.

Repeating the above procedure recursively we either prove that G is not CO, or we decompose G according to Theorem 1 (ii).

In the latter case not only do we prove that G is CO but we also obtain its cyclic orientation. In fact, there are exactly two feasible orientations, since the original single edge can be directed both ways. Let us prove by induction that in each further step there exists a unique extension of the obtained cyclic orientation. Indeed, a cyclic orientation of G' defines a unique cyclic orientation of $G = R(G', p, a, b)$, since a given orientation of the edge (a, b) defines a unique orientation of the path p; see Lemma 1.

In general, the number of cyclic orientations is equal to $2^{k+\ell}$ for a CO-graph that consists of k 2-connected components and ℓ extra edges that belong to none of them (in other words, these edges do not belong to simple cycles). Indeed, there exist exactly two orientations for each such edge and component.

2.3 A reformulation of the main Theorem

In [3], Theorem 1 is presented in another way. Given an arbitrary graph $G = (V, E)$, define a bipartite graph $B = B(G) = (V', E')$ as follows. To each chordless cycle C of G assign a vertex of type 1 and to each edge $e \in E$ assign a vertex of type 2. Let $V' = V_1 \cup V_2$ be the set of these vertices and let two vertices $v_1 \in V_1$ and $v_2 \in V_2$ be connected by an edge $e \in E'$ if and only if the chordless cycle corresponding to v_1 contains the edge corresponding to v_2.

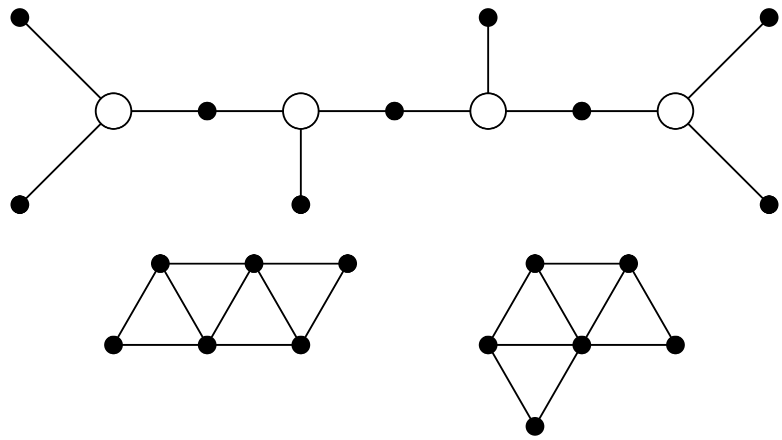
By definition, the obtained graph B is bipartite. It is also clear that $deg(v) \geq 3$ for every $v \in V_1$, since each cycle has at least 3 edges.

Theorem 2 *(i) A graph G is CO if and only if $B(G)$ is a forest.*

(ii) A graph G is 2-connected and CO if and only if $B(G)$ is a tree.

After a few remarks, we will show that Theorems 1 and 2 are equivalent.

First, let us note that two non-isomorphic 2-connected CO-graphs may generate the same tree. An example is given in the following Figure, where vertices of types 1 and 2 are colored white and black, respectively.



Let us also note that all leaves of B are from V_2, since $deg(v) \geq 3$ for every $v \in V_1$.

It follows from Theorem 2 that any two chordless cycles of G may have at most one common edge; otherwise a simple 4-cycle would appear in $B(G)$.

As we already mentioned, given an arbitrary graph G and a simple cycle C (chordless or not) in it, all vertices and edges of C belong to exactly one 2-connected component of G. In other words, the 2-connected components of G are in one-to-one correspondence with the connected components of $B(G)$. Hence, the second claim of Theorem 2 implies the first one.

Let us show that Theorem 2 (ii) is equivalent to Theorem 1 (ii) and (iii). To see this we follow the recursion of Theorem 1. We proceed by induction on the number of recursive steps. On the first step, given a single edge (a, b), we add a simple path p connecting a and b and get a simple (chordless) cycle C. The corresponding graph B is a star, where C is its center and the edges of C are its leaves.

In general, given a 2-connected CO-graph $G = (V, E)$, we extend it to $G' = R(G, p, a, b)$ as follows: choose an arbitrary edge $(a, b) \in E$ and add to G a simple path p of length $|p| \geq 2$ connecting a and b. By this, we add to G one more chordless cycle C formed by p and (a, b), and $|p|$ new edges of the path p.

By the induction hypothesis, there is a unique tree B whose vertices $V = V_1 \cup V_2$ are the chordless cycles and the edges of G. Let us add to B a star, whose center is C, one vertex is (a, b), and the remaining vertices are the edges of p; they are new leaves of the obtained tree B'. Obviously, B' corresponds to G'. Thus, we complete the induction and conclude that Theorems 1 and 2 are equivalent. We will prove Theorem 2 in Section 5.

3 CO-graphs, 2-trees, planar, and series-parallel graphs

In this section we derive more corollaries of Theorems 1 and 2.

It is well known (and obvious) that every 2-tree with n vertices has exactly $2n-3$ edges. By this and Theorem 1, we obtain the following bounds for CO-graphs:

(4) $|Edg| \leq 2|Ver| - 3,$

and furthermore, by (3),

(5) $|Cyc| \leq |Ver| - 3 + |Con|.$

In particular, for a 2-connected CO-graph

(6) $|Cyc| \leq |Ver| - 2.$

Thus, the number of edges of a CO-graph is linear in the number of its vertices, whereas the number of its chordless cycles is less than the number of its vertices.

A graph is called *planar* if it can be drawn in the plane without crossings. A graph is called *series-parallel* if it contains no subgraph isomorphic to a subdivision of K_4. By the Kuratowsky theorem, every series-parallel graph is planar. By Theorem 1, every CO-graph is series-parallel.

A family of graphs $\mathcal{F}$ is called *anti-monotone* if $G' \in \mathcal{F}$ whenever $G \in \mathcal{F}$ and G' is a subgraph of G. Obviously, the families of planar graphs and series-parallel graphs are anti-monotone. Yet, the family of CO-graphs is not. For example, the complete bipartite graph $K_{2,3} = (V, E)$, where $V = \{a_1, a_2; b_1, b_2, b_3\}$ and $E = \{(a_i, b_j); i = 1, 2; j = 1, 2, 3\}$ is not CO, yet, by adding to $K_{2,3}$ one more edge (a_1, a_2) one obtains a CO-graph. Both these observations follow from Theorem 1.

A graph $G = (V, E)$ from a family $\mathcal{F}$ is called *edge-maximal* in $\mathcal{F}$ if $G' = (V, E')$ is not in $\mathcal{F}$ whenever E' is a proper superset of E.

Theorem 3 *The following five properties of a graph G are equivalent:*

(a) *G is an edge-maximal CO-graph;*
(b) *G is an edge-maximal series-parallel graph;*
(c) *G is a 2-tree;*
(d) *G is series-parallel and $|Edg(G)| = 2|Ver(G)| - 3$;*
(e) *G is CO and $|Edg(G)| = 2|Ver(G)| - 3$.*

Proof . It is well known that the properties (b),(c), and (d) are equivalent. Hence, it is enough to show that (a),(c), and (e) are equivalent too. This follows from Theorem 1. Indeed, if each path p in the recursion consists of two edges then all three claims (a),(c), and (e) hold; if a longer path appears at least once then all three claims fail. $\square$

Remark 1 *It is well known that for planar graphs the following similar properties are equivalent:*

(f) *G is an edge-maximal planar graph;*
(g) *G is a triangulated planar graph;*
(h) *G is planar and $|Edg(G)| = 3|Ver(G)| - 6$.*

Remark 2 *It is known that the number of labelled 2-trees on n vertices is equal to $\binom{n}{2}(2n-3)^{n-4}$.*

4 Subdivisions of K_4 and $K_{2,3}$

Obviously, 4-clique K_4 is not CO. Hence, if G is CO then

(7) G does not contain K_4.

In the next section we will prove that each CO-graph $G = (V, E)$ has the following property:

(8) Given two vertices $a, b \in V$ and three paths between a and b such that they have no other common vertices, except a and b, then a and b are adjacent in G; that is, $(a, b) \in E$.

We will refer to (8) as the *three paths property.* In particular, (8) implies that no subdivision of $K_{2,3}$ is CO, since three paths in (8) form such a subdivision. Yet, unlike K_4, it can be a subgraph of a CO-graph. Indeed, adding the edge (a, b) to the three paths of (8), one gets a CO-graph. We already mentioned this in Section 3 to prove that the family of CO-graphs is not anti-monotone.

Furthermore, (8) does not imply (7), since K_4 satisfies (8). However, any non-trivial subdivision of K_4 does not satisfy (8). Hence, (7) and (8) imply that G contains no subdivision of K_4. In other words, CO-graphs are series-parallel. Let us also recall that series-parallel graphs are 3-colorable. This follows from the Hadwiger Conjecture for k=4. In general, this conjecture claims that a graph G is $(k-1)$-colorable whenever G contains no k-clique K_k as a minor. For $k = 4$ this was proved by Hadwiger in [4], and in fact, even earlier, by Wagner in [7].

In the next section we will prove the following theorem.

Theorem 4 *A graph G is CO if and only if G is K_4-free and it satisfies the three path property; that is, (1) $\Leftrightarrow$ (7) $\wedge$ (8).*

Conjecture 1 *The following generalization of the three path property could be of interest. For any integer $n \geq 2$ let us introduce a family $\mathcal{F}_n$ satisfying the following two properties:*

(i) G does not contain (a subdivision of) K_{n+1} and

(ii) given two vertices $a, b \in V$ and n paths between a and b that, except a and b, have no other pairwise common vertices, then a and b are adjacent, that is, $(a, b) \in E$.

Obviously, $\mathcal{F}_n$ consists of all forests and CO-graphs for $n = 2$ and 3 respectively. Is it true that $\chi(G) \leq n$ for each $G \in \mathcal{F}_n$? For $n \leq 3$ it is.

5 Proofs

Let us recall that, by Theorem 2, if G is CO then

(9) $B(G)$ is a forest.

In this section we prove the chain of implications (1) $\Rightarrow$ (7) $\wedge$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1) for an arbitrary graph G. By this we complete the proofs of Theorems 2 and 4 and we already know that Theorems 1 and 2 are equivalent.

Lemma 2 *CO-graphs contain no subdivisions of K_4; that is,* (1) $\Rightarrow$ (7).

Proof is indirect. Obviously, K_4 itself is not CO, and hence (1) $\Rightarrow$ (4). Assume that a CO-graph G contains a subdivision $\mathcal{K}_4$ of K_4 as a subgraph, not necessarily an induced one. Without loss of generality, let us assume that this subdivision is minimal; that is, each subdivision of K_4 in G contains at least as many vertices as $\mathcal{K}_4$. Let v_0, v_1, v_2, v_3 be 4 vertices of K_4 and $p(v_i, v_j)$ be 6 paths between them, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Except v_0, v_1, v_2, v_3, these paths have no other common vertices, and they are chordless, by minimality of $\mathcal{K}_4$. Yet, there must be chords in $\mathcal{K}_4$, since otherwise G is not CO. Moreover, already 3 (out of 4) chordless cycles of $\mathcal{K}_4$ (containing v_0) cannot be cyclically oriented.

Case 1. There is no chord through v_0 in $\mathcal{K}_4$. Let us show that in this case there exist 3 chordless cycles through v_0 that are not CO. Consider 3 cycles $C_1 = (p(v_0, v_2), p(v_2, v_3), p(v_3, v_0))$, $C_2 = (p(v_0, v_3), p(v_3, v_1), p(v_1, v_0))$, and $C_3 = (p(v_0, v_1), p(v_1, v_2), p(v_2, v_0))$. They may have chords. However, a chord divides C_i in two cycles C'_i and C''_i one of which, say C'_i, contains v_0. Clearly, there is a chord in C_i such that C'_i is chordless and 3 cycles C'_i, $i \in \{1, 2, 3, \}$ are not CO.

Case 2. There is a chord through v_0 in $\mathcal{K}_4$. In this case, by minimality, $\mathcal{K}_4$ is a "wheel", i.e. it contains the cycle $C = (p(v_1, v_2), p(v_2, v_3), p(v_3, v_1))$, vertex v_0 and several, $k \geq 4$, chords between v_0 and C. By minimality of $\mathcal{K}_4$, cycle C is chordless and each path $(p(v_0, v_i)$, $i \in \{1, 2, 3\}$, is of length 1, that is, just one edge. There are $k + 1$ chordless cycles: C and k cycles through v_0. Obviously, they cannot be cyclically oriented. If k is odd then already k cycles through v_0 are not CO. □

Lemma 3 *CO-graphs satisfy the three paths property; that is,* (1) $\Rightarrow$ (8).

Proof is indirect. Assume that there is a CO-graph G that contains a subdivision $\mathcal{K}_{2,3}$ of $K_{2,3}$, or in other words, G contains 2 vertices a and b and 3 paths p_1, p_2, p_3 between them such that: (i) each path is of length at least 2, (ii) except a and b, these paths have no other common vertices. We also suppose that (iii) (a, b) is not a chord in $\mathcal{K}_{2,3}$. Again, without loss of generality, we can assume minimality of $\mathcal{K}_{2,3}$. Then paths p_1, p_2, p_3 are chordless. If there is a chord between two distinct paths then we get a subdivision of K_4, in contradiction to Lemma 2. Otherwise, if there is no chord in $\mathcal{K}_{2,3}$, then 3 cycles $C_1 = (p_2, p_3)$, $C_2 = (p_3, p_1)$, and $C_3 = (p_1, p_2)$ are chordless but not CO. □

Lemma 4 *Two chordless cycles of a CO-graph G may have at most one edge in common.*

Proof . A simple case analysis shows that if there are two edges in common then G does not have the three paths property and hence cannot be CO. □

It is also easy to show that, more precisely, two chordless cycles of a CO-graph may have in common either (i) an edge, or (ii) a vertex, or (iii) nothing. In case (i) we call these two cycles *adjacent*. Clearly, in this case, the corresponding two vertices of type 1 are adjacent to the same vertex of type 2 in $B(G)$.

Lemma 5 *Chordless cycles of a CO-graph G cannot form a cycle.*

Proof . Assume indirectly that chordless cycles $C_1, \ldots, C_n$ form a cycle; that is, for each $i \in \{1, \ldots, n\}$ cycles C_i and C_{i+1} have a common edge e_i and the corresponding n vertices of type 1 and n vertices of type 2 form an alternating $2n$ cycle in $B(G)$. (As usual we set $n + 1 = 1$ and $1 - 1 = 0 = n$.)

Case 1. For some $i \in \{1, \ldots, n\}$ the cycle C_i is of length at least 4. Let (a, c) be a common edge of C_i and C_{i+1} and (b, d) be a common edge of C_i and C_{i-1}. Since $|C_i| \geq 4$, we can assume that a and b are not adjacent (though c and d may coincide). Clearly, there are 3 vertex disjoint paths between a and b and hence the three paths property does not hold for G.

Case 2. All n cycles are triangles. Consider a pair of adjacent cycles $C_i = (a, c, d)$ and $C_{i+1} = (b, c, d)$. Clearly, a and b are not adjacent, since otherwise G contains a K_4. It is also clear that there are 3 vertex disjoint paths between a and b and the three paths property does not hold for G in this case either. $\square$

Obviously, the last two Lemmas imply that $B(G)$ is a forest whenever G is CO. The inverse implication holds too.

Lemma 6 *G is CO whenever $B(G)$ is a forest; that is,* $(9) \Rightarrow (1)$.

Proof is constructive. If $B(G)$ is a forest then we get a cyclic orientation of G as follows. An orientation of an edge e in G uniquely defines the orientations of all chordless cycles of G which contain e. Vice versa, an orientation of a chordless cycle C in G trivially defines the orientations of all its edges. Thus traversing $B(G)$ we get a cyclic orientation of G. Since $B(G)$ is a forest, we can always avoid contradictory orientations. $\square$

Acknowledgments. I am thankful to Andrei Zelevinsky who brought to my attention the CO-graphs and some unpublished results in this area, to David Speyer for the reference to his preprint, to Endre Boros for helpful discussions, and to Katie D'Agosta and Diogo Andrade for technical assistance.

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