STANDARD 11 — PATTERNS, RELATIONSHIPS, AND FUNCTIONS

K-12 Overview

All students will develop an understanding of patterns, relationships, and functions and will use them to represent and explain real-world phenomena.

Descriptive Statement

Patterns, relationships, and functions constitute a unifying theme of mathematics. From the earliest age, students should be encouraged to investigate the patterns that they find in numbers, shapes, and expressions, and, by doing so, to make mathematical discoveries. They should have opportunities to analyze, extend, and create a variety of patterns and to use pattern-based thinking to understand and represent mathematical and other real-world phenomena. These explorations present unlimited opportunities for problem solving, making and verifying generalizations, and building mathematical understanding and confidence.

Meaning and Importance

Mathematics is often regarded as the *science of patterns*. When solving a complex problem, we frequently suggest to students that they try to work on simpler versions of the problem, observe what happens in a few specific cases — that is, *look for a pattern* — and use that pattern to solve the original problem. This *pattern-based thinking*, using patterns to analyze and solve problems, is an extremely powerful tool for doing mathematics. Students who are comfortable looking for patterns and then analyzing those patterns to solve problems can also develop understanding of new concepts in the same way. Most of the major principles of algebra and geometry emerge as generalizations of patterns in number and shape. For example, one important fact in geometry is that: *For a given perimeter, the figure with the largest possible area that can be constructed is a circle*. This idea can be discovered informally by students in the middle grades by examining the pattern that comes from a series of constructions and measurements. Students can be given a length, say 24 centimeters, for the perimeter of all figures to be created. Then they can construct and measure or compute the areas of a series of regular polygons: an equilateral triangle, a square, and a regular hexagon, octagon, and dodecagon (12 sides). The pattern that clearly emerges is that as the number of sides of the polygon increases — that is, as the polygon becomes more “circular”— the area increases.

All of the content standards are interconnected, but this standard is one that is particularly closely tied to all of the others. This is because pattern-based thinking is regularly applied to content in numeration, geometry, operations, discrete mathematics, and the fundamentals of calculus. There is a very special relationship, though, between patterns and algebra. Algebra provides the language in which we communicate the patterns in mathematics. Early on in their mathematical careers, students must begin to make generalizations about patterns that they find, and they should learn to express those generalizations in mathematical terms.

K-12 Development and Emphases

Children become aware of patterns very early in their lives — repetitive daily routines and periodic phenomena are all around them. Breakfast is followed by lunch which is followed by dinner which is followed by bedtime and then the whole thing is repeated again the next day. Each one of the three little pigs says to the wolf, at exactly the expected moment, *Not by the hair on my chinny-chin-chin!* In the primary grades, children need to build on those early experiences by **constructing, recognizing, and extending patterns** in a variety of contexts. Numbers and shapes certainly offer many opportunities, but so do music, language, and physical activity. Young children love to imitate rhythmic patterns in sound and language and should be encouraged to create their own. In addition, they should construct their own patterns with manipulatives such as pattern blocks, attribute blocks, and multilink cubes and should be challenged to extend patterns begun by others. Identifying attributes of objects, and using them for **categorization and classification**, are skills that are closely related to the ability to create and discover patterns and need to be developed at the same time.

Young students should frequently play games which ask them to follow a sequence of rules or to **discover a rule** for a given pattern. Sequences which begin as counting patterns soon develop into rules involving arithmetic operations. Children in the primary grades, for example, will make the transition from *2, 4, 6, 8, ...* as a counting by twos pattern to the rule *Add 2* or "*+2*." The calculator is a very useful tool for making this connection since it can be used for counting up or counting down by any constant amount. Students can be challenged to guess the number that will come up next in the calculator's display and then to explain the pattern, or rule, to the class.

At a slightly higher level, **input-output** activities which require recognition of relationships between one set of numbers (the "IN" values) and a second set (the "OUT" values) provide an early introduction to **functions**. One of these kinds of activities, the *function machine* games, is a favorite among first through fifth graders. In these, one student has a rule in mind to transform any number suggested by another student. The first number is inserted into the imaginary *function machine* and another number comes out the other side. The rule might be *plus 7*, or, *times 4 then minus 3*, or even *the number times itself*. The class's task is to discover the rule by an examination of the input-output pairs. In the intermediate grades, students can simulate the *function machine* with a computer spreadsheet *secretly* programmed to take the number typed in the first column and *transform* it into another number that is placed in the second column.

Slightly older students begin to work with patterns that can be used to **solve problems** within mathematics and from the real world. There should also be a more deliberate focus on **relationships involving two variables**. An exploration of the relationship between the number of teams in a round robin tournament and the total number of games that must be played, or between a number of coins to be flipped and the total number of possible outcomes, provides a real-world context for pattern-based thinking and informal work with functions. Graphing software and graphing calculators are extremely valuable at this level to help students visualize the relationships they discover.

At the secondary level, students are able to bring more of the tools of algebra to the task of analyzing and representing patterns and relationships. Thus we expect all students to be able to construct as well as to recognize symbolic representations such as $y = f(x) = 4x+1$. They should also develop an understanding of the many other representations and applications of functions as well as of a greater variety of functional relationships. Their work should extend to quadratic, polynomial, trigonometric, and exponential functions in addition to the linear functions they worked with in earlier grades. They should be comfortable with the symbols *f*, representing a rule, and *f(x)*, representing the value which *f* assigns to *x*.

The use of functions in modeling real-life and real-time observations also plays a central role in the high school mathematics experience. Line- and curve-fitting as approaches to the explanation of a set of experimental data help make mathematics come alive for students. Technology must play an important role in this process, since students are now able to graphically explore relationships more easily than ever before. Graphing calculators and computers must be made available to all students for use in these types of investigations.

IN SUMMARY, an important task for every teacher of mathematics is to help students recognize, generalize, and use patterns that exist in numbers, in shapes, and in the world around them. Students who have such skills are better problem solvers, have a better sense of the uses of mathematics, and are better prepared for work with algebraic functions than those who do not.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 11 — Patterns, Relationships, and Functions — Grades K-2

Overview

The development of *pattern-based thinking*, using patterns to analyze and solve problems, is an extremely powerful tool for doing mathematics, and leads in later grades to an appreciation of how functions are used to describe relationships. The key components of *pattern-based thinking* at the early grade levels, as identified in the K-12 Overview, are **recognizing, constructing, and extending patterns, categorizing and classifying objects,** and **discovering rules.**

> "Looking for patterns trains the mind to search out and discover the similarities that bind seemingly unrelated information together in a whole. . . . A child who expects things to 'make sense' *looks* for the *sense* in things and from this sense develops understanding. A child who does not see patterns often does not *expect* things to make sense and sees all events as discrete, separate, and unrelated."
> — Mary Baratta-Lorton (cited on p.112 of *About Teaching Mathematics* by Marilyn Burns)

Children in the primary grades develop an awareness of patterns in their environment. Those who are successful in mathematics expand this awareness into understanding and apply it to learning about the number system. Children who do not look for patterns as a means of understanding and learning mathematics often find mathematics to be quite difficult. Thus, it is critical in the early grades to establish an early predisposition to looking for patterns, creating patterns, and extending patterns.

Children should **recognize, construct and extend patterns** with pattern blocks, cubes, toothpicks, beans, buttons and other concrete objects. Children in kindergarten can recognize patterns in motion, color, designs, sound, rhythm, music, position, sizes, and quantities. They are very aware of sound and rhythm, and can clap out patterns that repeat, such as clap-clap-clap-pause, clap-clap-clap-pause, etc. They can sit in a circle and wear colored hats which make a pattern, such as red-white-blue, red-white-blue. One child can walk around the circle and tap successive children in an arm-shoulder-head pattern. The teacher may ask the class who the next person to be tapped on the head would be if the pattern were to be continued. In addition to repeating patterns, students should have experiences with expanding patterns. They can indicate such a pattern by using motion: skip-jump-turn around, skip-jump-jump-turn around, skip-jump-jump-jump-turn around, and so on. Songs are excellent examples of repetition of melody or of words, as well as of rhythmic patterns. Children's literature abounds with stories which rely on rhythm, rhyming, repetition and sequencing. As students move on to first and second grade, they should start to create their own patterns and develop pictorial and symbolic representations of those patterns. The transition will be from working with patterns using physical objects to using pictures, letters, and geometric figures in two and three dimensions, and then to using symbols, such as words and numbers, to represent patterns.

Categorization and classification are also important skills for students in the primary grades. Kindergartners should have numerous opportunities to sort, classify, describe, and order collections of many different types of objects. For example, students might be asked to sort attribute shapes, buttons, or boxes into two groups and explain why they sorted them as they did. This area offers an excellent opportunity for students to verbalize their thought processes and to integrate learning in mathematics and science as they sort natural objects such as shells, rocks, or leaves.

Discovering a rule and **input-output** games are two other settings in which primary children can enhance their work and their skills with patterns. The children might be asked to solve the *mystery of the crackers* as the teacher slowly and deliberately gives every boy two crackers and every girl four crackers one day during snack time. The inequity is addressed, of course, as soon as the children solve the mystery by discovering the rule that the teacher was using. On a different day, first graders can be told that they may request between 3 and 5 crackers for snack. But then each child is actually given two crackers less than his or her request. Again, as soon as the children verbalize the relationship between the request (input) and the portion allotted (output), they receive the missing crackers.

Establishing the habit of looking for patterns is exceedingly important in the primary grades. By studying patterns, young children develop necessary tools to become better learners of mathematics as well as better problem solvers. In addition, patterns help students to appreciate the beauty of mathematics and to make connections within mathematics and among mathematics and other subject areas.

Standard 11 — Patterns, Relationships, and Functions — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and in grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Reproduce, extend, create, and describe patterns and sequences using a variety of materials.**
   - Students make a collage with examples of patterns in nature.
   - Students create visual patterns with objects, colors, or shapes using materials such as buttons, macaroni, pattern blocks, links, cubes, attrilinks or attribute blocks, toothpicks, beans, or teddy bear counters. They challenge other students to describe or extend their patterns.
   - Students sort objects such as leaves, buttons, animal pictures, and blocks, using categories corresponding to characteristics like number of holes, number of sides, shapes, or thickness.
   - One child walks around the outside of a circle and taps successive children in a head-shoulder-shoulder-head pattern. The teacher asks who the next person to be tapped on the head would be if the pattern were to be continued. The children sing and act out the song, *Head, shoulders, knees and toes*.
   - Students describe patterns made from circles, triangles, and squares, and select the next shape in the pattern.
   - Students make patterns with letters and extend the sequence.
   - As an assessment task, students use letters to translate patterns they have created with objects — for example, RRBRRB for a Unifix pattern of red-red-blue-red-red-blue, or ABBCABBC for a shape pattern of square - circle - circle - triangle - square - circle - circle - triangle.
   - Students connect the dots to make a picture by following a number sequence, such as 2, 4, 6, 8, ...
   - Students create *one more* and *one less* patterns.
   - Students create patterns with the calculator. They enter any number such as 10, and then add 1 for 10+1= = = ... The calculator will automatically repeat the function and display 11, 12, 13, 14, etc. Some calculators may need to have the pattern entered twice: 10+1=+1= = = ... Other calculators will need 1++10= = = ... Students may repeatedly add or subtract any number.
   - Students name things that come in pairs (or 4s or 5s): eyes, ears, hands, arms, legs, mittens, shoes, bicycle wheels, etc. They work in pairs to find how many people there are if there are 20 eyes.
   - Students count by 2, 5, or 10 using counters or creating color patterns with Unifix or Linker

cubes; they repeat this using skip counting on a number line.

- Students use skip counting or calculators to find multiples of numbers and then color them on the hundreds chart. Linking cubes or Unifix cubes can be used to build towers or trains with every other cube or every third cube a certain color to illustrate, recognize, and practice skip counting patterns.
- Students write their first name repeatedly on a 10x10 grid, and then color the first letter of their name to create a pattern. They discuss the patterns formed.
- Students identify the same pattern in a variety of contexts. For example, black-white-black-white is like sit-stand-sit-stand and ABAB and up-down-up-down and straight-curve-straight-curve.
- Students identify patterns on a calendar using pictures or numerals. For example, in November, even dates might be marked with a snowflake, and odd dates with a picture of a turkey. Or, they might mark each date with the day of the week.
- Students create a pattern using various rubber stamp blocks or picture designs.
- Students use or create patterns with geometric figures (circles, triangles, squares, pentagons, hexagons, etc.) and record how many of each shape exist after each repeating cluster.
- Students create a mosaic design (tessellation) made of different shapes using objects such as pattern blocks. They color congruent shapes of a mosaic design with the same color.

2. Use tables, rules, variables, open sentences, and graphs to describe patterns and other relationships.

- Students complete a table given several starting numbers and a verbal rule.
- Kindergartners look at *Anno's Counting House* by Mitsumasa Anno to see if they can figure out the pattern that is used in moving from one set of pages to the next. The people in this book move, one by one, from one house to another.
- Students describe the pattern illustrated by the numbers in a table by using words (e.g., one more than), and then the teacher helps them to represent it with symbols in an open sentence (☐ = ♡ + 1).
- Students use colored squares to make a graph showing the multiples of 3 and relate this to a table and an expression involving a variable, such as 3 x ☐.

3. Use concrete and pictorial models to explore the basic concept of a function.

- Students study the pictures in *Anno's Math Games II* by Mitsumasa Anno. As they do, they try to figure out what happens to the objects as the elves put them into the magic machine. Sometimes the number of objects doubles, sometimes the objects grow eyes, and sometimes the objects turn into circles.
- Students put numbers into *Max the Magic Math Machine* and read what comes out. (The teacher acts as Max.) Then they describe what Max is doing to each number. The teacher pays careful attention to the students' responses to assess their levels of understanding.
- Students investigate a hole-making machine that puts 4 holes into buttons. They make a table that shows the number of buttons put into the machine and the total number of holes

that must be made in them. Then they write a sentence that describes how the total number of holes changes as new buttons are added.

- Students play *Guess my Rule*. The teacher gives them a starting number and the result after using the rule. She continues giving examples until students discover the rule.
- Students count the number of pennies (or nickels) in 1 dime, 2 dimes, 3 dimes and record their results in chart form. They study the patterns and discuss the *rules* observed.
- Students consider the cost of two or three candies if one candy costs one dime. They make a chart using the information.
- Students count the number of lifesavers in an assorted pack. They make a table showing the number of each color and the total number in one pack. Then, assuming all of the packs are the same, they make a table showing the total number of each color for 2 packs, 3 packs, 4 packs, and so on. They check their results with packs of lifesavers, which in general, have the same number of each color.

4. Observe and explain how a change in one physical quantity can produce a corresponding change in another.

- Students discuss how ice changes to water as it warms. They talk about how it snows in January or February but rains in April or May.
- Students plant seeds and watch them grow. They write about what they see and measure the height of their plants as time passes. They discuss how changes in time bring about changes in the height of the plants. They also talk about how other factors might affect the plants, such as light and water.

5. Observe and recognize examples of patterns, relationships, and functions in other disciplines and contexts.

- Students go on a *pattern hunt* around the classroom and the school, discussing the patterns they find.
- Students sing and act out songs like “Rattlin’ Bog” (Bird on the leaf, and the leaf on the tree, and the tree in the hole, and the hole in the ground, . . .) and “Old MacDonald Had a Farm.”
- In reading, students recognize patterns in rhythm, in rhyming, in syllables and in sequencing. Stories such as *Ten Black Dots* by Donald Crews, *Five Little Monkeys Jumping on a Bed* by Eileen Christelow, *Jump, Frog, Jump* by Robert Kalan, *The Little Red Hen*, and Dr. Seuss books offer such opportunities. Visual patterns can be shown using picture representations for children’s books such as *1 Hunter* by Pat Hutchins, *Rooster’s Off to See the World* by Eric Carle, *The Patchwork Quilt* by Valerie Flournoy, and *The Keeping Quilt* by Patricia Polacco.
- Students identify every third letter of the alphabet; every fourth letter, etc. They use those sets of letters to see what words they can make.
- Students choose a day. Using a calendar, they identify the name of the next day, of the previous day, and also the name of the day two days (or more) before and after. They select a date, and give the date of the next day and of the previous day, the name of the month, of the next month, and of the previous month. They give the name of the date two days before

and after, and three days (or more) before and after.

- Students graph daily weather patterns, showing sunny, cloudy, rainy or snowy days. Then they discuss monthly or seasonal patterns.
- In social studies, students identify traffic patterns such as how many cars, trucks, or buses pass the front of the school during five minutes at different times of the day. They keep records for five days, organizing the information in chart form.
- In art, students observe patterns in pictures, mosaics, tessellations, and Escher-like drawings, as well as in wallpaper, fabric, and floor tile designs.

6. Form and verify generalizations based on observations of patterns and relationships.

- Students draw pictures of faces and make a table that shows the number of faces and the number of eyes. The teacher writes a sentence on the board that the class composes, describing the patterns that they find.
- Students observe that there are 12 eggs in a carton of eggs. These are called a dozen. They explain how to find the number of eggs in 2 cartons, 3 cartons, and so on.
- Students write a sentence or more telling about the patterns they have observed in a particular activity. They may use pictures to describe or generalize what they have observed. For example, after students have colored multiples of a certain number on the hundreds chart, they write about the geometric pattern they observe on the chart.

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Flournoy, Valerie. *The Patchwork Quilt*. New York: Dial Books, 1985.

Hutchins, Pat. *1 Hunter*. New York: Greenwillow Books, 1982.

Kalan, Robert. *Jump, Frog, Jump*. New York: Greenwillow Books, 1981.

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Seuss, Dr. Most Dr. Seuss books exhibit appropriate patterns.

The Little Red Hen. Many versions are available.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post

additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 11 — Patterns, Relationships, and Functions — Grades 3-4

Overview

The development of *pattern-based thinking*, using patterns to analyze and solve problems, is an extremely powerful tool for doing mathematics, and leads in later grades to an appreciation of how functions are used to describe relationships. The key components of *pattern-based thinking* at the early grade levels, as identified in the K-12 Overview, are **recognizing, constructing, and extending patterns, categorizing and classifying objects discovering rules,** and **working with input—output situations.**

In grades 3 and 4, students begin to learn the importance of investigating a pattern in an organized and systematic way. Many of the activities at these grade levels focus on creating and using tables as a means of analyzing and reporting patterns. In addition, students in these grades begin to move from learning about patterns to learning with patterns, using patterns to help them make sense of the mathematics that they are learning.

Students in grades 3-4 continue to **construct, recognize, and extend patterns**. At these grade levels, pictorial or symbolic representations of patterns are used much more extensively than in grades K-2. In addition to studying patterns observed in the environment, students should use manipulatives to investigate what happens in a pattern as the number of terms is extended or as the beginning number is changed. Students should also study patterns that involve multiplication and division more extensively than in earlier grades. Students continue to investigate what happens with patterns involving money, measurement, time, and geometric shapes. They should use calculators to explore patterns.

Students in these grades continue to **categorize and classify** objects. Now categories can become more complex, however, with students using two (or more) attributes to sort objects. For example, attribute shapes can be described as red, large, red and large, or neither red nor large. Classification of naturally-occurring objects, such as insects or trees, continues to offer an opportunity for linking the study of mathematics and science.

Students in grades 3 and 4 are more successful in playing **discover a rule** games than younger students and can work with a greater variety of operations. Most students will still be most comfortable, however, with one-step rules, such as *multiplying by 3* or *dividing by 4*.

Third and fourth graders also continue to work with **input-output situations.** While they still enjoy putting these activities in a story setting (such as *Max the Magic Math Machine* which takes in numbers and hands out numbers according to certain rules), they are now able to consider these situations in more abstract contexts. Students at this age often enjoy pretending to be the machine themselves and making up rules for each other.

In grades 3 and 4, then, students expand their study of patterns to include more complex patterns based on a greater variety of numerical operations and geometric shapes. They also work to organize their study of patterns more carefully and systematically, learning to use tables more effectively. In addition, they begin to apply their understanding of patterns to learning about new mathematics concepts, such as multiplication and division.

Standard 11 — Patterns, Relationships, and Functions — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4..

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Reproduce, extend, create, and describe patterns and sequences using a variety of materials.**
    - Students make a pattern book that shows examples of patterns in the world around them.
    - Students use pattern blocks, attribute blocks, cubes, links, buttons, beans, toothpicks, counters, crayons, magic markers, leaves, and other objects to create and extend patterns. They might describe a pattern involving the number of holes in buttons, the number of sides in a geometric figure, the shape or the thickness of objects.
    - Students use sequences of letters or numbers to identify the patterns they have created.
    - Students investigate the sum of the dots on opposite faces of an ordinary die and find they always add up to 7.
    - Students solve two-dimensional attribute block patterns where, for instance, each column is a different shape and each row is a different color. They should be able to choose the block that fills in the missing cell in such patterns.
    - Students count by 2, 3, 4, 5, 6, 10 and 12 on a number line, on a number grid, and on a circle design.
    - Students begin with numbers between 50 and 100 and count backwards by 2, 3, 5, or 10.
    - Students create patterns with the calculator: They enter any number such as 50, and then repeatedly add or subtract 1 or 2 or 3 etc. If, for example, they enter 50+1=== ... , the calculator will automatically repeat the function and display 51, 52, 53, 54, ... Some calculators may need to have the pattern entered twice: 50+1=+1=== ... . Others may need 1++ 50=== ... .
    - Students begin with a number less than 10, double it, and repeat the doubling at least five times. They record the results of each doubling in a table and summarize their observations in a sentence.
    - Students read *Anno's Magic Seeds* by Mitsumasa Anno. In it, a wizard gives Jack two seeds and tells him that if he eats one, he won't be hungry for a year and if he plants the other one, two new seeds will be formed. Jack continues in this way for awhile and then tries other schemes that produce even more new seeds. The students work in groups to make charts and tables to show how many seeds Jack has at given points in time. As an individual assessment assignment, students are asked to find how many seeds Jack has after ten years using one of the discovered patterns and to support their answers in writing and with tables.
    - Students supply the missing numbers on a picture of a ruler which has some blanks. Then

they explore how to find the missing numbers between any two given numbers on a number line. They extend this to larger numbers; they might label each of five intervals from 200 to 300 or each of four intervals from 1,000 to 2,000.

- Students investigate number patterns using their calculators. For example, they might begin at 30, repeatedly add 6, and record the first 10 answers, making a prediction about what the calculator will show before they hit the equals key. Or they might begin at 90 and repeatedly subtract 9.

2. Use tables, rules, variables, open sentences, and graphs to describe patterns and other relationships.

- As a regular assessment activity, done during the year whenever new numerical operations have been explored, students fill in *guess my rule* tables like those shown below. Sometimes they are given the rule and sometimes they are asked to find the rule.

times 2	
6	?
9	?
2	?
7	?

plus 12	
34	?
58	?
?	37
?	12

?????	
12	4
27	9
9	3
15	?

- Students describe the pattern illustrated by the numbers in a table by using words (e.g., twice as much as) and then represent it with symbols in an open sentence (□ = 2 x ♡).
- On a coordinate grid, students plot coordinate pairs consisting of a number and the product of the number times 3. They join them with a line, making a line graph. They relate this to a table, and write the rule as an expression involving a variable, such as 3 x □.
- Students repeatedly add (or subtract) multiples of 10 to (from) a 3-digit starting number. They describe the pattern orally and write it symbolically as, for example, 357, 337, 317, 297, ...
- Students work in groups to solve problems that involve organizing information in a table and looking for a pattern. For example, *If you have 12 wheels, how many bicycles can you make? How many tricycles? How many bicycles and tricycles together?* Using objects or pictures, children make models and organize the information in a table. They discuss whether they have looked at all of the possibilities systematically and describe in words the patterns they have found. They write about the patterns in their journals and, with some assistance, develop some symbolic notation (e.g., 2 wheels for each bike and 3 wheels for each trike to get 12 wheels all together might become 2xB + 3xT = 12).

3. Use concrete and pictorial models to explore the basic concept of a function.

- Students use buttons with two or four holes and describe how the total number of holes is related to the number of buttons.
- Students use multilink cubes or base ten blocks to build rectangular solids. They count how many cubes tall their structure is, how many cubes long it is, and how many cubes wide it is. Then they count the total number of cubes in their structure. They record all of this

information in a table and look for patterns.

- Students take turns putting numbers into *Max the Magic Math Machine*, reading what comes out, and finding the rule that tells what Max is doing to each number. A student acts as Max each time. Appropriate rules to use in grades 3 and 4 involve multiplication and division.

4. Observe and explain how a change in one physical quantity can produce a corresponding change in another.

- Students use cubes to build a one-story "house" and count the number of cubes used. They add a story and observe how the total number of cubes used changes. They explain how changing the number of stories changes the number of cubes used to build the house.
- Students measure the temperature of a cup of water with ice cubes in it every fifteen minutes over the course of a day. They record their results (time passed and temperature) in a table and plot this information on a coordinate grid to make a broken line graph. They discuss how the temperature changes over time and why.
- Students plant seeds in vermiculite and in soil. They observe the plants as they grow, measuring their height each week and recording their data in tables. They examine not only how the height of each plant changes as time passes but also whether the seeds in vermiculite or soil grow faster.

5. Observe and recognize examples of patterns, relationships, and functions in other disciplines and contexts.

- Students go on a *scavenger hunt* for patterns around the classroom and the school. They are given a list of verbal descriptions of specific patterns to look for, such as a *pattern using squares* or *an ABAB pattern*. They use cameras to make photographs of the patterns that they find.
- Students read *The Twelve Days of Summer* by Elizabeth Lee O'Donnell and Karen Lee Schmidt. Using the same pattern as the song *The Twelve Days of Christmas*, the authors tell the story of a young girl on vacation by the ocean. On the first day, she sees "a little purple sea anemone," on the eighth, "eight crabs a-scuttling," and so on. Since she sees everything that she has previously seen on every succeeding day, the book offers the obvious question *How many things did the little girl see today?*
- Students learn about the different time zones across the country. They describe the number patterns found in moving from east to west, and vice versa.
- Students read books such as *Six Dinner Sid* by Inga Moore or *The Greedy Triangle* by Marilyn Burns. They explore the patterns and relationships found in these books.
- Students study patterns in television programming. For example, they might look at the number of commercials on TV in an hour or how many cartoon shows are on at different times of the day. They discuss the patterns that they find as well as possible reasons for those patterns.

6. Form and verify generalizations based on observations of patterns and relationships.

- Students measure the length of one side of a square in inches. They find the perimeter of one square, two squares (not joined), three squares, and so on. They make a table of values and describe a rule which relates the perimeter to the number of squares. They predict the perimeter of ten squares.
- Students use their calculators to find the answers to a number of problems in which they multiply a two-digit number by 10, 100, or 1000. Looking at their answers, they develop a "rule" that they think will help them do this type of multiplication without the calculator. They test their rule on some new problems and check whether their rule works by multiplying the numbers on the calculator.

References

Anno, Mitsumasa. *Anno's Magic Seed.* New York: Philomel Books, 1995.

Burns, Marilyn. *The Greedy Triangle.* New York: Scholastic Inc., 1994.

O'Donnell, Elizabeth Lee, and Karen Lee Schmidt. *The Twelve Days of Summer.* Morrow Junior Books, 1991.

Moore, Inga. *Six Dinner Sid.* New York: Simon and Schuster, 1993.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 11 — Patterns, Relationships, and Functions — Grades 5-6

Overview

In grades K-4 students have been encouraged to view patterns in the world around them and to use their observations to explore numbers and shapes. In grades 5-6 students will expand their use of patterns, incorporating variables and using patterns to help them solve problem. The key components of *pattern-based thinking*, as identified in the K-12 Overview, involve **exploring, analyzing, and generalizing patterns**, and **viewing rules and input/output situations as functions.**

Patterns, relationships and functions will become a powerful problem solving strategy. In many routine problem solving activities, the student is taught a rote method which will lead to a solution. Thus, when faced with a problem of the same type he or she just uses that method to get the solution. In the real world, however, real problems are not usually packaged as nicely as textbook problems. The information is vague or fuzzy, some of the information needed to solve the problem might be missing, or there might be extraneous information on hand. In fact, a rote method might not exist to solve the problem. Such problems are generally referred to as non-routine problems.

Students who are faced with non-routine problems, and have no standard method for solving them, often simply give up, because they do not know how to get started. The ability to discover and analyze patterns becomes an important tool to help students move forward. When the students start to collect data and look for a pattern in order to solve a problem, they are often uncertain about what they are looking for. As they organize their information into charts or tables and start to analyze their data, sometimes, almost like magic, patterns begin to appear, and students can use these patterns to solve the problem.

Patterns help students develop an understanding of mathematics. Whenever possible, students in grades 5-6 should be encouraged to use manipulatives to **create, explore, discover, analyze, extend and generalize patterns** as they encounter new topics throughout mathematics. By dealing with more sophisticated patterns in numerical form, they begin to lay a foundation for more abstract algebraic concepts. Looking for patterns helps students tie concepts together, gain a greater conceptual understanding of the world of mathematics, and become better problem solvers.

Students in the middle grades should also continue to work with **categorization and classification**, particularly in the context of new mathematical topics, although much less emphasis should be given to these activities. For example, as students learn about fractions and mixed numbers, they must identify fractions as being less than one or more than one, as being in lowest terms or not. They also apply categorization and classification skills in geometry, as they distinguish between different types of geometric figures and learn more about the properties of these figures.

Students in grades 5-6 should begin using letters to represent variables as they do activities in which they are asked to **discover a rule**. They should also begin working with rules that involve more than one operation. Students at this level describe patterns that they see by using diagrams and pictorial representations of a mathematical relationship; some students will be more comfortable starting with manipulatives and then using a pictorial representation. Students should record their findings in words, in tables, and in symbolic equations.

Students in grades 5-6 should begin thinking of **input/output** situations as **functions.** They should recognize that a function machine takes in a number (or shape), operates with a consistent rule, and provides a predictable outcome. They should begin to use letters to represent the number going in and the number coming out, although considerable assistance from the teacher may be needed.

Throughout their work with patterns, students in grades 5-6 should use the calculator as a tool to facilitate computation and allow time for higher level thinking. Teachers should explore its capabilities with the students and encourage its use, so that students become proficient.

Standard 11 — Patterns, Relationships, and Functions — Grades 5-6

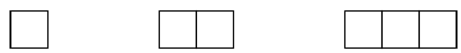
Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

7. Represent and describe mathematical relationships with tables, rules, simple equations, and graphs.

- Students let the length of one side of a square be 1 unit. They then find the perimeter of one square, two squares connected along an edge, three squares connected along their edges, and so forth, as shown below.

  They make a table of values and use it to determine a function rule which describes the pattern. They understand the rule $P = 2 \times s + 2$, where *s* is the number of squares, and they use it to predict the perimeter of ten squares.

- Students use a geoboard to model squares with sides of 1, 2, 3, and 4 units. They determine the area and discover the rule for the area, $A = s^2$, given that *s* is the length of the side of a square.

- Students use multilink cubes or base ten blocks to build rectangular solids. They find the volume of the rectangular solids (either by counting the cubes or by developing their own shortcuts), and record the length, width, and height of each solid along with its volume in a table. They use this information to discover a rule or formula for finding the volume of a rectangular solid.

- Students find the number of primes between 1 and 100 using a hundreds chart and applying the process of the Sieve of Eratosthenes. That is, they first cross out all multiples of 2, then all multiples of 3, then all multiples of 5, and so on; the numbers which remain are the primes.

- Students play the *secret number* calculator game. One student enters a secret number into the calculator by dividing the number by itself (e.g., 17 ÷ 17). She then asks her opponent to guess the number. Each guess is entered into the calculator and then the equals sign is pressed; the calculator shows the result of dividing the guess by the secret number. For example, if 17 is the secret number and 34 is guessed, then the student enters 34 = and sees 2 on the calculator. Play continues until 1 is shown on the calculator, so that the opponent has guessed the secret number. (Some calculators will need to have different keys pressed for the same result, such as 17 ÷÷ 34 =.)

- Students look for the numbers which are palindromes (remain the same value when the digits are reversed) between any given pair of numbers. They decide which years in the 21st century will be the first 5 palindromes.

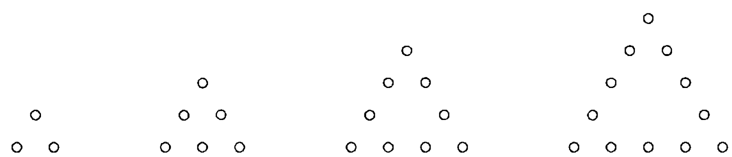
- Students determine how much money is earned hourly for a job mowing lawns or babysitting. They find the amount earned for working different numbers of hours. They organize the data in chart or table form. They look for a pattern and write simple equations; for example, the sentence *For babysitting or mowing lawns, I get $5 per hour.* translates into the equation $E = 5 \times h$ (**E**arnings equal five times the number of **h**ours worked).

- Students use patterns to help them find the value of a point on a number line between two whole numbers when the number line is divided into fractional or decimal parts.

- Students investigate patterns involving arithmetic operations that can be generalized to a mathematical expression with a variable. For example:

1. Choose any number	☆	n
2. Multiply your number by 6	☆☆☆☆☆☆	$6 \times n$
3. Add 12 to the result	☆☆☆☆☆☆ VVVVVV	$6 \times n + 12$
4. Take half	☆☆☆ VVV	$3 \times n + 6$
5. Subtract 6	☆☆☆	$3 \times n$
6. Divide by 3	☆	n
7. Write your answer		n

- Students use lima beans or other counters to create *trinumbers*, the number of beans used for the triangles below. They try to predict what the tenth trinumber will be and then use this result to develop the expression ($3 \times n$) for the nth trinumber.

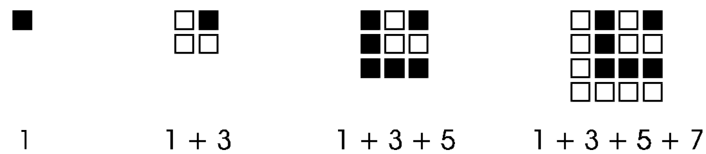


8. Understand and describe the relationships among various representations of patterns and functions.

- Using a 4x4 geoboard or dot paper, students create various sized squares. For each square, they record the length of its side and its area in a table. Then they show their results as a bar graph (side vs. area), as ordered pairs (side, area), as a verbal rule ("the area of a square is the length of its side times itself"), and as an equation ($A = s \times s$, where s is the length of the side). They repeat this for rectangles of varying sizes, recording the length and width and corresponding area in a table. Students discover the pattern and develop the formula for the area of a rectangle, $A = l \times w$, by inspecting the numbers in the table.

- Students cut out squares from graph paper, recording the length of the side of the square and the number of squares around the border of the square. They look for a pattern that will allow them to predict the number of unit squares in the border of a 10 x 10 square and then a 100 x 100 square. They describe their pattern in words. The teacher then helps them to develop a formula ($4 \times n$) 4 for finding the number of unit squares in the border of an $n \times n$ square.

- Students explore patterns involving the sums of consecutive odd integers (1, 1 + 3, 1 + 3 + 5, 1 + 3 + 5 + 7, ...) by using unit squares to make Ls to represent each number and then nesting the Ls, as in the diagram below:



  Then they make a table that shows how many Ls are nested and the total number of unit squares used. They look for a pattern which will help them predict how many squares will be needed if 10 Ls are nested (i.e., if the first 10 odd numbers are added together). They make a prediction and describe how they generated their prediction (e.g., when you add the first 3 odd numbers, it makes a square that is three units on a side, so when you add the first 10 odd numbers, it should make a square that is ten units on a side and you will need 100 squares). They share their solution strategies with each other and develop one (or more) expressions, like *n* x *n*, that can be used to find the sum of the first *n* odd numbers.

9. Use patterns, relationships, and functions to model situations and to solve problems, in mathematics and in other subject areas.

- Students use calculators to study fractions whose decimal expressions repeat, and predict what digit is in any given place.
- Students look at what happens when a ball is hit at a 45 angle on a rectangular pocket billiards table. They make a table that shows dimensions of various tables, initial positions of the ball, and which pocket, if any, the ball eventually goes into. They look for patterns which will help them predict which pocket the ball will go into for other situations.
- Students work in groups to decide how to make a supermarket display of boxes of SuperCrunch cereal. The boss wants the boxes to be in a triangular display which is 10 boxes high and one box deep. Each box is 12 inches high and 8 inches wide. The students use patterns to help them decide how many boxes they would need and whether this is a practical way to display the cereal.
- Students look at the fee structure for crossing a toll bridge near them, such as the Walt Whitman bridge into Pennsylvania or the George Washington bridge into New York, and use patterns to help them decide whether it makes sense for someone who works on the other side of the bridge for 12 days each month to buy a commuter sticker.

10. Analyze functional relationships to explain how a change in one quantity results in a change in another.

- Students predict what size container is needed to hold pennies if, on the first day of a 30-day month, they put in one penny and double the number of pennies each succeeding day. After making their predictions, they calculate how many days it will take to fill that container, and how many containers like that they would actually need for the whole month.

- Students make a chart that helps them understand the charges for a taxi ride when the taxi charges $2.75 for the first 1/4 mile and $.50 for each additional 1/4 mile. They look at rides of different lengths and figure out how much each trip would cost. Then they write an explanation of how they found the cost.
- Students look for a pattern between the temperature in degrees Fahrenheit and degrees Celsius and write an explanation of that relationship.

11. Understand and describe the general behavior of functions.

- As a regular assessment exercise, students fill in *Function Machine* tables like those shown below. Sometimes they are given the rule and sometimes they are asked to find the rule.

x	y
6	12
9	18
2	?
7	?
y = 2x	

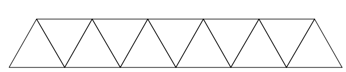
x	y
5	18
7	?
?	38
3	?
y = 4x–2	

x	y
1	4
10	31
?	16
7	?
???	

- Students investigate graphs without numbers. For example, they may study a graph that shows how far Yasmin has walked on a trip from home to her friend's house and back, where time is shown on the horizontal axis and the distance covered is on the vertical axis. Students tell a story about her trip, noting that where the graph is horizontal, she has stopped for some reason.

12. Use patterns, relationships, and linear functions to model situations in mathematics and in other areas.

- Students start with a single equilateral triangle with side of length 1 and find its perimeter. Then they add a second such triangle, matching sides exactly, to make a train or a wall and find its perimeter. They continue adding triangles, as in the diagram below, and find the perimeters of the resulting figures. The students try to predict the perimeter of ten triangles in a wall and then look for a function rule which describes the pattern — for example, if n is the number of triangles, then $P = n + 2$.



- Students look for as many different ways to make change for 50¢ as they can find. They make a table showing their results listed in an organized fashion and explain why they think they have found all of the possibilities.
- Students investigate what happens when they do arithmetic on a 12-hour clock. They find that 3 + 11 = 2 and that 4 – 6 = 10. They understand that 5, 17, and 29 are all equivalent to 5, and connect this to the remainder obtained when dividing each by 12.
- Students develop a patchwork quilt design using squares and isosceles right triangles to make a 12 inch by 12 inch patch. They use patterns to help them decide how many pieces of

each size are needed in order to complete a 3 foot by 5 foot quilt.

- Students are given the following open-ended assessment problem: *Who will win the 100 meter race between Pat and his older sister, Terry? Pat runs at an average of 3 meters per second, while Terry runs at an average of 5 meters per second. Since Pat is slower, he gets a 25-meter head start. Use a table or a graph to help you find out who will be the winner. Then write an explanation of how you solved the problem and explain what head-start you think Pat should have.*

13. Develop, analyze, and explain arithmetic sequences.

- Students explore and try to explain the sequence made by the numbers of diagonals in a series of polygons with increasing numbers of sides. For example, a triangle has no diagonals, a square has 2 diagonals, a pentagon has 5 diagonals, a hexagon has 9 diagonals, and so on. They examine the sequence 0, 2, 5, 9, try to extend it, and justify their conclusion.
- Students read *The King's Chessboard* by David Birch. In this old folk tale, which has been told many times in many languages, a king is undecided about a gift to give to one of his advisors. Finally he decides that, on one day, the advisor will receive a single grain of rice on the first square of a chessboard. On the next day, that amount will be doubled and two grains will be placed on the second square. On day three, four grains will be put on the third square, and so on, doubling every day until the entire board is filled. The students use calculators to figure out the king's indebtedness on the tenth day, the twentieth day, and so on.

References

Bird, David. *The King's Chessboard.* Puffin Pied Piper Books, 1988.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 11 — Patterns, Relationships, and Functions — Grades 7-8

Overview

The key components of *pattern-based thinking*, as identified in the K-12 Overview, involve **exploring, analyzing, and generalizing patterns**, and **viewing rules and input/output situations as functions.** In grades 7-8, the importance of studying patterns continues with an emphasis on representing and describing relationships with tables and graphs and on the development of rules using variables. Patterns also become important now in the analysis of statistics and the development of geometric relationships. Graphing calculators and computers are helpful in illustrating the usefulness of symbols and in making symbolic relationships more tangible. Although the symbolism and notation used become more algebraic at these grade levels (e.g., $A = 4s$ instead of $A = 4 \text{ x } s$), students should still be encouraged to model many patterns with concrete materials. Engineers, scientists, architects, and other researchers all build working models of projects for analysis and demonstration.

Students in these grades should also be given ample opportunity to analyze patterns, to discover the relevant features of the patterns, and to construct understandings of the concepts and relationships involved in the patterns. From these investigations, students should develop the language necessary to communicate their ideas about the patterns and should learn to differentiate among the variety of patterns they have studied (that is, to **categorize** and to **classify** them). They will apply their understanding of patterns as they learn about such topics as exponents, rational numbers, measurement, geometry, probability, and functions.

Seventh and eighth graders continue to **discover rules** for mathematical relationships and for quantifiable situations from other subject areas. In particular, students should focus on **relationships involving two variables.** Students should analyze how a change in one quantity results in a change in another. They further need to develop their understanding of the general behavior of **functions** and use these to model a variety of phenomena.

Students should be encouraged to solve problems by looking for patterns that involve words, pictures, manipulatives, and number descriptions. These situations naturally lead to the use of variables and informal algebra in solving problems.

As seen in prior grades, computers and graphing calculators provide many benefits for students in investigating mathematical concepts and problems. These tools make mathematics accessible to more students because they enable the students to analyze what they can see rather than requiring them to develop mental images or manipulate situations symbolically from the outset. Furthermore, technology enables students to calculate rapidly and to investigate conclusions immediately, freeing them from the limitations imposed by cumbersome and time-consuming computations.

Standard 11 — Patterns, Relationships, and Functions — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

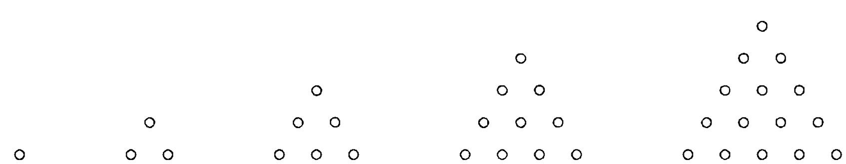
Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

7. Represent and describe mathematical relationships with tables, rules, simple equations, and graphs.

- Students use calculators to investigate which fractions have decimal equivalents that terminate and which repeat. They summarize their findings in their math journals.
- Students study patterns made by the units digit in the expansion of powers of a number. For example, what is the units digit of 9^{18}? The pattern 9^1, 9^2, 9^3, . . . yields either a 1 or a 9 in the units place. Students record their findings in a table or as a graph on rectangular coordinate paper. They write a paragraph justifying their answer. They then similarly investigate the patterns made by the units digit in the expansion of the powers of other one-digit numbers.
- Students consider what happens if you start with two bacteria on a kitchen counter and the number of bacteria doubles every hour. They make a table and graph their results, noting that the graph is not linear.

8. Understand and describe the relationships among various representations of patterns and functions.

- Students arrange bowling pins in the shape of equilateral triangles of various sizes, as shown in the diagram below. They make a table showing the number n of rows in each triangle and the number b of bowling pins in each triangle. The numbers in the second column — 1, 3, 6, 10, ... — are called the *triangular numbers*. They find a rule expressing this relationship $p = n(n + 1) /2$, by putting two triangles of the same size side by side, counting the total number of bowling pins, and dividing by 2.



- Using a 5x5 geoboard or dot paper, students create various sized parallelograms. For each parallelogram, they record the length of the base (b), the height of the parallelogram (h), and the area of the parallelogram (A), found by counting squares. The students look for a relationship among the numbers in the three columns of their table, express this relationship as a verbal rule, and then write the rule in symbolic form.

- Students investigate how many stools with three legs and how many chairs with four legs can be made using 48 legs. They may use objects or draw pictures to make models of the solutions. They look for patterns in the numbers and display their results in a table, as ordered pairs graphed on the rectangular coordinate plane, as a rule like $3s + 4c = 48$, and as an equation like $s = 16 – 4c/3$, which gives the number of stools as a function of the number of chairs. They describe the pattern and how they found it in writing.

- Students create their own designs using iteration. They may use patterns such as spirolaterals or write a program in Logo on the computer. They use simple equations to iterate patterns. For example, they use the equation y = x + 1 and start with any x value, say 0. The resulting y value is 1. Using this as the new x value yields a 2 for y. Using this as the next x gives a 3, and so on. The related values can be organized in a table and the ordered pairs graphed on a rectangular coordinate system. Students note that the graph is a straight line and use this to predict other values. Then students use a slightly different equation, y = .1x + .3. Again, starting with an x value of 0 they find the resulting y value of .3. Using this as the new x value gives a value of .33 for y. Repeating this process yields the series of y values .3, .33, .333, ... which get closer and closer to 1/3.

9. Use patterns, relationships, and functions to model situations and to solve problems in mathematics and in other subject areas.

- Students analyze a given series of terms and fill in the missing terms. Patterns include various arithmetic (repeating patterns) and geometric (growing patterns) sequences and other number and picture patterns. Students develop an awareness of the assumptions they are making. For example, given the sequence 0, 10, 20, 30, 40, 50, one might expect 60 to be next; but not on a football field, where the numbers now decrease!

- Students compare different pay scales, deciding which is a better deal. For example, is it better to be paid a salary of $250 per week or to be paid $6 per hour? They create a table comparing the pay for different numbers of hours worked and decide at what point the hourly rate becomes a better deal.

- Students supply missing fractions between any two given numbers on a number line. They might label each of eight intervals between 1 and 2, or they might label the next 16 intervals from 23 1/2 to 24. They extend this to decimals, labeling each missing number in increments of .1 or .01. For example, students might label each of five intervals between 59.34 and 59.35.

- Students decide how many different double-dip ice cream cones can be made from two flavors, three flavors, and so on up to Baskin and Robbins' 31 flavors. They arrange the information in a table. They discuss whether one flavor on top and another on the bottom is a different arrangement from the other way around, and how that would change their results. They also discuss a similar problem (see Standard 14 and the 5-6 Vignette *Pizza Possibilities* in the First Four Standards): *How many different types of pizzas can be made using different toppings?*

- Students predict how many times they will be able to fold a piece of paper in half. Then they fold a paper in half repeatedly, recording the number of sections formed each time in a table. They find that the number of folds physically possible is surprisingly small (about 7). The students try different kinds of paper: tissue paper, foil, etc. They describe in writing any patterns they discover and generate a rule for finding the number of sections after *10*, *20*, or *n* folds. They also graph the data on a rectangular coordinate plane using integral values. They extend this problem to a new situation by finding the number of ancestors each person had ten generations ago and also to the problem of telling a secret to two people who each tell two people, etc.

10. Analyze functional relationships to explain how a change in one quantity results in a change in another.

- Students investigate how increasing the temperature measured in degrees Celsius affects the temperature measured in degrees Fahrenheit and vice versa. They collect data using water, ice, and a burner. They use their data to develop a formula relating Celsius to Fahrenheit, summarize the formula in a sentence, and graph the values they have generated.
- Students investigate how the temperature affects the number of chirps a cricket makes in a minute.
- Students investigate the effect of changing the radius or diameter of a circle upon its circumference by measuring the radius (or diameter) and the circumference of circular objects. They graph the values they have generated, notice that it is close to a straight line, and describe the relationship they have found in a paragraph. Then they develop a symbolic expression that describes that relationship.
- Students investigate the effect on the perimeters of given shapes if each side is doubled or tripled. They summarize their findings.
- Students investigate how the areas of rectangles change as the length is doubled, or the width is doubled, or both are doubled. They discuss their findings.
- Students work on problems like this one from the New Jersey Department of Education's *Mathematics Instructional Guide (p. 7-69)*: *Two of the opposite sides of a square are increased by 20% and the other two sides are decreased by 10%. What is the percent of change in the area of the original square to the area of the newly formed rectangle? Explain the process you used to solve the problem.*
- Students investigate how the areas of triangles change if the base is kept the same, but the height is repeatedly increased by one unit.
- Students stack a given number of unit cubes in various ways and find the surface areas of the structures they have built. They sketch their figures and discuss which of the figures has the largest surface area and which has the smallest, and justify their conclusions.
- Students make models of cubes using blocks or other manipulatives, and investigate how the volume changes if the length, width, and height are all doubled.
- Using a spreadsheet, students investigate how adding (or subtracting) values to given data can affect the mean, median, mode, or range of the data. They discuss how various other changes to the data would affect the mean, the median, the mode, or the range.

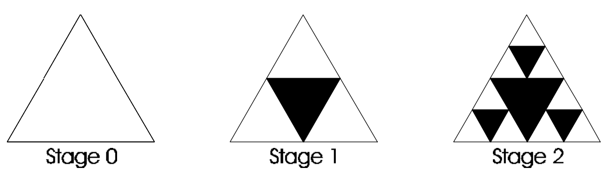
11. Understand and describe the general behavior of functions.

- Students investigate graphs without numbers. For example, they may study a graph that shows how far Olivia has walked on a trip from home to the store and back, where time is shown on the horizontal axis and the distance covered is on the vertical axis. Students tell a story about her trip, noting that where the graph is horizontal, she has stopped for some reason. In addition, their stories account for those parts of the graph that are steeper, by explaining why Olivia is walking faster (e.g., she is running from a dog), and those parts of the graph that are not as steep, by explaining why Olivia is walking slower (e.g., she is going up a hill).

- Students use probes and graphing calculators or computers to collect data involving two variables for several different science experiments (such as measuring the time and distance that a toy car rolls down an inclined plane, or the temperature of a beaker of water when ice cubes are added). They look at the data that has been collected in tabular form and as a graph on a coordinate grid. They classify the graphs as straight or curved lines and as increasing (direct variation), decreasing (inverse variation), or mixed. For those graphs that are straight lines, the students try to match the graph by entering and graphing a suitable equation.
- Given several non-linear functions, such as $y = x^2$, $y = 3x^2$, $y = x^2 + 1$, $y = x^3$, or $y = 16/x$, students create a table of values for each and use graphing calculators to graph them.

12. Use patterns, relationships, and linear functions to model situations in mathematics and in other areas.

- Groups of students pretend that they work for construction companies bidding on a federal project to build a monument. The monument is to be built from marble cubes, with each cube being one cubic foot. The monument is to have a "triangular" shape, with one cube on top, then two cubes in the row below, then three cubes, four cubes, and so on. The monument is to be 100 feet high. The students make a chart and look for a pattern to help them predict how many cubes they will need to buy so that they can include the cost of the cubes in their bid.
- Students look at the Sierpinski triangle as an example of a fractal. Stage 0 is an unshaded triangle. To get Stage 1, you take the three midpoints of the sides of the unshaded triangle, connect them, and shade the new triangle in the middle. To get Stage 2, you repeat this process for each of the unshaded triangles in Stage 1. This process continues an infinite number of times. The students make a table that records the number of unshaded triangles at each stage, look for a pattern, and use their results to predict the number of unshaded triangles there will be at the tenth (3^9) and twentieth (3^{19}) stages.



- Students use the constant function on the calculator to determine when an item will be on sale for half price. If the price goes down by a constant dollar amount each week, then they record successive prices, such as 95 – 15 = = = . . . (or 15 – – 95 = = = on other calculators). If the price is reduced by a certain percent each week, then they use the constant function on the calculator to obtain successive discounts as percents by multiplying. For example, if a $95 item is reduced by 10% each week, they key in 95 x .9 = = = . . . (or as .9 x x 95 = = = . . . on other calculators).

- Using a temperature probe and a graphing calculator or computer, students measure the temperature of boiling water in a cup as it cools. They make a table showing the temperature at five-minute intervals for an hour. Then they graph the results and make observations about the shape of the graph, such as *the temperature went down the most in the first few minutes* or *it cooled more slowly after more time had passed*, or *it's not a linear relationship*. The students also predict what the graph would look like if they continued to collect data for another twelve hours.

- Students use coins to simulate boys (tails) and girls (heads) in a family with five children. They make a list of all of the possible combinations, use patterns to help them organize all of the possibilities, and find the probability that all five children are girls or that exactly three are girls. As a question on a test, they are asked to react to an argument between Pam and Jerry, a couple who want to have four children. Jerry thinks that they will probably end up with two boys and two girls, while Pam thinks that they will probably wind up with an unequal number of boys and girls.

- Students make Ferris wheel models from paper plates, with notches representing the cars. They use the models to make a table showing the height above the ground of a person on a ferris wheel at specified time intervals, determined by the time needed for the next chair to move to loading position. After collecting data through two or three complete turns of the wheel, they make a graph of time versus height. In their math notebooks, they respond to questions about their graphs: *Why doesn't the graph start at zero? What is the maximum height? Why does the shape of the graph repeat?* The students learn that this graph represents a periodic function.

13. Develop, analyze, and explain arithmetic sequences.

- Students use the following chart of postal rate history to make a graph of the increases and then to try to predict what the cost will be to mail a one-ounce letter in the year 2001.

Cost to Mail a One-Ounce Letter Since 1917

Date	Cost
1917	3 cents
1919	2 cents
1932	3 cents
1958	4 cents
1963	5 cents
1968	6 cents
1971	8 cents
1974	10 cents

Date	Cost
1975	13 cents
1978	15 cents
1981	18 cents
1981	20 cents
1985	22 cents
1988	25 cents
1991	29 cents
1995	32 cents

- Students describe, analyze, and extend the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... , where each term is the sum of the two preceding terms. They investigate applications of this sequence in nature, such as sunflower seeds, the fruit of the pineapple, and the rabbit problem. They create their own Fibonacci-like sequences, using different starting numbers.
- Students read Isaac Asimov's short story *Endlessness* and write book reports to convey their reactions.

References

Asimov, Isaac. *Endlessness.* (in *Literature: Bronze*, 2nd Ed.) Englewood Cliffs, NJ: Prentice Hall, 1991.

New Jersey Department of Education. *Mathematics Instruction Guide.* D. Varygiannes, Coord. January, 1996.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 11 — Patterns, Relationships, and Functions — Grades 9-12

Overview

Patterns, relationships, and functions continue to provide a unifying theme for the study of mathematics in high school. Pattern-based thinking throughout the earlier grades, as described in the K-12 Overview, and the informal investigations begun in the middle grades have prepared students to make extensive use of both the concept of a function and functional notation. Students should describe the relationships found in concrete situations with algebraic expressions, formulas and equations, as well as with tables of input-output values, with graphs, and with written statements.

Students in high school **construct, recognize, and extend patterns** as they encounter new areas of the mathematics curriculum. For example, students in algebra recognize patterns when multiplying binomials, and students in geometry utilize patterns in similar triangles. Students in high school should also analyze a variety of different types of sequences, including both arithmetic and geometric sequences, and express their behavior using functional notation.

High school students continue to **categorize and classify** objects, especially in the context of learning new mathematics. For example, in studying geometry, they classify various lines and line segments as chords or secants or tangents to a given circle. In studying algebra, they distinguish linear relationships from non-linear relationships.

The **function** concept is one of the most fundamental unifying ideas of modern mathematics. Students begin their study of functions in the primary grades, as they observe and study patterns in nature and create patterns using concrete models. As students grow and their ability to abstract matures, students investigate patterns using concrete models, and then abstract them to form rules, display information in a table or chart, and write equations which express the relationships they have observed. In high school, students move to expand their knowledge of functions as a natural outcome of the earlier discussion of patterns and relationships. Concepts such as domain and range are formalized and the $f(x)$ notation is introduced as a natural extension of initial informal experiences.

Students frequently have difficulty with the concept of a function, possibly because of its many interpretations. The *formal ordered-pair definition* of a function, while perhaps the most familiar to many teachers, is also the least understood and possibly the most abstract way of approaching functions (Wagner and Parker, 1993). Looking at functions as *correspondences between two sets* seems to be more easily grasped while facilitating the introduction of the concepts of domain and range. Visualizing functions as *graphs* which satisfy the vertical line test provides an extremely accessible way of representing functions, especially when graphing calculators and computers are used. Students entering high school should already be familiar with functions as *input-output processes* through the use of function machines. They should also have encountered functions given by *rules or formulas* involving independent and dependent variables. Students moving on to calculus also need to view functions as *objects of study* in themselves.

The correspondence between all of these interpretations of the concept of a function may not be very clear to students, and so attention should be drawn explicitly to the different ways of understanding functions, and how together they provide a more complete understanding of the concept. For example, while discussing sequences, students should explore how they can be considered as functions using the correspondence model, the rule model, the input-output model, the graph model, and the ordered pairs model.

High school students should spend considerable time in analyzing **relationships involving two variables**, and should understand how dependent and independent variables are used. Beginning with concrete situations (possibly involving social studies or science concepts), students should collect and graph data (often using graphing calculators or computers), discover the relationship between the two variables, and express this relationship symbolically. Students need to have experiences with situations involving linear, quadratic, polynomial, trigonometric, exponential, and rational functions as well as piecewise-defined functions and relationships that are not functions at all.

High school students should use functions extensively in solving problems. They should frequently be asked to analyze a real-world situation by using patterns and functions. They should extend their understanding of relationships involving two variables to using functions with several dependent variables in mathematical modeling.

Throughout high school, students continue to work with patterns by collecting and organizing data in tables, by graphing the relationships among variables, and by discovering and describing these relationships in formal, written, and symbolic form.

Reference

Wagner, S., and S. Parker. "Advancing Algebra" in *Research Ideas for the Classroom: High School Mathematics*, P. Wilson, Ed. New York: Macmillan Publishing Company, 1993.

Standard 11 — Patterns, Relationships, and Functions — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

14. Analyze and describe how a change in the independent variable can produce a change in a dependent variable.

- Students investigate the relationship between stopping distance and speed of travel in a car. The students gather data from the driver's education manual, graph the values they have found, note that the relationship is linear, and look for an equation that fits the data.
- Students investigate the effect on the perimeters of given shapes if each side is doubled or tripled. They summarize their findings in writing and symbolically.
- Students investigate how the area of a parallelogram changes as the length of the base is doubled, or the height is doubled, or both are doubled. They repeat the experiment for tripling and quadrupling each measurement. They discuss their findings and represent them symbolically.
- Students compare two fare structures for taxis: one in which the taxi charges $2.75 for the first 1/4 mile and $.50 for each additional 1/4 mile, and one in which $4.25 is charged for the first 1/4 mile and $.20 for each additional 1/8 mile. They develop tables, graph specific points, and generate equations to describe each situation. They find which trips cost more for each fare structure and when both will result in the same cost.
- Students investigate patterns of growth, such as compound interest or bacterial growth, with a calculator. They make a table showing how much money is in a savings account (if none is withdrawn) after one quarter, two quarters, and so on, for ten years. They represent their findings graphically, note that this is not a linear relationship (although simple interest is linear), and write an equation describing the relationship between the amount P deposited initially, the interest rate r, the number n of times that interest is paid each year, the number of years y, and the total T available at the end of that time period: $T = (1 + r/n)^{ny}(P)$.

15. Use polynomial, rational, trigonometric, and exponential functions to model real world phenomena.

- Students model population growth and decline of people, animals, bacteria and decay of radioactive materials, using the appropriate exponential functions.
- Students use a sound probe and a graphing calculator or computer to collect data on sound waves or voice patterns, and graph these data noting that these patterns are represented by trigonometric functions. They use a light probe to collect data on the relationship between the brightness of the light and its distance from the light source, and analyze the graph provided by the calculator.

- Students use M&Ms to model decay. They spill a package of M&Ms on a paper plate and remove those with the M showing, recording the number of M&Ms removed. They put the remaining M&Ms in a cup, shake, and repeat the process until all of the M&Ms are gone. They plot the trial number versus the number of M&Ms remaining and note that the graph represents an exponential function. They try different equations until they find one that they think fits the data pretty well. They verify their results using a graphing calculator.

- Students work on this HSPT-like problem from the New Jersey Department of Education's *Mathematics Instructional Guide*: *The Granda Theater has a special rate for groups of 10 or more people: $40 for the first 10 people and $3 for each additional person. Which of the following expressions tells the amount that a group of 10 or more will have to pay if n represents the number of people in the group, where* $n \geq 10$*: a. 40 + 3n, b. (40 + 3)n, c. 40 + 3(n + 10), or d. 40 + 3(n – 10)?*

- Students learn about the Richter Scale for measuring earthquakes and about the pH measurement of a solution, noting how exponents are built into these measurements. For example, a pH of 4 is 10 times more acidic than a pH of 5 and 100 times more acidic than a pH of 6.

- Students work in groups to investigate what size square to cut from each corner of a rectangular piece of cardboard in order to make the largest possible open-top box. They make models, record the size of the square and the volume for each model, and plot the points on a graph. They note that the relationship is not linear and make a conjecture about the maximum volume, based on the graph. The students also generate a symbolic expression describing this situation and check to see if it matches their data by using a graphing calculator.

16. Recognize that a variety of phenomena can be modeled by the same type of function.

- Different groups of students work on problems with different settings but identical structures. For example, one group determines the number of collisions possible between two, three and four bumper cars at an amusement park and develops an equation to represent the number of possible collisions among *n* bumper cars (assuming that no two bumper cars collide more than once). Another group investigates the number of possible handshakes between 2, 3, and 4 people, and develops an equation to represent the number of handshakes for *n* people. A third group discusses the total number of sides and diagonals possible in a triangle, a quadrilateral, and a pentagon, and develops an equation that gives the total number of sides and diagonals for an *n*-sided polygon. A fourth group looks at the number of games required for a tournament if each team plays every other team only once, while a fifth considers connecting telephone lines to houses. Each group presents its problem, its approach to solving the problem, and its solution. Then the teacher leads the class in a discussion of the similarities and differences among the problems. Students note the similarities between the approaches used by the different groups and that they all came up with the general expression $n(n-1)/2$.

- Students investigate a number of situations involving the equation $y = 2^x$. They look at how much money would be earned by starting out with a penny on the first day and doubling the amount on each successive day. They discuss what happens if they start with two bacteria and the number of bacteria doubles every half hour. They consider the total number of pizzas possible as more and more toppings are added. They consider the number of subsets for a given set. They fold a sheet of paper repeatedly in half and look at how many sections are created after each fold.

- Students look for connections among problem situations involving temperature in Celsius and Fahrenheit, the relationship of the circumference of a circle to its diameter, the relationship between stopping distance and car speed, between money earned and hours worked, between distance and time if the rate is kept constant, and between profit and price per ticket.

17. Analyze and explain the general properties and behavior of functions, and use appropriate graphing technologies to represent them.

- As regular parts of their assessments, students make up graphs to represent specific problem situations, such as the cost of pencils that sell at two for a dime, the temperature of an oven as a function of the length of time since it was turned on, their height from the ground as they ride a ferris wheel as a function of the amount of time since they got on, the time it takes to travel 100 miles as a function of average speed, or the cost of mailing a first-class letter based on its weight in ounces.

- Students use a string of constant length, say 30 inches, and list all possible lengths and widths of rectangles with integral sides which have this perimeter. They determine the perimeter and area for each rectangle. Then they make three graphs from their data: length vs. width, length vs. perimeter, and length vs. area. They look for equations to describe each graph, determine an appropriate range of values for each variable, and then graph the functions using graphing calculators or computers. The rectangle of maximum area, a square, does not have integral values, but can be found using the trace function or algebraic procedures. Students also investigate the area of a circle made with the same string and compare it to the areas of the rectangles.

- Students take on the role of "forensic mathematicians," trying to determine the height of a person whose femur was 17 inches long. They measure their own femurs and their heights, entering the class data into a graphing calculator or computer and creating a scatterplot. They note that the data are approximately linear, so they use the built-in linear regression procedures to find the line of best fit and then make their prediction.

18. Analyze the effects of changes in parameters on the graphs of functions.

- Students investigate the characteristics of linear functions. For example, in $y = kx$, how does a change in k affect the graph? In $y = mx + b$, what is the role of b? Does k in the first equation serve the same purpose as m in the second? Students use the graphing calculator to investigate and verify their conclusions.

- Students investigate the effects of a dilation and/or a horizontal or vertical shift on the algebraic expression of various types of functions. For example, how does moving a graph up 3 units affect its equation?

- Students look at the effects of changing the coefficients of a quadratic equation on its graph. For example, how is the graph of $y = 4x^2$ different from that of $y = x^2$? How is $y = .2x^2$ different from $y = x^2$? How are $y = x^2 + 4$, $y = x^2 \quad 4$, $y = x^2 \quad 4x$, and $y = x^2 \quad 4x + 4$ each different from $y = x^2$? How is $y = \sin 4x$ different from $y = 4 \sin x$? Students use graphing calculators to look at the graphs and summarize their conjectures in writing.

- Students study the behavior of functions of the form $y = ax^n$. They investigate the effect of *a* on the curve and the characteristics of the graph when *n* is even or odd. They use the graphing calculator to assist them and write a sentence summarizing their discoveries.

19. Understand the role of functions as a unifying concept in mathematics

- Students in all mathematics classes use functions, making explicit connections to what they have previously learned about functions. As students encounter a new use or meaning for functions, they relate it to their previous understandings.
- Students use recursive definitions of functions in both geometry and algebra. For example, they define $n!$ recursively as $n! = n\cdot(n-1)!$ They use recursion to generate fractals in studying geometry. They may use patterns such as spirolaterals, the Koch snowflake, the Monkey's Tree curve, the Chaos Game, or the Sierpinski triangle. They may use Logo or other software to iterate patterns, or they may use the graphing calculator. In studying algebra, students consider the equation $y = .1x + .6$, starting with an *x*-value of .6, and find the resulting *y*-value. Using this *y*-value as the new x-value, they then calculate its corresponding *y*-value, and so on. (The resulting values are .6, .66, .666, .6666, ... — providing closer and closer approximations to the decimal value of 2/3!) Students investigate the results of iterations which are other starting values for the same function; the results are surprising! They use other equations and repeat the procedure. They graph the results with a graphing calculator, adjusting the range values to permit viewing the resulting *y*-values. (See *Fractals for the Classroom* by H.-O. Peitgen, et al.)

References

Peitgen, Heinz-Otto, et al. *Fractals for the Classroom: Strategic Activities, Volume One and Two.* Reston, VA: NCTM and New York: Springer- Verlag, 1992.

Wagner, S. and S. Parker. "Advancing Algebra" in *Researching Ideas for the Classroom: High School Mathematics,* P. Wilson, Ed. New York: Macmillan Publishing Co., 1993.

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