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Extended $(P_5, \overline{P}_5)$-free graphs

by

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ABSTRACT

Let G and H be graphs. A *substitution* of H in G instead of a vertex $v \in V(G)$ is the graph $G(v \to H)$, which consists of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$.
For a hereditary class of graphs $\mathcal{P}$, the *substitutional closure* of $\mathcal{P}$ is defined as the class $\mathcal{P}^*$ consisting of all graphs which can be obtained from graphs in P by repeated substitutions.
We characterize the substitutional closure $(P_5 \cup K_1, \overline{P}_5 \cup K_1)$-free graphs in terms of forbidden induced subgraphs.
The weighted stability problem is polynomially solvable for this class.

1 Introduction

The *neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = N(x)$ of all vertices in G that adjacent to x.

Definition 1. *Let G and H be graphs. A* substitution of H in G instead of a vertex *$v \in V(G)$ is the graph $G(v \to H)$ consisting of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$.*

Definition 2. *For a class $\mathcal{P}$ of graphs, its* substitutional closure *$\mathcal{P}^*$ consists of all graphs that can be obtained form $\mathcal{P}$ by repeated substitutions, i.e., $\mathcal{P}^*$ is generated by the following rules:*

(S1) *$\mathcal{P} \subseteq \mathcal{P}^*$;*

(S2) *if $G, H \in \mathcal{P}^*$ and $v \in V(G)$, then $G(v \to H) \in \mathcal{P}^*$.*

Let $\mathrm{ISub}(G)$ be the set of all induced subgraphs of a graph G (considered up to isomorphism). A class of graphs $\mathcal{P}$ is called *hereditary* if $\mathrm{ISub}(G) \subseteq \mathcal{P}$ for every $G \in \mathcal{P}$.

For a set of graphs Z, the class of *Z-free* graphs consists of all graphs G such that $\mathrm{ISub}(G) \cap Z = \emptyset$.

Proposition 1. *If $\mathcal{P}$ is a hereditary class then $\mathcal{P}^*$ is also a hereditary class.*

Open Problem 1 (De Simone [1]). *For a hereditary class $\mathcal{P}$ given by a set Z of forbidden induced subgraphs, find a forbidden induced subgraph characterization of the substitutional closure $\mathcal{P}^*$.*

Now we discuss Reducing Copath Method proposed by I. Zverovich [4] for solving Problem 1 for an arbitrary hereditary class.

Definition 3. *A set $W \subseteq V(G)$ is called* homogeneous *in a graph G if*

(H1) *$2 \le |W| \le |V(G)| - 1$, and*

(H2) *$N(x) \backslash W = N(y) \backslash W$ for each $x, y \in W$.*

According to (H2), a homogeneous set W defines a partition

$$W \cup W^+ \cup W^- = V(G)$$

such that

- every vertex of W is adjacent to every vertex of W^+ (notation $W \sim W^+$), and
- every vertex of W is non-adjacent to every vertex of W^- (notation $W \not\sim W^-$).

By (H1), $W^+ \cup W^- \neq \emptyset$ for every homogeneous set W.

Definition 4. *A graph without homogeneous sets is called* prime*.*
A graph H is called a (primal) extension *of a graph G if*

(E1) *G is an induced subgraph of H, and*

(E2) *H is a prime graph.*

Definition 5. *An extension H of G is* minimal *if there are no extensions of G in the set* $\mathrm{ISub}(H)\backslash\{H\}$*.*
We denote by $\mathrm{Ext}(G)$ *the set of all minimal extensions of a graph G.*

Definition 6. *For a set of graphs Z we put*

$$\mathrm{Ext}(Z) = \bigcup_{G \in Z} \mathrm{Ext}(G),$$

and we define Z^o as the set of all minimal graphs in $\mathrm{Ext}(Z)$ *with respect to the partial order 'to be an induced subgraph'.*

Proposition 2. *If Z is the set of all minimal forbidden induced subgraphs for a hereditary class $\mathcal{P}$ then Z^o is the set of all minimal forbidden induced subgraphs for $\mathcal{P}^*$.*

The notation $x \sim y$ (respectively, $x \not\sim y$) means that x and y are adjacent (respectively, non-adjacent). For disjoint sets X and Y, the notation $X \sim Y$ (respectively, $X \not\sim Y$) means that every vertex of X is adjacent to (respectively, non-adjacent) to every vertex of Y. In case of $X = \{x\}$ we also will write $x \sim Y$ and $x \not\sim Y$ instead of $\{x\} \sim Y$ and $\{x\} \not\sim Y$, respectively.

Definition 7. *Let G be an induced subgraph of a graph H, and let W be a homogeneous set of G.*
We define a reducing W-copath (with respect to G) *in H as a sequence*

$$R = (u_1, u_2, \ldots, u_t), t \geq 1,$$

of pairwise distinct vertices of $V(H)\backslash V(G)$ satisfying the following conditions:

(R1) *there exist vertices $w_1, w_2 \in W$ such that*

> **(R1a)** $u_1 \sim w_1$*, and*
>
> **(R1b)** $u_1 \not\sim w_2$*;*

(R2) *for each $i = 2, 3, \ldots, t$ either*

> **(R2a)** $u_i \sim u_{i-1}$ *and* $u_i \not\sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$*, or*

(R2b) *$u_i \not\sim u_{i-1}$ and $u_i \sim W \cup \{u_1, u_2, \dots, u_{i-2}\}$* [when $i = 2$, $\{u_1, u_2, \dots, u_{i-2}\} = \emptyset$];

(R3) *for every $i = 1, 2, \dots, t-1$*

(R3a) *$u_i \sim W^+$, and*

(R3b) *$u_i \not\sim W^-$;*

(R4) *either*

(R4a) *$u_t \not\sim x$ for a vertex $x \in W^+$, or*

(R4b) *$u_t \sim y$ for a vertex $y \in W^-$.*

Theorem 1 (Zverovich [4]). *Let H be an extension of its induced subgraph G, and let W be a homogeneous set of G. Then there exists a reducing W-copath (with respect to G) in H.*

2 A characterization of the substitutional closure of $(P_5 \cup K_1, \overline{P}_5 \cup K_1)$-free graphs

The graph $P_5 \cup K_1$ consists of the path P_5 with vertex set $\{a, b, c, d, e\}$ and an isolated vertex f. Similarly, the graph $\overline{P}_5 \cup K_1$ consists of the complement $\overline{P}_5$ of P_5 with vertex set $\{a, b, c, d, e\}$ and an isolated vertex f.

Giakoumakis and Rusu [2] probed that the weighted stability number problem is polynomially solvable for $(P_5, \overline{P}_5)$-free graphs. It follows that this problem also can be solved in polynomial time for $(P_5 \cup K_1, \overline{P}_5 \cup K_1)$-free graphs as well as for their substitutional closure.

Now we define a set of graphs

$$\mathcal{Z} = \mathcal{A}_1 \cup \mathcal{B}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_2 \cup \mathcal{A}_3 \cup \mathcal{B}_3$$

as follows.

- Each graph in $\mathcal{A}_1$ consists of one of the graphs $A_1, A_2, \dots, A_{18}$ (Figure 1) and an additional vertex f that is adjacent to u_1.

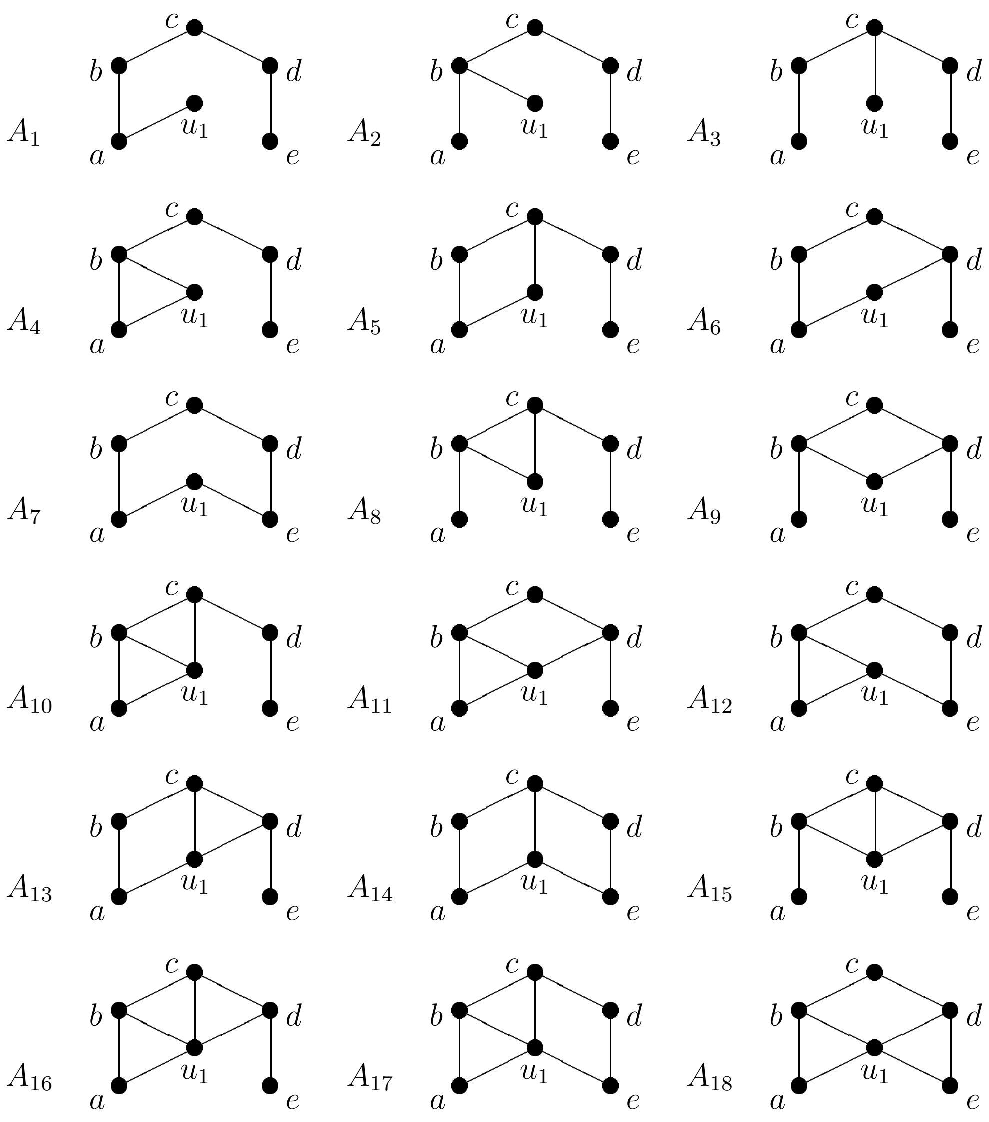

Figure 1. The graphs A_i

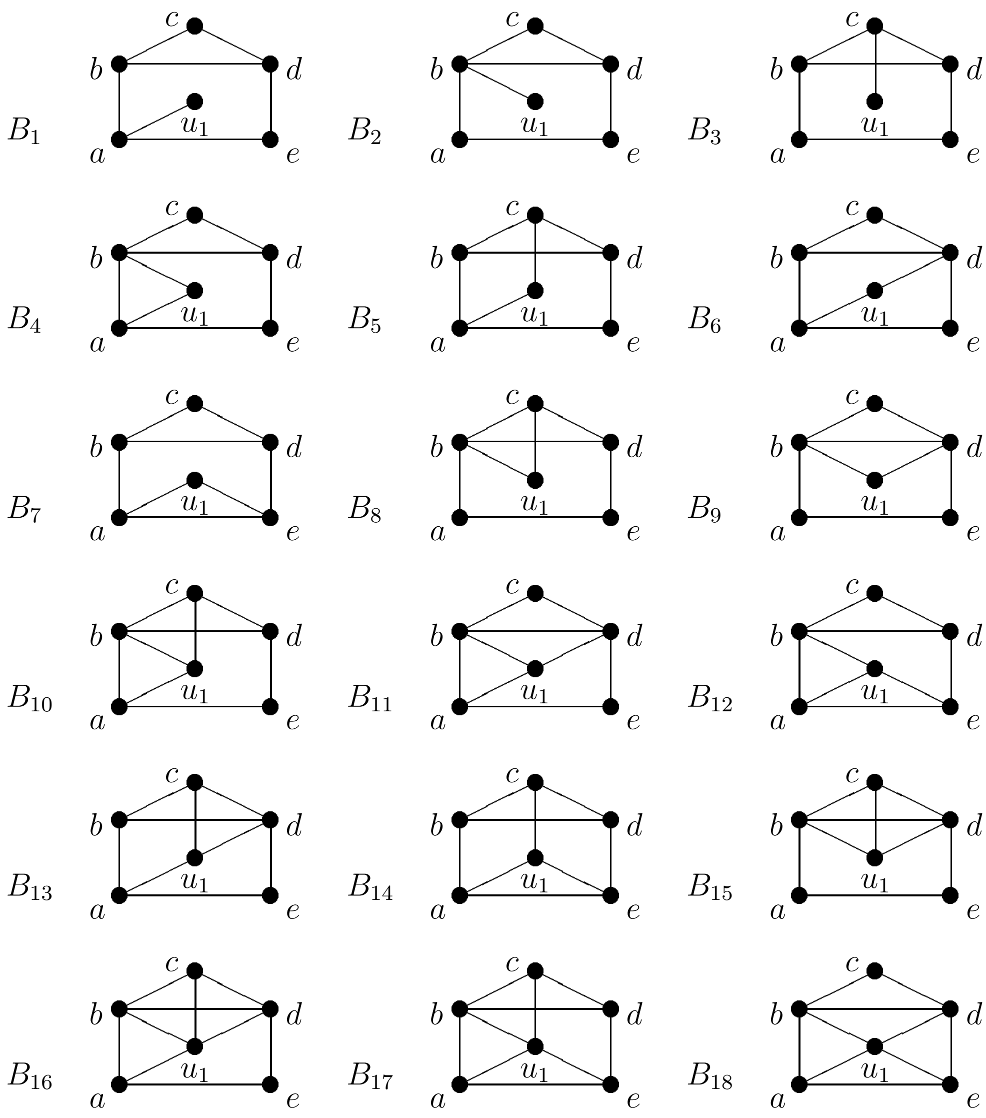

Figure 2. The graphs B_i

- Each graph in $\mathcal{B}_1$ consists of one of the graphs $B_1, B_2, \ldots, B_{18}$ (Figure 2) and an additional vertex f that is adjacent to u_1.
- Each graph in $\mathcal{A}_2$ consists of one of the graphs A_3, A_8, A_{16} (Figure 1) and two additional vertices f and u_2 with $N(f) = \{u_2\}$ and $N(u_2) = \{a, b, c, d, e, f\}$.
- Each graph in $\mathcal{B}_2$ consists of one of the graphs B_2, B_{11}, B_{18} (Figure 2) and two additional vertices f and u_2 with $N(f) = \{u_2\}$ and $N(u_2) = \{a, b, c, d, e, f\}$.
- Each graph in $\mathcal{A}_3$ consists of one of the graphs A_1, A_2, A_3, A_4, A_8 (Figure 1) and three additional vertices f, u_2 and u_3 with $N(f) = \{u_3\}$, $N(u_2) = \{a, b, c, d, e\}$ and $N(u_3) = \{a, b, c, d, e, f, u_1\}$.
- Each graph in $\mathcal{B}_3$ consists of one of the graphs $B_1, B_2, B_3, B_4, B_7, B_8, B_9, B_{15}$ (Figure 2) and three additional vertices f, u_2 and u_3 with $N(f) = \{u_3\}$, $N(u_2) = \{a, b, c, d, e\}$ and $N(u_3) = \{a, b, c, d, e, f, u_1\}$.

Thus, $\mathcal{A}_1$ contains 18 graphs, $\mathcal{B}_1$ contains 18 graphs, $\mathcal{A}_2$ contains 3 graphs, $\mathcal{B}_2$ contains 3 graphs, $\mathcal{A}_3$ contains 7 graphs, and $\mathcal{B}_3$ contains 8 graphs. Thus, $|\mathcal{Z}| = 55$.

Theorem 2. *The substitutional closure of $(P_5 \cup K_1, \overline{P}_5 \cup K_1)$-free graphs is defined by the set $\mathcal{Z}$ of its forbidden induced subgraphs.*

Proof. Every graph in $G \in \mathcal{Z}$ has the set $\{a, b, c, d, e, f\}$ that induces either $P_5 \cup K_1$ or $\overline{P}_5 \cup K_1$. Also, G is a prime graph. It follows that G is a minimal forbidden induced subgraph for the substitutional closure of $(P_5 \cup K_1, \overline{P}_5 \cup K_1)$-free graphs.

Now let G be a minimal forbidden induced subgraph for the substitutional closure of $(P_5 \cup K_1, \overline{P}_5 \cup K_1)$-free graphs. In particular, G contains a set $\{a, b, c, d, e, f\}$ that induces $H \in \{P_5 \cup K_1, \overline{P}_5 \cup K_1\}$ (f is an isolated vertex f of H). The set $W = \{a, b, c, d, e\}$ is homogeneous in H. Since G is a prime graph, there exists a reducing W-copath $R = (u_1, u_2, \ldots, u_t)$ in G.

We have $W^+ = \emptyset$ and $W^- = \{f\}$. According to (R4), u_t is adjacent to f.

If $t = 1$ then $u_1 = u_t$ satisfies (R1), i.e., u_1 is adjacent to some vertex of W, and u_1 is non-adjacent to some vertex of W. All possible variants (up to isomorphism) for the subgraph induced by $W \cup \{u_1\}$ are show in Figure 1 and Figure 2. In fact, $W \cup \{f, u_1\}$ induces a graph G' from $\mathcal{A}_1$. By minimality of G, $G = G'$, and we are done.

Thus, we may assume that $t \geq 2$.

Claim 1 Suppose that $t \geq 2$ and R has a vertex u_i ($i \in \{2, 3, \ldots, t\}$) that satisfies (R2a). Then $W \cup \{u_i\}$ induces a graph H' isomorphic to H, and $(u_1, u_2, \ldots, u_{i-1})$ is a reducing W-copath with respect to H'.

Proof. The condition (R2a) implies that $u_i \not\sim W$. Hence $H' \cong H$. Also, by (R2a) $u_i \sim u_{i-1}$, i.e., the condition (R4) holds for u_{i-1} (with respect to H'). It remains to note that each of the vertices $u_1, u_2, \dots, u_{i-2}$ is non-adjacent to u_{i-1} (by (R2a)), and therefore they satisfy (R3) (with respect to H'). $\square$

By Claim 1, we may assume that all the vertices $u_2, u_3, \dots, u_t$ of R satisfy (R2b).

Claim 2 Suppose that $t \geq 4$ and all the vertices of R satisfy (R2a). Then there exist non-adjacent vertices $x, y \in W$ such that $u_1 \sim x$ and $u_1 \not\sim y$, the set $W' = \{x, y, u_1, u_2, u_3\}$ in duces $\overline{P}_5$, and $(u_4, u_5, \dots, u_t)$ is a reducing W'-copath with respect to the subgraph $\overline{P}_5 \cup K_1$ induced by $W' \cup \{f\}$.

Proof. By (R1), u_1 is adjacent to some vertex $w_1 \in W$ and u_1 is non-adjacent to some vertex $w_2 \in W$. It follows that both $X = W \cap N(u_1)$ and $Y = W \backslash N(u_1)$ are nonempty. Note that $X \sim Y$ is impossible, since W induces either P_5 or $\overline{P}_5$. Thus, we can choose non-adjacent vertices $x \in X$ and $y \in Y$.

By (R3), $u_i \not\sim f$ $(i = 1, 2, \dots, t-1)$. Finally, according to (R2a), $u_i \not\sim u_{i-1}$ and $u_i \sim \{x, y, u_1, u_2, \dots, u_{i-2}\}$ for each $i = 2, 3, \dots, t$. $\square$

Claim 2 implies that we need only to consider $t \in \{2, 3\}$. Recall that all the vertices in $R \backslash \{u_1\}$ satisfy (R2b).

Case 1 $t = 2$ and $H \cong P_5$.

If $W \cup \{u_1\}$ induces one of A_3, A_8, A_{16} then $W \cup \{f, u_1, u_2\}$ induces a graph from $\mathcal{A}_2$.

If $W \cup \{u_1\}$ induces one of $A_1, A_2, A_4, A_5, A_5, A_7, A_9, A_{10}, A_{12}, A_{13}, A_{14}, A_{15}$ then we can delete a vertex z from W such that the remaining graph $A_i - z$ is isomorphic to P_5. Then the set $(W \backslash \{z\}) \cup \{f, u_1, u_2\}$ induces a graph from $\mathcal{A}_1$.

If $W \cup \{u_1\}$ induces one of A_{11}, A_{17}, A_{18} then we can delete a vertex z from W such that the remaining graph $A_i - z$ is isomorphic to $\overline{P}_5$. Then the set $(W \backslash \{z\}) \cup \{f, u_1, u_2\}$ induces a graph from $\mathcal{B}_1$.

Case 2 $t = 2$ and $H \cong \overline{P}_5$.

If $W \cup \{u_1\}$ induces one of B_2, B_{11}, B_{18} then $W \cup \{f, u_1, u_2\}$ induces a graph from $\mathcal{B}_2$.

If $W \cup \{u_1\}$ induces one of B_1, B_3, B_4, B_8 then we can delete a vertex z from W such that the remaining graph $B_i - z$ is isomorphic to P_5. Then the set $(W \backslash \{z\}) \cup \{f, u_1, u_2\}$ induces a graph from $\mathcal{A}_1$.

If $W \cup \{u_1\}$ induces one of $B_5, B_6, B_7, B_9, B_{10}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}$ then we can delete a vertex z from W such that the remaining graph $B_i - z$ is isomorphic to $\overline{P}_5$. Then the set $(W \backslash \{z\}) \cup \{f, u_1, u_2\}$ induces a graph from $\mathcal{B}_1$.

Case 3 $t = 3$ and $H \cong P_5$.

If $W \cup \{u_1\}$ induces one of A_1, A_2, A_3, A_4, A_8 then $W \cup \{f, u_1, u_2, u_3\}$ induces a graph from $\mathcal{A}_3$.

Otherwise there we can delete two vertices z, z' from W such that the remaining graph $A_i - \{z, z'\}$ is isomorphic to $\overline{P}_5$. Then the set $(W \backslash \{z, z'\}) \cup \{f, u_1, u_2, u_3\}$ induces a graph from $\mathcal{B}_2$.

Case 4 $t = 3$ and $H \cong \overline{P}_5$.

If $W \cup \{u_1\}$ induces one of $B_1, B_2, B_3, B_4, B_7, B_8, B_9, B_{15}$ then $W \cup \{f, u_1, u_2, u_3\}$ induces a graph from $\mathcal{B}_3$.

Otherwise there we can delete two vertices z, z' from W such that the remaining graph $A_i - \{z, z'\}$ is isomorphic to $\overline{P}_5$. Then the set $(W \backslash \{z, z'\}) \cup \{f, u_1, u_2, u_3\}$ induces a graph from $\mathcal{B}_2$. □

Let $V = \{a, b, c, d, e\}$.

- A Chair is the graph with vertex set V and edge set $\{ab, bc, cd, ce\}$.
- A Kite be the graph with vertex set V and edge set $\{ab, bc, cd, ce, de\}$.
- A Parachute is the graph with vertex set V and edge set $\{ab, bc, cd, ae, be, ce, de\}$.
- A Butterfly is the graph with vertex set V and edge set $\{ab, cd, ae, be, ce, de\}$.

Lozin [3] proved that the stability number of (Chair, Kite, Parachute, Butterfly)-free graph can be found in polynomial time. We improve this result in two directions simultaneously, namely we consider the weighted stability number in a wider class.

Corollary 1. *The weighted stability number of (Chair, Kite, Parachute)-free graph can be found in polynomial time.*

Proof. It is easy to see that all graphs in $\mathcal{Z}$, except the path $P_7 \in \mathcal{A}_1$, has at least one of Chair, Kite, or Parachute as an induced subgraph.

Let G be a (Chair, Kite, Parachute)-free graph. If G in the substitutional closure of $(P_5 \cup K_1, \overline{P}_5 \cup K_1)$-free graphs, then we are done. Otherwise there is an induced subgraph $H \cong P_7$ in G. We denote by C the component of G containing H.

We claim that C is either a path or a cycle. To see that, let P be the longest induced path/cycle in C. If $C \neq P$ then there is a vertex $x \in V(C) \backslash V(P)$ such that x is adjacent to a non-pendant vertex if P (otherwise P would be not longest). Then we choose a subpath P' of order 6 in P with the same property: x is adjacent to a non-pendant vertex if P'. It can be directly checked that $V(P') \cup \{x\}$ induces a graph that contains at least one of Chair, Kite, or Parachute as an induced subgraph. □

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