STANDARD 13 — ALGEBRA

K-12 Overview

All students will develop an understanding of algebraic concepts and processes and will use them to represent and analyze relationships among variable quantities and to solve problems.

Descriptive Statement

Algebra is a language used to express mathematical relationships. Students need to understand how quantities are related to one another, and how algebra can be used to concisely express and analyze those relationships. Modern technology provides tools for supplementing the traditional focus on algebraic techniques, such as solving equations, with a more visual perspective, with graphs of equations displayed on a screen. Students can then focus on understanding the relationship between the equation and the graph, and on what the graph represents in a real-life situation.

Meaning and Importance

Algebra is the language of patterns and relationships through which much of mathematics is communicated. It is a tool which people can and do use to model real situations and answer questions about them. It is also a way of operating with concepts at an abstract level and then applying them, often leading to the development of generalizations and insights beyond the original context. The use of algebra should begin in the primary grades and should be developed throughout the elementary and secondary grades.

The algebra which is appropriate for all students in the twenty-first century moves away from a focus on manipulating symbols to include a greater emphasis on conceptual understanding, on algebra as a means of representation, and on algebraic methods as problem-solving tools. These changes in emphasis are a result of changes in technology and the resulting changes in the needs of society.

The vision proposed by this *Framework* stresses the need to prepare students for a world that is rapidly changing in response to technological advances. Throughout history, the use of mathematics has changed with the growing demands of society as human interaction extended to larger groups of people. In the same way that increased trade in the fifteenth century required businessmen to replace Roman numerals with the Hindu system and teachers changed what they taught, today's education must reflect the changes in content required by today's society. More and more, the ability to use algebra in describing and analyzing real-world situations is a basic skill. Thus, this standard calls for *algebra for all students*.

What will students gain by studying algebra? In a 1993 conference on *Algebra for All*, the following points were made in response to the commonly asked question, "*Why study algebra?*"

- Algebra provides methods for moving from the specific to the general. It involves discovering the patterns among items in a set and developing the language needed to think about and communicate it to others.
- Algebra provides procedures for manipulating symbols to allow for understanding of the world around us.
- Algebra provides a vehicle for understanding our world through mathematical models.
- Algebra is the science of variables. It enables us to deal with large bodies of data by identifying variables (quantities which change in value) and by imposing or finding structures within the data.
- Algebra is the basic set of ideas and techniques for describing and reasoning about relations between variable quantities.

Standard 8 (Numerical Operations) addressed the need for us to rethink our approach to paper-and-pencil computation in light of the availability of calculators; the need to examine the dominance of paper-and-pencil symbolic manipulation in algebra is just as important. The development of manipulatives, graphing calculators, and computers have made a more intuitive view of algebra accessible to all students, regardless of their previous mathematical performance. These tools permit and encourage visual representations which are more readily understood. No longer need students struggle with abstract concepts presented with very few ties to real-life situations. Rather, the new view of algebra offers real situations for students to examine, to generalize, and to represent in ways which facilitate the asking and answering of meaningful questions. Moreover, inexpensive symbolic processors perform algebraic manipulations, such as factoring, quickly and easily, reducing the need for drill and mastery of paper-and-pencil symbol manipulation.

K-12 Development and Emphases

Algebra is so significant a part of mathematics that its foundation must begin to be built in the very early grades. It must be a part of an entire curriculum which involves creating, representing, and using quantitative relationships. In such a curriculum, algebraic concepts should be introduced in conjunction with the study of patterns and developed throughout each student's mathematical education. The earlier students are exposed to informal algebraic experiences, the more willing they will be to use algebra to represent **patterns**.

The concept of representing **unknown quantities** begins with using symbols such as pictures, boxes, or blanks (i.e., $3 + \square = 7$). It is vital that students recognize that the symbol that is used to represent an unknown quantity has meaning. The only way this can be accomplished is to consistently relate the use of unknowns to actual situations; otherwise, students lack the ability to judge whether their answers make sense.

As students develop their understanding of arithmetic operations, they need to investigate the patterns which arise. Some of these patterns (which are commonly called **properties**) should be initially expressed in words. As the students develop more facility with variables, the properties can be expressed in symbolic form.

In the middle grades, problem situations should provide opportunities to generalize patterns and use additional symbols such as names and literal variables (letters). This development should continue throughout the remainder of the program, ensuring that the relationship between the variables (unknowns) and the quantities they represent is consistently stressed. Middle school students should extend their ability to use algebra to generalize patterns by exploring different types of relationships and by formalizing some of those relationships as **functions**. They should explore and generalize patterns which arise from nature, including non-linear relationships. As students move into the secondary grades, the graphing calculator and graphing software provide tools for examining relationships between x-intercepts and roots, between turning points and maximum or minimum values, and between the slope of a curve and its rate of change. As the student continues through high school, similar experiences should be provided for other functions, such as exponential and polynomial functions; these functions should be introduced using situations to which students can relate.

The use of algebra as a tool to **model real world situations** requires the ability to represent data in tables, pictures, graphs, equations or inequalities, and rules. Through exploration of problems and patterns, students are provided with opportunities to develop the ability to use concrete materials as well as the representations mentioned above. Having students use multiple representations for the same situation helps them develop an understanding of the connections among them. The opportunity to verbally explain these different representations and their connections provides the foundation for more formal expressions.

A fundamental skill in algebra is the **evaluation of expressions** and **the solution of equations and inequalities**. This process will be easier to understand if it is related to situations which give them meaning. Expressions, equations, and inequalities should arise from students' exploration in a variety of areas such as statistics, probability, and geometry. Elementary students begin constructing and solving open sentences. The use of concrete materials and calculators allow them to explore solutions to real-life situations. Gradually, students are led to expand these informal methods to include graphical solutions and formal methods. The relationship between the solutions of equations and the graphs of the related functions must be stressed regularly.

IN SUMMARY, there are algebraic concepts and skills which all students must know and apply confidently regardless of their ultimate career. To assure that all children have access to such learning, algebraic thinking must be woven throughout the entire fabric of the mathematics curriculum.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 13 — Algebra — Grades K-2

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

Students begin their study of algebra in grades K-2 by learning about the use of pictures and symbols to represent variables. They look at **patterns** and describe those patterns. They begin to look for **unknown numbers** in connection with addition and subtraction number sentences. They **model** the relationships found in real-world situations by writing number sentences that describe those situations. At these grade levels, the study of algebra is very much integrated with the other content standards. Children should be encouraged to play with concrete materials, describing the patterns they find in a variety of ways.

People tend to learn by identifying patterns and generalizing or extending them to some conclusion (which may or may not be true). A major emphasis in the mathematics curriculum in the early grades should be the opportunity to experience numerous **patterns**. The development of algebra as a language should build on these experiences. The ability to extend patterns falls under Standard 11 (Patterns and Functions), but having students communicate their reasoning is also an algebra expectation. Initially, ordinary language and concrete materials should be used for communication. As students grow older and patterns become more complex, students should develop the ability to use tables and pictures or symbols (such as triangles or squares) to represent numbers that may change or are **unknown** (variable quantities).

The primary grades provide an ideal opportunity to lay the foundation for the development of the ability to represent situations using **equations or inequalities** (open sentences) and solving them. Students can be asked to communicate or represent relationships involving concrete materials. For example, two students might count out eight chips and place them on a mat. One of the students then places a margarine tub over some of the counters and challenges the other student to figure out how many chips are hidden under the tub. A more complex situation might involve watching the teacher balance a box and two marbles with six marbles. The students draw a picture of the situation, and try to decide how many marbles would balance the box by physically removing two marbles from each side of the balance. In a problem involving an inequality, students might be asked to find out how many books Jose has if he has more than three books but fewer than ten. Situations from the classroom and the students' real experiences should provide ample opportunities to construct and solve such open sentences.

As operations are developed, students need to examine **properties** and make generalizations. For example, giving students a set of problems which follow the pattern $3 + 4$, $4 + 3$, $1 + 2$, $2+1$, etc. should provide the opportunity to develop the concept that order does not affect the answer when adding (the commutative property). After students understand that these properties are not necessarily true for all operations (e.g., $5 - 2$ is not equal to $2 - 5$), the teacher should mention that the properties are important enough to be given names. However, the focus of this work should be on using the properties of operations to make work easier rather than on memorizing the properties and their names.

Students in grades K-2 spend a great deal of time developing meaning for the arithmetic operations of addition, subtraction, multiplication, and division. As they work toward understanding these concepts, they focus on developing **mathematical models** for concrete problem situations. The number sentences that they write to describe these problem situations form a foundation for more sophisticated mathematical models.

Standard 13 — Algebra — Grades K-2

Indicators and Activities

The cumulative progress indicators for grades K-2 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and in grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. Understand and represent numerical situations using variables, expressions, and number sentences.

- Students represent a problem situation with an open sentence. For example: *If there are 25 students in the class and Marie brought 26 cookies for snack, how many will be left over? (26 – 25 = ?)* Another example might be: *We have 10 cups left in the package and there are 25 children in the class, so how many more cups do we need to get? (10 + ? = 25)*
- Students read *The Doorbell Rang* by Pat Hutchins. They act out the story and realize that many different combinations of students can share 12 cookies equally.
- Students make a table relating the number of people and the number of eyes. They use a symbol such as a stick figure to represent the number of people and a cartoon drawing of an eye to represent the number of eyes and then express the relationship between them.

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2. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and number sentences, and translate from one to another.

- Students in groups are given a container to which they add water until its height is 5 centimeters, measured with Cuisennaire rods. They add marbles to the container until the height of the water is 6 centimeters. They continue adding marbles, recording each time the number of marbles it takes to raise the water level one centimeter. They describe the relationship between the number of marbles added and the height of the water.
- In regular assessment activities, students look at a series of pictures which form a pattern. They draw the next shape, describe the pattern in words, and explain why they chose to draw that shape.
- Using a calculator, students play *Guess My Rule.* The lead student enters an expression such as *5+4* and presses the = key; she shows only the answer to her partner. The second student tries to guess the rule by entering different numbers, one at a time, pressing the = key after each number. The calculator, after each = is pressed, should show the sum of the entered number and the second addend (in this case, 4). (Some calculators perform this function differently; see the user's manual for instructions.) When the second student thinks she knows the pattern (in this case, *adding 4*), she makes a guess. The pattern is written in words and then as a rule using a picture or symbol for the variable (the number which the second student enters).

- Placing four different-colored cubes in a can, students predict which color would be drawn out most often if each child draws one cube without looking. The teacher helps the students keep track of their results by making a chart with the colors on the horizontal axis and the number of times a color is drawn on the vertical axis. As students select cubes, an "x" is placed above the color drawn, forming a frequency diagram. After several turns, the students describe the patterns they see in the graph.
- Students read *Ten Apples Up on Top!* by Theo Le Sieg and discuss the mathematical comparisons and equations that appear in the story.

3. Understand and use properties of operations and numbers.

- Students are given five computational problems to solve. They are permitted to use the calculator on only two of them. Two of the problems are related to another two by operation properties (e.g., $3 + 2$ and $4 + 6$ are related to $2 + 3$ and $6 + 4$ by the commutative property) and the last involves a property of number such as adding *0*. Students share their thought processes in a follow up discussion.
- The second grade teacher has a box containing slips of paper with open sentences such as $25 - 8 = \square$ or $15 + \square = 23$. Students draw out a slip and tell or write a story which would involve a situation modeled by the sentence.
- Students discover that, since the order of the numbers when adding them is not important, they can solve a problem like $3 + 8$ by starting with 8 and counting up 3, as well as by starting with 3 and counting up 8.
- In their math journals, students write their reactions to the following situation:

  Sally just used her calculator to find out that 324 + 486 was equal to 810. In another problem, she must find the answer to 486 + 324. What should she do? Why?

4. Construct and solve open sentences (example: $3 + \square = 7$) that describe real-life situations.

- Kindergarten students play the *hide the pennies* game. The first player places a number of pennies (say 7) on the table and lets the other player count them. The first player covers up a portion of the pennies, and the second player must determine how many are covered. They may represent the situation with markers or pictures to help them. Some second-grade students are ready to write a number sentence that describes the situation.
- Students are given a bag with Unifix cubes. They are told that the bag and 2 cubes balance 7 cubes. They use a balance scale to find how many cubes are in the bag.

References

Hutchins, Pat. *The Doorbell Rang*. Mulberry Books, 1986.

Le Sieg, Theo. *Ten Apples Up on Top!* New York, NY: Random House, 1961.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 13 — Algebra — Grades 3-4

Overview

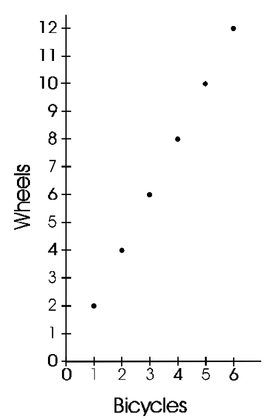
Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

In grades K-2, students use pictures and symbols to represent variables, generalize patterns verbally and visually, and work with properties of operations. Although the formality increases in grades 3 and 4, it is important not to lose the sense of play and the connection to the real world that were present in earlier grades. As much as possible, real experiences should generate situations and data which students attempt to generalize and communicate using ordinary language. Students should explain and justify their reasoning orally to the class and in writing on assessments using ordinary language. When introducing a more formal method of communicating, such as the language of algebra, it is helpful to revisit some of the situations used in previous grades.

Since algebra is the language of patterns, the mathematics curriculum at this level needs to continue to focus on **patterns**. The use of letters to represent **unknown quantities** should gradually be introduced as a replacement for pictures and symbols. The use of **function** machines permits the introduction of letters without the need to move to formal symbolic algebra. Since they have had the opportunity to experience real function machines such as the calculator or a gum bank, where one penny yields two pieces of gum, the notation of function machines should make sense.

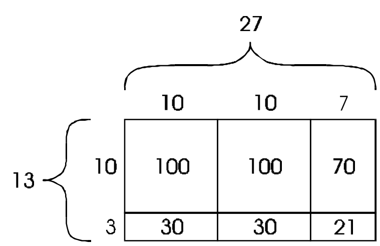
Here the box is thought of as the function machine *times 2* which takes in a number "a" and produces a number "b" which is *twice* "a." Students can use such symbols to communicate their generalization of patterns. They put two or more machines together making a composite function; for example, they can follow the *times 2* machine with an *add 3* machine. They determine not only what each input produces but also what input would produce a given output.

Students should continue to communicate their generalizations of patterns through ordinary language, tables, and concrete materials. Graphs should be introduced as a method for quickly and efficiently representing a pattern or function. They should develop graphs which **represent real situations** and be able to describe patterns of a situation when shown a graph. For example, when given the graph at the right which shows the relationship between the number of bicycles and wheels in the school yard, they should be able to describe the relationship in words.



Students in grades 3 and 4 should continue to use **equations and inequalities** to represent real situations. While variables can be introduced through simple equations such as $35 \div n = 5$, students should be viewing variables as place holders similar to the open boxes and pictures they have already used. At these grade levels, they need not use variables in more complicated situations. Given a situation such as determining the cost of each CD if 5 of them plus $3 tax is $23, they should be permitted to represent it in whatever way they feel comfortable. Students should be able to use, explain, and justify whatever method they wish to solve equations and inequalities. Some may continue to use concrete materials for some situations; they might count out 23 counters, set aside 3 for the tax, and divide the remainder into 5 equal piles of 4. Others might try different numbers until they find one that works. Some students may write $23 - 3 = 20$ and $20 \div 5 = 4$. Still others may want to relate this to function machines and figure out what had to go in for $23 to come out. It is important for students to see the diversity of approaches used and to discuss their interrelationships.

Students should continue to examine the **properties of operations** and use them whenever they would make their work easier. There are some excellent opportunities for providing a foundation for algebraic concepts in these grades. For example, explaining two-digit multiplication by using the area of a rectangle (see figure below illustrating 13 x 27) provides the student with a foundation for multiplication of binomials, the distributive property, and factoring. While the teacher at this grade level should focus on the development of the multiplication algorithm, the teacher of algebra several years later will be able to build on this experience of the student.



Standard 13 — Algebra — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Understand and represent numerical situations using variables, expressions, and number sentences.**

   - Students do comparison shopping based on items that are for sale in multiples. For instance: *If chewing gum is sold at 3 packs for 85 cents (p = 85/3), is that a better or a worse buy than a single pack for 30 cents (p = 30)?* They sort their examples into groups where the multiple buy is a better deal, the same deal, or a worse deal than the single package deal.
   - One student has been folding origami cranes to send to Hiroshima for Peace Day in August. He brings the 47 cranes that he has folded so far to class and asks for help to fold many more. The class decides to have each of the 26 students fold one crane each week for the rest of the school year. The teacher asks groups of students to find a way to determine how many cranes will be in the collection after some given number of weeks. She starts off the discussion by having students list the numbers for the first few weeks:

     $$47,$$
     $$47 + 26,$$
     $$47 + 26 + 26,$$
     $$47 + 26 + 26 + 26,$$
     and so on.

     They figure out whether they can reach 500 cranes by the end of the year.

2. **Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and number sentences, and translate from one to another.**

   - Students compare two allowance plans. Plan A provides an allowance of $5 the first week and adds $1 each week. Plan B starts with 1¢ and doubles the allowance each week. Using calculators, students make tables listing the number of the week, the amount of allowance under Plan A and the amount of the allowance under plan B. They complete several rows of the table so that they understand what is happening with each plan and see that Plan B soon overcomes Plan A. They might want to try to use physical objects such as centimeter cubes to demonstrate the behavior of the two plans visually.
   - Each student is given an even number of square tiles and asked to use them all to make a rectangle with two columns. Students are asked to notice that the heights of the rectangles are different for different starting numbers of tiles. They collect the data into a table, giving each student's name, the number of tiles used, and the height of the rectangle. They understand that the number of tiles is the area and can figure out the height of the rectangle if

they know the number of tiles that are used — that is, they can verbalize that the height of the rectangle is half the total number of tiles.

- Students read *1 Hunter* by Pat Hutchins, wherein a determined hunter looks and looks and looks for animals but sees none, even though the reader can clearly see 2 elephants, 3 giraffes, 4 ostriches, and so on, up to 10 parrots. They are asked how many animals in all the hunter was unable to see. Students use graphs, concrete materials, pictures, and number sentences to express their understanding of the situation.

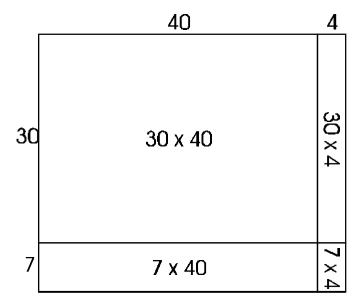
- Students play *Guess My Rule* by suggesting inputs and having the *rule-maker* (the teacher or a student) put the corresponding outputs into a table like this one:

Input	Output
3	7
1	4
16	19
.	.
.	.

Students should always be challenged to show they understand the rule by giving a verbal explanation of it. Partially filled-in *Guess My Rule* tables are a good assessment technique to evaluate the students' inductive reasoning power and their level of comfort with arithmetic operations.

3. Understand and use properties of operations and numbers.

- When the students are introduced to two-digit by two-digit multiplication, they begin with a problem of finding the area of a rectangular field which is 37 feet by 44 feet. They know they need to multiply the numbers to find the area, but they don't know how to multiply without calculators. The teacher draws a rectangle and uses a line to divide the width into two regions which are 30 feet and 7 feet. She does the same with the length, cutting it into lengths 40 feet and 4 feet. This divides the rectangle into four smaller rectangles (*30×40, 30×4, 7×40, 7×4*) all of which are multiplications the students can do.



37 x 44 = (30 + 7)(40 + 4)

- Lea and Suzanne discovered a method for multiplying even numbers by six easily. Their method, applied to the example *6×24*, is:

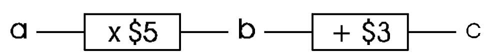
Cut the other even number in half	12
Add a zero	120
Add the number	120+24=144

When they told their classmates their discovery, they were stumped when they were asked why it worked. The teacher, grasping the *teachable moment,* divided the class into groups and challenged them to do a few examples using the girls' method and try to figure out and explain why it worked.

- Students and teacher together work through Robert Froman's book, *The Greatest Guessing Game: A Book about Dividing* to reinforce their notions of division.
- Students explain that they solved a problem like $300 - 56$ mentally by first subtracting *50* and then subtracting *6*, since that is the same as subtracting *56*. They also do $25 \times 7 \times 4$ by first multiplying 25×4 and then multiplying by *7*. Such simplifications will give a good foundation for later work in algebra.

4. Construct and solve open sentences (example: 3 + □ = 7) that describe real-life situations.

- In an assessment situation, groups of students are asked to describe in words the situation of four people sharing a five dollar bill found on the way to school, and then to transform it to symbolic form using pictures, symbols or letters.
- Students want to help the New Jersey environment and raise money at the same time. They discover that in two bordering states (New York and Delaware), plastic soda bottles can each be turned in for a 5¢ refund. They write an equation which represents the amount of money they will receive for *b* bottles. Students answer questions such as *How much money will we get for 25 bottles?* and *How many bottles will we need to make $10?*
- Students are presented with a function machine representing the situation of buying music tapes for $5 each through the mail and paying a $3 shipping and handling charge for the order. They answer questions such as *How much would it cost for 5 tapes?* and *How many tapes were bought if the bill was $43?*



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Fromer, Robert. *The Greatest Guessing Game: A Book About Dividing*. New York, NY: Thomas Y. Crowell Publishers, 1978.

Hutchins, Pat. *1 Hunter*. New York, NY: Greenwillow Books, 1982.

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Standard 13 — Algebra — Grades 5-6

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

It is important that students continue to have *informal* algebraic experiences in grades 5 and 6. Students have previously had the opportunity to generalize patterns, work informally with open sentences, and represent numerical situations using pictures, symbols, and letters as variables, expressions, equations, and inequalities. At these grade levels, they will continue to build on this foundation.

Algebraic topics at this level should be integrated with the development of other mathematical content to enable students to recognize that algebra is not a separate branch of mathematics. Students must understand that algebra is an expansion of the arithmetic and geometry they have already experienced and a tool to help them describe situations and solve problems.

Students should use algebraic concepts to investigate situations and solve interesting mathematical and **real world** problems. There should be numerous opportunities for collaborative work. Since algebra is the language for describing **patterns**, students should have regular and consistent opportunities to discuss and explain their use of these concepts. They should write generalizations of situations in words as well as in symbols. To provide such opportunities, the activities should move from a concrete situation or representation to a more abstract setting. Students at this level can begin using standard algebraic notation to represent known and **unknown quantities and operations**. This should be developed gradually, moving them from the previous symbols in such a way that they can appreciate the power and elegance of the new notation.

Students need to learn how variables are different from numbers (a variable can represent many numbers simultaneously, it has no place value, it can be selected arbitrarily) and how they are different from words (variables can be defined in any way we want and can be changed without affecting the values they represent). Students need to see variables (letters) used as names for numbers or other objects, as unknown numbers in equations, as a range of unknown values in inequalities, as generalizations in pattern rules or formulas, and as characteristics to be graphed (independent and dependent variables).

An **algebraic expression** involves numbers, variables, and operations such as $2b$, $3x - 2$, or $c - d$. In fifth and sixth grade, students should begin to become familiar with the common notational shortcut of omitting the operation sign for multiplication, so that when $b=3$, $2b$ equals 6 and not 23. Thus they recognize that there are slightly different rules for reading expressions involving variables than those involving only numbers.

Students in grades 5 and 6 should focus on understanding the role of the equal sign. Because it is so often used to signal the answer in arithmetic, students may view it as a kind of operation sign — a “write the answer" sign. They need to come to see its role as a relation sign, balancing two equal quantities. Students should develop the ability to **solve simple linear equations** using manipulatives and informal methods. With the appropriate background, students at grades 5 and 6 have the ability to find the solution of an equation, such as *7* for $x+5=12$, whether they use manipulatives, a graph, or any other method. It is imperative that in the discussion of the solution of an equation, the many methods in obtaining that solution are described.

Students in grades 5 and 6 should use concrete materials, such as algebra tiles, to help them develop a visual, geometric understanding of algebraic concepts. For example, students can represent the expression $3x - 2$ by using three strips and two units. They should make graphs on a rectangular coordinate system from data tables, analyze the shape of the graphs, and make predictions based on the graphs. Students should have opportunities to plot points, lines, geometric shapes, and pictures. They should use variables to generalize the formulas they develop in studying geometry (e.g., $p = 4s$ *for a square or* $A = l \times w$ *for a rectangle*). Students should be able to describe movements of objects in the plane through horizontal and vertical slides (translations). They should experiment with probes which generate the graphs of experimental data on computers or graphing calculators. The majority of this work will be with graphs that are straight lines (linear functions), but students should have some experience seeing other shapes of graphs as well; in particular, when dealing with real data and probes, many times the graph will not be linear.

Standard 13 — Algebra — Grades 5-6

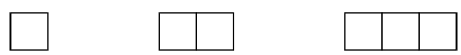
Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

5. Understand and use variables, expressions, equations, and inequalities.

- Students find the perimeter of one square, two squares connected along an edge, three squares connected along their edges, and so forth, as shown below. The length of one side of a square is assumed to be one unit.

  They make a table of values and use it to determine a function rule which describes the pattern. They understand the rule $P = 2 \times s + 2$, and use it to predict the perimeter of ten squares.
- Students write a Logo program to draw a rectangle of any dimensions using the variables :LENGTH and :WIDTH.
- In a health unit, students are studying the dietary needs for maintaining healthy bodies. The teacher provides guidelines for the maximum amounts of fat and cholesterol at any meal. Each group of students chooses two foods for a meal. They determine the fat content per unit of each of the foods and the cholesterol amount per unit and make inequalities which relate these unit values, the number of units, and the given maximum amounts. The students determine possible combinations of the amounts of the two foods whose fat and cholesterol would still be acceptable. The teacher uses a function graphing computer program or graphing calculator to represent visually the acceptable amounts.

6. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and standard algebraic notation.

- Each group of students is given a *Mr.* or *Mrs. Grasshead* (i.e., a sock filled with dirt and grass seed which sits in a dish of water). They create a name for their grasshead and begin a diary, recording the number of days that have passed and the height of the grass. At the end of specified time periods, they discuss the changes in the height, the average rate of change over the time period, and the overall behavior of the grass growth. Each group makes a graph of height versus the number of days. The students note whether the graph is close to a straight line.
- Students find the number of tiles around the border of a floor 10 tiles long and 10 tiles wide by looking at smaller square floors, making a table, and identifying a pattern. They describe their pattern in words and, with assistance from the teacher, develop the expression $(4 \times n) +$

4 for the number of border tiles needed for an $n \times n$ floor.

- Students play *Guess My Rule* by suggesting input and having the *rule-maker* (the teacher or a student) put the corresponding output into a table like this one:

Input	Output
3	9
1	5
16	35
.	.
.	.
.	.
n	2n+3

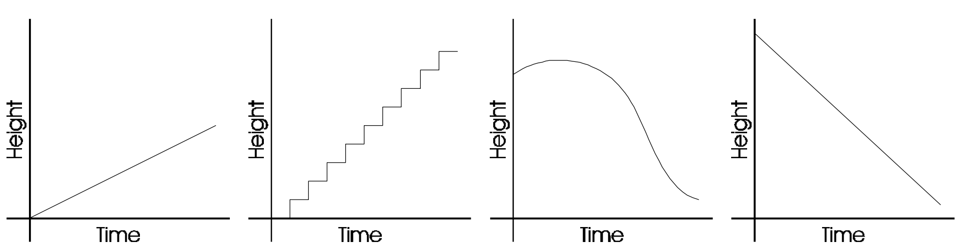
Students should always be challenged to show they understand the rule by filling in the last row for an input of *n*. Partially filled-in *Guess My Rule* tables are also a good assessment technique to evaluate students' inductive reasoning power and their ability to use standard algebraic notation to express relationships.

- Students use play money to act out the following situation and solve the problem.

  A man wishes to purchase a pair of slippers marked $5. He gives the shoe salesman a $20 bill. The salesman does not have change for the bill so he goes to the pharmacist next door and gets a $10 and two $5 bills. He gives the customer his change and the man leaves. The pharmacist enters shortly after and complains the $20 was counterfeit. The shoe salesman gives her $20 and gives the counterfeit bill to the FBI. How much did the shoe salesman lose?

- Students place 8 two-color chips in a paper cup and toss them ten times, recording the number of red and yellow sides showing on each toss. For each red chip that shows, they lose $1. For each yellow, they win $1. For each toss, the students write a number sentence that shows their win or loss for that toss. For example, after tossing 3 yellows and 5 reds, their sentence would read 3   5 =   2. Afterwards, the students look for patterns in the number sentences that they have written. They discuss these patterns and then write about them in their notebooks.

- In both classroom and assessment situations, students interpret simple non-numeric graphs and decide what kinds of relationships they demonstrate. For example: *Which of the following graphs would show the relationship between the height of a flag and time as a boy scout raised the flag on a flagpole?*



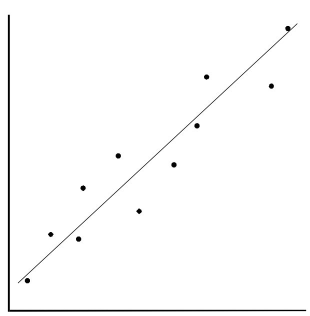
- Students work through the *Powers of the Knight* lesson that is described in the Introduction to this *Framework.* They learn that doubling the number of coins on consecutive squares of a chessboard results in a rapidly increasing sequence of numbers — the powers of 2.
- Students read *Anno's Mysterious Multiplying Jar* by Mitsumasa Anno and try to analyze and represent the numerical patterns shown using variables.

7. Use graphing techniques on a number line to model both absolute value and arithmetic operations.

- Students use number lines to demonstrate addition of integers. They point to the number representing the amount currently in the bank and then slide their finger in the appropriate direction (right for deposits and left for withdrawals) over the distance indicated by the second amount. As they slide their finger, they use arrows to track their movements over the number line, and the teacher keeps track of the operations using positive and negative integers. Through this dual representation, students begin to understand the relationship between the addition of integers and movement along the number line.
- Students are given a variety of objects whose dimensions they must determine. They are given a number line marked from 5 to 27 to simulate a broken yardstick. Students work in pairs to develop a process for determining the lengths of each of the objects to the nearest unit. After they have a workable method and have written an explanation of it in their journal, the teacher replaces the tape with another which is marked from -10 to 10. The students repeat their effort. This process helps them develop an understanding of subtraction of integers and the relationship between the operation of subtraction and distance on the number line.
- Addition and subtraction of signed numbers is explored using two-colored disks and a number line. Red is used to represent positive numbers and yellow is used to represent negative numbers. When given a problem such as $-3 + 5$ the students place 3 yellows and 5 reds on the table. They pair up as many red and yellow disks as they can and remove them from the table. In the case of the example, 3 red and yellow disks would be paired and removed, leaving 2 red disks which represents the sum, +2.

8. Analyze tables and graphs to identify properties and relationships.

- Students use tables or two-color chips to help them solve the following problem: *A classroom has 25 lockers in a row. The first person to enter the room opened every locker. The second person closed every other locker beginning with the second locker. The third person started with the third locker and changed every third locker from open to closed or closed to open. This continued until 25 people had passed through the room. Which lockers would be open after the 25th person walked into the room?*
- A plastic rectangular shape is exhibited on the overhead. The lengths of both sides of the image, and the distance from the screen to the overhead are measured. The overhead is moved and the process is repeated so that measurements are taken at six to ten different distances. One group of students is responsible for determining



the relationship between the distance from the screen and the length of one side of the image. A second group is responsible for studying the relationship between the distance from the screen and the area of the image. Each group makes a scatterplot of its data and eyeballs a line of best fit using a piece of spaghetti. They then use the graph to answer questions about the relationship between distance and length or distance and area. They also develop a summary statement describing the relationship.

- Students make pendulums using strings of length 64, 32, 16, and 8 cm with a washer at one end and a screw eye or ruler at the other. The strings are swung from a constant height and the number of swings in 30 seconds is recorded. A graph is made plotting the number of swings against the string length. Students study the results and determine if there might be a pattern they could continue. They attempt to answer questions such as: *Will the number of swings ever reach zero?* (This activity is a good one to repeat at later grades since the relationship appears linear but when very short lengths and very long lengths are used, it becomes clear that it is actually a quadratic relationship.)

- Students are given the times of the Olympic 100-meter freestyle swimming winners both in the men's event and the women's event. Using different colors for the two genders, they produce a scatterplot and use a piece of spaghetti to eyeball a line of best fit for each set of data. They use their lines to determine times in the years not given (when no Olympics were held) and to predict times in the years beyond those they were given. They also determine if the data supports the assertion that the women will some day swim as fast as the men and predict from their lines when that would happen.

9. Understand and use the rectangular coordinate system.

- Students are paired to play a game similar to battleship in which they attempt to determine where the two lines their opponent has drawn intersect. Both students draw axes which go from 10 to 10 in both the x- and y-directions. They sit so that neither can see the other's paper. The first player draws two lines which intersect at a point with integer coordinates and colors the four regions different colors. The second player gives the coordinates of a point. The first player responds with the color of the region the point is in, or that the point is on a line, or that it is *the* point! The second player keeps a record of his guesses on his axes and continues guessing until the chosen point is determined.

- Students keep track of the high and low temperatures for a month in two different colors on a graph. The horizontal axis represents the day of the month and the vertical axis represents the temperature. At the end of the month, they connect the points making two broken-line graphs. They use their graphs to discuss the temperature variations of that month and to determine the overall "high" and "low" for the month.

- Students consider what happens if they start with two bacteria and the number of bacteria doubles every hour. They make a table showing the number of hours that have passed and the number of bacteria and then plot their results on a coordinate graph.

- Students draw broken-line pictures in a cartesian plane and identify the coordinates of critical points in the pictures. Their partners attempt to re-create the picture using the coordinates of the critical point and verbal descriptions of how the critical points should be connected.

10. Solve simple linear equations using concrete, informal, and graphical methods, as well as appropriate paper-and-pencil techniques.

- Students use algebra tiles to solve an equation. For example, they represent the equation $3x + 2 = 5x$ by placing three strips and two units on one side of a picture of a balance beam and five strips on the other side. They decide that the balance will stay even if they take the same number of objects off both sides, so they take three strips off both sides and have two units balanced with two strips. Then they correctly decide that 1 unit must balance one strip.

- Students want to use their class fund as a donation to a town in Missouri devastated by the summer floods. They agree that each of the 26 students in the class is going to contribute 25¢ a week. The fund already contains $7. The students develop the expression *\$6.50×W+\$7* as the amount of money in the fund at the end of W weeks. The teacher asks them how much they would like to send to the town, and the students agree on $100. The teacher then asks them to write an equation which would say that the amount of money after W weeks was $100. The students write *\$6.50×W + \$7 = \$100*. Finally, the students try a number of different strategies for finding out what W should be. Some of the students use calculators in a guess-and-check method. Some students go to the computer and use a spreadsheet to generate the amounts for different weeks until the total is more than $100. Others express the relationship as a composite function using function machines and then use the inverse operations to subtract $7 and then divide by $6.50. They use their calculators to carry out the division. The teacher discusses all these methods and introduces the traditional algebraic shorthand method for solving the problem.

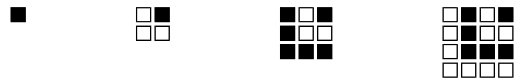
11. Explore linear equations through the use of calculators, computers, and other technology.

- Using a motion sensor connected to a graphing calculator or computer, the class experiments with generating graphs which represent the distance from the sensor against time. They discover that if they walk at a fixed rate the graph is a straight line. They try walking away from the sensor at different fixed rates of speed to determine what effect the speed has on the line. They start at different distances from the sensor to see the effect that has. They try walking toward the sensor and standing still. Students discuss the relationships between the lines they are generating and the physical activity they do. As an assessment, the teacher has individual students walk so as to generate a straight line. The students are then asked to write in their journals what someone would have to do to produce a line which was less steep. After closing their journals, individual students are provided an opportunity to verify their conclusions using the graphing calculator.

- After measuring several students' heights and the length of the shadows they produce, the data is entered in a spreadsheet, computerized statistics package, or graphing calculator. A scatterplot is formed from the data and the students see that the plot is approximately linear. The technology is used to produce a line of best fit which the class uses to determine heights of unknown objects (such as a flagpole) and the length of the shadow of objects with known heights.

12. Investigate inequalities and nonlinear equations informally.

- Students explore patterns involving the sums of the odd integers *(1, 1+3, 1+3+5, ...)* by using small squares to make Ls to represent each odd number and then nesting the Ls. They make a table that shows how many Ls are nested and the total number of squares used.



They look for a pattern that will help them predict how many squares will be needed if 10 Ls are nested (i.e., if the first 10 odd numbers are added together). They make a prediction and describe the basis for their prediction (e.g., when you added the first 3 odd numbers, and placed the three Ls together, they formed a square that was three units on a side, so when you add the first 10 odd numbers, that should make a square that is ten units on a side and whose total area is 100 squares.) They share their solution strategies with each other and develop an expression that can be used to find the sum of the first n odd numbers (i.e., $n \times n$).

- Students set up a table listing the length of the sides of various squares (x) and their areas (y). Some students use the centimeter blocks to help them find the values. The teacher completes a table of values in a function graphing computer package on the class computer which has an LCD panel for overhead projection or on an overhead version of a graphing calculator. When the students have finished completing the table, the teacher turns on the overhead and displays her table. The students check their answers and ask questions. The teacher graphs the data on the computer or calculator, and the students use the graph to answer questions such as *If the side was 3.5 cm, what would the area be?* and *If the area was 60 square centimeters, what would the side be?* The teacher uses the trace function to identify the points being discussed.
- Students explore inequality situations such as: *I have $150. How many more weeks would I need to save my $15 allowance to buy a stereo that costs $200?* They represent the relationship as an inequality, both in words and in symbols, and use play money, base ten blocks, graphs, or trial and error to solve the problem.

13. Draw freehand sketches of, and interpret, graphs which model real phenomena.

- Students keep track of how far they are from home during one specified day. They draw a graph which represents the distance from home against the time of day and write an explanation of their graph in relation to their actual activities on that day.
- Students are presented with a graph representing a student's monthly income from performing lawn care for people over the past year. The graph shows no income during the months of November, December, and March. They write a story which explains the behavior of the graph in terms of the need for services over the course of the year.

References

Anno, Mitsumasa. *Anno's Mysterious Multiplying Jar.* Philomel Books, 1983.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post

additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 13 — Algebra — Grades 7-8

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

Students in grades 7 and 8 continue to explore algebraic concepts in an informal way. By using physical models, data, graphs, and other mathematical representations, students learn to generalize number **patterns** to model, represent, or describe observed physical patterns, regularities, and problems. These informal explorations help students gain confidence in their ability to abstract relationships from contextual information and use a variety of representations to describe those relationships. Manipulatives such as algebra tiles provide opportunities for students with different learning styles to understand algebraic concepts and manipulations. Graphing calculators and computers enable students to see the behaviors of **functions** and study concepts such as slope.

Students need to continue to see algebra as a tool which is useful in describing mathematics and solving problems. The algebraic experiences should develop from **modeling situations** where students gather data to solve problems or explain phenomena. It is important that all concepts are presented within a context, preferably one meaningful to students, rather than through traditional symbolic exercises. Once a concept is well-understood, the students can use traditional problems to reinforce the algebraic manipulations associated with the concept.

Many activities which are used in earlier grades should be revisited as students become more sophisticated in their use of algebra. At the same time, activities used in later grades can be incorporated on an informal basis. For example, students in earlier grades might have gathered the heights and armspans and attempted to generalize the relationship between them in words. As students became familiar with the rectangular coordinate system, they might have generalized the relationship using a scatterplot and fitting a line to the data. In seventh and eighth grade, students might be taught how to find the median-median line to determine the line of best fit and use that line to solve problems. In later grades, when students have learned to find the slope of a line through two points symbolically, they can determine the equation of the median-median line. (The median-median line, available on many calculators which have statistics capabilities, is found by dividing the data points on the x-y plane into three equal sets, grouped by x-value, finding a single point for each set whose coordinates are the medians of the respective coordinates of the points in the set, connecting the first and third points by a straight line, and shifting this line 1/3 of the way toward the second point.)

Students should have numerous opportunities to develop an understanding of the relationship between a function and its graph. A limited number of functions should be plotted by hand, but students should also use technology to graph functions. While linear relationships should be the focus, inequalities and nonlinear functions should be explored as well. Students should develop an understanding of the relationship between solutions of equations and graphs of functions. For example, the solution of the equation $3x - 4 = 5$ can be found by plotting $y = 3x - 4$, tracing along the function until a y-value of 5 is found, and then determining the corresponding x-value. Students should develop the ability to find solutions using the trace function of graphing calculators and computer graphing programs and discuss how it assists in solving equations. They should also have opportunities to use spreadsheets as a method for representing and solving problems.

Students should be able to **evaluate expressions** using all forms of real numbers when calculators are available. They should have developed an understanding of the importance of the algebraic order of operations and be able to correctly evaluate expressions using it. It is imperative that students understand that they cannot blindly accept answers produced on the calculator. They should recognize that a standard four-function calculator does not use the standard order of operations. They should recognize that even with a scientific calculator, operations such as the division of two binomial quantities requires the use of parentheses.

Students should refine their ability to solve **simple linear equations** (i.e., $ax+b=cx+d$). Students may continue to use informal, concrete, and graphic methods but should begin to link these methods to more formal symbolic methods. As students have opportunities to explore interesting problems, applications, and situations, they need to be encouraged to reflect on their explorations and summarize concepts, relationships, processes, and facts that have emerged from their discussions. Developing a suitable notation to describe these conclusions leads naturally to a more formal, more symbolic view of algebra.

Standard 13 — Algebra — Grades 7-8

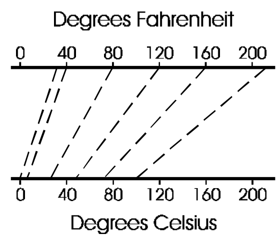
Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

5. Understand and use variables, expressions, equations, and inequalities.

- Students make a model of the relationship between Celsius and Fahrenheit temperatures. They represent the relationship as a formula, and check the formula against two known data points — 0 C = 32 F and 100 C = 212 F. Students use their formula to convert between Celsius and Fahrenheit temperatures.



- Students examine the following situation, making a table for the first few months to gain an understanding of the pattern involved in the problem.

  Juanita opened a checking account and deposited $500. She works as a part-time engineer's assistant in a local firm and will receive a check for $130 on the 1st and 15th of each month. She intends to take $40 from each paycheck for cash expenses and then deposit the remainder. On the 15th of each month, she will write a check for $220 to cover the cost of her car payment.

  The students develop an equation that describes how much money is in the account on the 1st and the 15th of each month. They use their equation to determine the amount for various months in the future as well as to find out when Juanita will overdraw the account. Students use their equations and the information they have found to write a letter to Juanita explaining why her plan is not financially sound and what she might do to correct it.

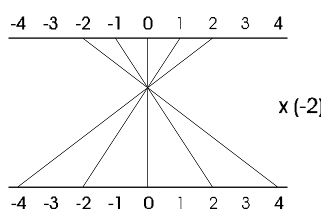
6. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and standard algebraic notation.

- The seventh grade is preparing for a skiing trip. The interdisciplinary team has decided to integrate the planning for the trip into all the courses. One of the items being discussed in math class is the number of buses that will be required. Since the actual number of people is not yet determined, the situation is best modeled with variables and unknowns. Knowing that each bus holds 35 people, students develop a table based on 5 or 6 different numbers of people on the trip, in an attempt to find a pattern. Some students who require more concrete operations to develop a sense of the pattern use unit cubes and decimal rods to represent the situation with different numbers of people. The group works together to develop a graph based on their findings. The discussion begins with one person suggesting graphing the points *(35,1)*, *(70,2)*, and *(105,3)* and then connecting them with a straight line. The teacher does this (in a way that can be readily erased later) and asks the class if there is any problem with this solution. More discussion leads to understanding that the graph would be made up of discrete points since there cannot be fractions of people and the graph would not be a straight line but a series of steps 35 people long and going up by 1 bus. Students are able to verbalize the rule but are curious as to how it would be represented symbolically. The teacher shows them the symbol for the step function, x .

- Students construct squares on each side of right triangles on their geoboards, then find the area of each square. They record their results in a table and look for a pattern, leading them to "discover" the Pythagorean Theorem.

7. Use graphing techniques on a number line to model both absolute value and arithmetic operations.

- Students use a graphic representation of the mapping resulting from a multiplication by a negative number (such as -2 as shown below) to explain why the order of the inequality reverses when both sides are multiplied by that negative number.



They explain that while the original sequence of inputs is ascending *{-2,-1,0,1,2}*, the images are descending *{4,2,0,-2,-4}*.

- Presented with an absolute value equation such as *x - 5 * 3, the students use the idea that this means identifying all the points which are 3 units or less from 5. They represent the solution set with dots at 2 and 8 and a line connecting them.

- Students represent multiplication of integers on the number line as repeated addition. They use the idea that 3×4 is $4 + 4 + 4$, and, on the number line, it would be represented by three segments, each four units in length, placed end to end starting at zero. The students write in their journals how they would picture $3 \times (-4)$. After reviewing their responses and clarifying concerns, they discuss that $(-3) \times (-4)$ would be the opposite of $3 \times (-4)$.

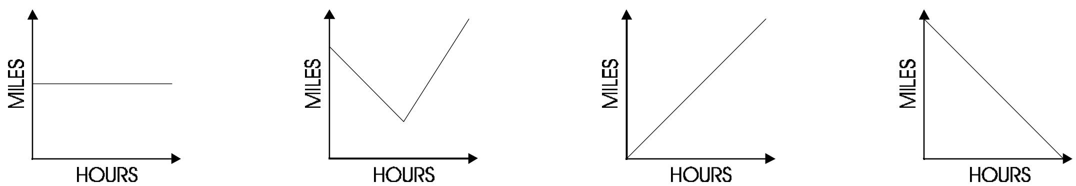
8. Analyze tables and graphs to identify properties and relationships.

- Students work on the handshake problem (*How many handshakes would there be in a group of 25 people if each person shook hands with every other person exactly once?)* by considering smaller groups of people and recording the results for these smaller groups in a table. Students identify the pattern and develop an expression which relates the number of people to the number of handshakes. Some students place 25 points on paper, forming a polygon, and begin drawing the segments which connect the points to generate a geometric pattern.

- Students explore the relationship between the number of sides of a regular polygon and the total number of diagonals that can be drawn in that polygon. They organize their work in a table, graph the data, and write a general rule that could be applied to the *n*th polygon. They connect this problem to the handshake problem.

- Students work in groups on the following EWT-like problem from the *Mathematics Instructional Guide* published by the New Jersey State Department of Education:

  The Hiking Club takes a long walk every Saturday. If they hike at a constant speed, which graph shows the relationship between the distance they cover and the time it takes them to cover the distance?



- Students perform an experiment in which they determine how far a toy car rolls from the end of a ramp as the height of the ramp changes. They gather the data, make a scatterplot, and fit a line to the data using the median-median line method. They use the graph to answer additional questions regarding the situation. Some students find the slope and y-intercept of the graph and use these to determine the equation of the line. The class then uses the equation to check its answers to the questions.

9. Understand and use the rectangular coordinate system.

- Students draw the quadrilateral ABCD, where the coordinates of the vertices are A(-3,2), B(4,7), C(2,-3), and D(-5,-6). They produce the figure that results from a size change of ½ and then slide that quadrilateral left 3 units and down 5.
- Students understand that the point (5,-3) is the intersection of two lines which have the equations $x=5$ and $y=-3$. They can identify quickly lines with equations such as $x=5$ and $y=-3$. They can identify the half planes and intersections of half planes identified by inequalities such as $x>5$ and $y< -3$.
- Students study the relationship between perimeters and areas of rectangles. Some students keep the perimeter constant and study the changes in area while others keep the area constant and study the changes in perimeter. Both groups plot their results as graphs and look for patterns.
- Students use squares or grid paper to study the relationship between the radius of a circle and its area (found by counting squares on centimeter grid paper). They graph their data and use it to predict the area of a circle of radius *r*.
- Students play *Green Globs* on the computer, entering equations and trying to hit as many globs as possible with them.

10. Solve simple linear equations using concrete, informal, and graphical methods, as well as appropriate paper-and-pencil techniques.

- Presented with a picture of a balance scale showing objects with unknown weights on both sides as well as known weights (e.g., $3x + 12 = 7x + 4$), students identify the standard algebraic equation related to the picture, describe in words how it would be solved using the concrete objects on the balance scale, and record their actions symbolically.

- Students use algebra tiles to solve an equation like $3(2x + 5) = 21$. They first place 21 units on the right and three groups of two strips and five units on the left. They then note that this is the same as saying 6 strips and 15 units balance 21 units $(6x + 15 = 21)$. Then they take 15 units off both sides, leaving 6 strips balanced with 6 units ($6x = 6$). They conclude that one strip must equal one unit $(x = 1)$.

- As part of a regular exam, students write a verbal explanation describing the relationship between the function $y=2x+4$, its graph, and the equation $2=2x+4$.

11. Explore linear equations through the use of calculators, computers, and other technology.

- The students perform an experiment to answer the question about how long it would take a *wave* to go around Veterans' Stadium. They gather data by timing *waves* done by a various numbers of people, plot the data, and determine the line of best fit using the median-median line method. They determine the number of people that could sit around Veterans Stadium, but they discover that, unlike the previous data sets, they cannot use the graph to answer the question directly. The teacher explains that they need to determine an equation for the line. She has the students investigate functions of the form $y=mx+b$ using a graphing calculator in order to develop the idea that m represents the slope and b represents the y-intercept.

- Using a motion detector connected to a computer or graphing calculator, the teacher has students walk so that the distance from the detector plotted against time is a straight line. The teacher gives directions to students such as: *Walk so that the line has a positive slope, Walk so that the slope is steeper than the last line, Walk so that the line has a slope of 0,* or *Walk so that the line has a negative slope.*

12. Investigate inequalities and nonlinear equations informally.

- Presented with the information that the Cape May-Lewes Ferry has space for 20 cars, and a bus takes up the space of 3 cars, students are asked to draw a graph which represents how many cars and how many buses can be taken across on one trip. Students use variables to represent the unknowns *(x for cars and y for buses)* and develop the inequality x+3y   20 as a model for the situation. Recognizing that the solutions have to be whole numbers, they identify the points whose coefficients are non-negative integers and in the first quadrant on or below the line.

- Students use a motion detector, connected to a calculator or computer, to record the motion of a ping-pong ball tossed by a small catapult. The motion detector is on the floor below the trajectory of the ball. Students note that the graph of distance against time is not linear. They experiment with different initial velocities and different release points to see how these affect the graph.

13. Draw freehand sketches of, and interpret, graphs which model real phenomena.

- Students are asked to draw a sketch of the graph which would describe a person's distance off the ground during a ride on a ferris wheel which had a radius of 60 feet. Some students just draw a curve that looks similar to a sine curve. Others put more detail into their drawing showing the step function behavior which occurs as people get on and get off and that there are limited revolutions permitted.
- Presented with a graph showing the population of frogs in a local marsh over the past ten years, students generate hypotheses for why the curve has the shape it does. They check their hypotheses by talking with a local biologist who has studied the marsh over this time period.

References

New Jersey State Department of Education. *Mathematics Instructional Guide*. D. Varygiannes, Coord. Trenton: 1996.

Software

Green Globs and Graphing Equations. Sunburst Communications.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 13 — Algebra — Grades 9-12

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions,** and **solving equations and inequalities.**

With the foundation developed in the K-8 program, students should be able to be successful in most secondary algebra programs. However, instructional strategies should continue to focus on algebra as a means for representing and **modeling real situations** and answering questions about them. The traditional methods of teaching algebra have been likened to teaching a foreign language, focusing on grammar and not using the language in real conversation. Algebra courses and programs must encourage students to "speak the language" as well as use "proper grammar."

Algebraic understanding is necessary for all students regardless of the structure of the 9-12 program. Students in mathematics programs from technical/basic through college preparatory programs should learn a common core of algebra, with the remainder of the program based on their particular needs. All students should learn the same basic ideas. All students benefit from instructional methods which provide context for the content. Such an approach makes algebra more understandable and motivating.

Techniques for manipulating algebraic expressions remain important, especially for students who may continue into a calculus program. These can be woven into the curriculum or they might all be combined into separate courses labeled "algebra" taken by students who intend to pursue a mathematics-related career. No matter how this instruction is organized, however, instruction must produce students who understand the logic and purposes of algebraic procedures.

Students should be comfortable with **evaluating expressions** and with **solving equations and inequalities**, by whatever means they find most appropriate. They should understand the relationship between the graphs of functions and their equations. Prior to high school, they have focused predominantly on linear functions. In high school, students should gain more familiarity with nonlinear functions. They should develop the ability to solve equations and inequalities using appropriate paper-and-pencil techniques as well as technology. For example, they should be able to understand and solve quadratic equations using factoring, the quadratic formula, and graphing, as well as with a graphing calculator. They should recognize that the methods they use can be generalized to be used when functions look different but are actually composite functions using a basic type (e.g., $\sin^2 x + 3 \sin x + 2 = 0$ is like $x^2 + 3x + 2 = 0$); this method is sometimes called "chunking." This use of **patterns** to note commonalities among seemingly different problems is an important part of algebra in the high school.

Algebraic instruction at the secondary level should provide the opportunity for students to revisit problems. Traditional school problems leave students with the impression that there is one right answer and that once an answer is found there is no need to continue to think about the problem. Since algebra is the language of generalization, instruction in this area should encourage students to ask questions such as *Why does the solution behave this way?* They should develop an appreciation of the way algebraic representation can make problems easier to understand. Algebraic instruction should be rich in problems which are meaningful to students.

Algebra is the gatekeeper for the future study of mathematics and of science, social sciences, business, and a host of other areas. In the past, algebra has served as a filter, screening people out of these opportunities. For New Jersey to be part of a global society, it is important that 9-12 instruction in algebra play a major role in the culminating experiences of a twelve-year program that opens these gates for all.

Standard 13 — Algebra — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

14. Model and solve problems that involve varying quantities using variables, expressions, equations, inequalities, absolute values, vectors, and matrices.

- Students take data involving two variables in an area of interest to them from the World Almanac, construct a scatterplot, and predict the type of equation or function which would best model the data. They use a computerized statistics package or a calculator to fit a function of this type to the data. They choose from linear, quadratic, exponential or logarithmic methods and discuss how well the model fits as well as the limitations.
- Students use matrices to represent tabular information such as the print runs below for each of two presses owned by a book company. They then calculate a third matrix that indicates the growth in production of each press from 1987 to 1988 and discuss the meaning of the data contained in it.

1988	**Textbooks**	**Novels**	**Nonfiction**
Press 1	250,000	125,000	312,000
Press 2	60,000	48,000	90,000

1987	**Textbooks**	**Novels**	**Nonfiction**
Press 1	190,000	100,000	140,000
Press 2	45,000	60,000	72,000

  Some students perform the operations by hand while others explain how they would do it and then use their graphing calculator or a spreadsheet, or write a computer program, which accomplishes the task.

- Students use vectors to determine the path of a plane that was flying due north at 300 miles per hour while a wind was blowing from the southwest at a speed of 15 miles per hour. Students draw a diagram and use an algebraic approach.

- Students work in groups on the following HSPT-like problem from the *Mathematics Instructional Guide* published by the New Jersey State Department of Education:

  As you ride home from a football game, the number of kilometers you are away from home depends (largely) on the number of minutes you have been riding. Suppose that you are 13 km from home when you have been riding for 10 minutes, and 8 km from home when you have been riding for 15 minutes. (Assume that the distance varies linearly with time.) Make a graph with the vertical axis representing distance home and the horizontal axis representing time. Label your graph. Plot the data given as two points on your graph. About how long did it take (on average) to travel 1 km? About how far was the football game from your home? Explain your answer.

- Students work through the *Breaking the Mold* lesson that is described in the Introduction to this *Framework.* They use a science experiment involving growing a mold to learn about exponential growth of populations and compound interest.

- Students work through the *What's My Line* unit that is described in the Keys to Success chapter of this *Framework.* In this unit, students find a linear relationship between the length of a person's thigh bone and his height, and use this to estimate the height of a person whose thigh bone has been found.

- Students work through the *Ice Cones* lesson that is described in the First Four Standards of this *Framework.* In this lesson, an ice-cream vendor's problem of folding a circle into a cone of maximum volume is solved by expressing the volume as a function which is then displayed on an graphing calculator.

15. Use tables and graphs as tools to interpret expressions, equations, and inequalities.

- On a test, students are asked to determine the truth value of the statement "$\log x > 0$ for all positive real numbers x." One student remembers that log x is the exponent to which 10 must be raised to get x, and to get a number less than 1 would require a negative exponent. Another student picks some trial points, develops a chart of the points and their logarithms and discovers that when $x < 1$, $\log x < 0$. A third student graphs the function on his graphing calculator and sees that when $x < 1$, the graph of log x is below the x-axis.

- Faced with the problem of solving the inequality $x^2 - 3x \quad 4$, some students use the equation $x^2 - 3x - 4 = 0$ to determine the boundary points of the interval that satisfies the inequality. They factor the equation and find that these endpoints are -1 and 4. They place dots on the number line at those points, since they know the endpoints are included in the solution set, and then substitute 0 for x in the original inequality. When they find that the resulting statement is true, they shade the interval connecting the two points to obtain their solution $-1 \quad x \quad 4$. Some students determine the endpoints in the same way but roughly sketch a graph of the parabola $y = x^2 - 3x - 4$ and determine that since the inequality was , the problem was asking when the parabola was below the x-axis. Their graph indicates that this happens in the region between the two endpoints. Other students used their graphing calculators to graph the function $x^2 - 3x$ and used the trace function to determine when the graph was on or below the line $y=4$. All students, however, are able to use all these methods to solve the problem.

16. Develop, explain, use, and analyze procedures for operating on algebraic expressions and

matrices.

- Students use algebra tiles to develop procedures for multiplying binomials and factoring trinomials. They summarize these procedures in their math notebooks, applying them to the solution of real-world problems. They work through the *Making Rectangles* lesson described in the First Four Standards of this *Framework,* where they discuss which combinations of tiles can be formed into rectangles, and relate this question to factoring trinomials.

- Students work in groups on the following HSPT-like problem from the *Mathematics Instructional Guide* (p. 7-153) published by the New Jersey State Department of Education:

  Which of the following is NOT another name for 1?

  A. $\frac{x}{x}\,,\ x \quad 0$ **B.** $\frac{x+2}{2+x}\,,\ x \quad -2$

  C. $\frac{x \div 2}{x \div 2}\,,\ x \quad 0$ **D.** $\frac{x-2}{2-x}\,,\ x \quad 2$

- After many experiences with trying to determine appropriate windows for graphing functions on computers or graphing calculators, students develop an understanding of the need to know what the general behavior of a function will be before they use the technology. Students are then asked to explain how they could determine the behavior of the graph of the function below.

$$F(x) = \frac{x^3 + 5x^2 - 6x}{x^3 - 36x}$$

  Students factor the numerator and denominator to determine the values which make them zero and use those values to identify the x-intercepts and vertical asymptotes, respectively. They discuss the fact that the factors x and $x+6$ appear in both places and lead to a removable discontinuity represented by a hole in the graph. They discuss the end behavior of the function as approaching $y = 1$ and the behavior near the vertical asymptote of $x = 6$.

- Following a unit on combinations and binomial expansion, students make a journal entry discussing the power of Pascal's triangle in expanding powers of binomial expressions as compared to the traditional multiplication algorithm.

17. Solve equations and inequalities of varying degrees using graphing calculators and computers as well as appropriate paper-and-pencil techniques.

- Students are asked to find the solutions to $2^x = 3x^2$. Some students use a spreadsheet to develop a table of values. Once they find an interval of length 1 which contains a solution, they refine their numbers to develop the answer to the desired precision. Other students graph both $y = 2^x$ and $y = 3x^2$ using graphing calculators or computers and use the trace function to determine where they intersect. Other students graph the function $y = 3x^2 - 2^x$

and use the trace function to find the zeroes. Other students enter the function in their graphing calculator and check the table of values.

- As a portion of a final assessment, students are given one opportunity to place a cup which is supported 6 inches off the ground in such a position as to catch a marble rolled down a ramp. They perform the roll without the cup to locate the point where the marble strikes the ground. They measure the height of the end of the ramp above the ground and the distance from the point on the ground directly beneath the end of the ramp to the point where the marble struck the ground. They generate the quadratic function which models the path of the marble. Several students use different methods to ensure they have the correct function. Then they decide where they will have to place the cup by substituting 6 for the function value and determining the corresponding x-value. Students solve the equation using the quadratic formula, and the trace function on a graphing calculator, and proceed to place the cup and roll the ball only when the solutions produced by all of the methods agree.

18. Understand the logic and purposes of algebraic procedures.

- Students use matrices as arrays of information, so that the matrix below, for example, is recognized as representing the four vertices {(1,4), (5,6), (3, – 2), (– 2, – 2)} of a polygon. Reducing the polygon by ½ can then be represented by multiplying the matrix by the scalar ½ and moving the polygon to the right one unit can be represented by adding to it a 2 x 4 matrix whose top row consists of 1s and its bottom row of 0s.

$$\begin{bmatrix} 1 & 5 & 3 & -2 \\ 4 & 6 & -2 & -2 \end{bmatrix}$$

- Students read *Mathematics in the Time of the Pharaohs* by Richard Gillings to understand the development of Egyptian mathematics including computational procedures for dealing with direct and inverse proportions, linear equations, and trigonometric functions.
- Students explain in their journals how the identity matrix is like the number one.

19. Interpret algebraic equations and inequalities geometrically, and describe geometric objects algebraically.

- Students investigate the characteristics of the linear functions. For example: *In $y = kx$, how does a change in k affect the graph? In $y = mx + b$, what does b do? Does k in the first equation serve the same purpose as m in the second?* Students use the graphing calculator to investigate and verify their conclusions.
- Students investigate the effects of a dilation and/or a horizontal or vertical shift on the coefficients of a quadratic function. *For example: How does moving the graph up 3 units affect the equation? How does moving the graph right 3 unites affect the equation?*
- Students look at the effects of changing the coefficients of a quadratic equation on the graph. For example: *How is the graph of $y = 4x^2$ different from that of $y = x^2$? How is $y = .2x^2$ different from $y = x^2$? How are $y = x^2 + 4$, $y = x^2 - 4$, $y = x^2 - 4x$, and $y = x^2 -4x + 4$ each different from $y = x^2$?* Students use graphing calculators to look at the graphs and summarize their conjectures in writing. They also work through the *Building Parabolas* lesson described in the First Four Standards of this *Framework,* where students discuss the

general equation of a parabola and use *Green Globs* software to find equations of parabolas that pass through specified points.

- Students study the behavior of functions of the form $y = ax^n$. They investigate the effect of *a* on the curve and the characteristics of the graph when *n* is even or odd. They use the graphing calculator to assist them and write a sentence summarizing their discoveries.

- Students are asked to consider the following situation:

  A landscaping contractor uses a combination of two brands of fertilizers, each containing a different amount of phosphates and nitrates. In a package, brand A has 4 lb. of phosphates and 2 lb. of nitrates. Brand B contains 6 lb. of phosphates and 5 lb. of phosphates. On her current job, the lawn requires at least 24 lb. of phosphates and at least 16 lb. of nitrates. How much of each fertilizer does the contractor need?

  Students represent the given conditions as inequalities and use the intersection of their regions as the set of feasible answers.

- Students recognize that solving two equations simultaneously like 2x+y=5, 4x–y=1 amounts to finding the point of intersection of the two lines with equations y=–2x+5, y=4x–1. Similarly they recognize that solving a quadratic equation like $2x^2 - 3x - 5 = 0$ amounts to finding where the parabola $y = 2x^2 - 3x - 5$ crosses the x-axis.

References

Gillings, Richard. *Mathematics in the Time of the Pharoahs.*

New Jersey State Department of Education. *Mathematics Instructional Guide*. D. Varygiannes, Coord. Trenton: 1996.

Software

Green Globs and Graphing Equations. Sunburst Communications.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.