STANDARD 6 — NUMBER SENSE

K-12 Overview

All students will develop number sense and an ability to represent numbers in a variety of forms and use numbers in diverse situations.

Descriptive Statement

Number sense is defined as an intuitive feel for numbers and a common sense approach to using them. It is a comfort with what numbers represent, coming from investigating their characteristics and using them in diverse situations. It involves an understanding of how different types of numbers, such as fractions and decimals, are related to each other, and how they can best be used to describe a particular situation. Number sense is an attribute of all successful users of mathematics.

Meaning and Importance

Successful users of mathematics have good number sense. When someone chooses to use fractions in one situation and decimals in another because the respective operations are easier to perform or the results are easier to understand, that process is evidence of good number sense. When students continue work on a problem involving numbers until they recognize that their answers are reasonable in the context of the problem, they are using good number sense. When a student is comfortable with using an approximation to a number in certain situations, or understands that an approximation rather than an exact number might have been used, then that reflects good number sense. When students recognize that an answer is off by a factor of ten, or alternately that a decimal point has been misplaced, they are using good number sense.

Our students often do not connect what is happening in their mathematics classrooms with their daily lives. It is essential that the mathematics curriculum build on the sense of number that students bring with them to school. Problems and numbers which arise in the context of the students' world are more meaningful to students than traditional textbook exercises and help them develop their sense of how numbers and operations are used. Frequent use of estimation and mental computation are also important ingredients in the development of number sense, as are regular opportunities for student communication. Discussion of their own invented strategies for problem solutions helps students strengthen their intuitive understanding of numbers and the relationships between numbers.

A "sense-building mode" is best established when students are provided with opportunities to explore number relationships, are encouraged to question and to challenge, and are allowed to experiment to discover strategies and techniques of their own that ease the path to the solution of mathematical problems.

K-12 Development and Emphases

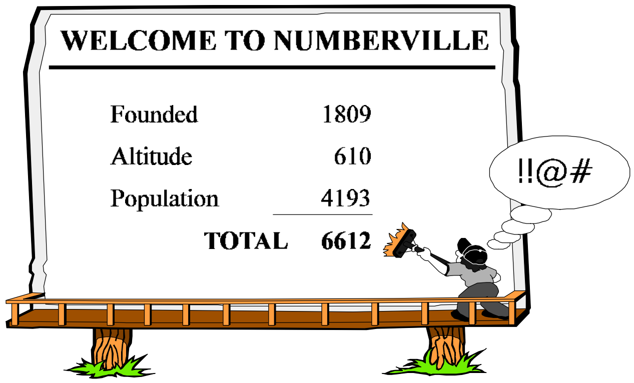
A necessary foundation for a strong number sense is the development of meaning for numbers, beginning with whole numbers, decimals, and fractions. Traditionally, this component of the curriculum has been called **numeration**, and it is vitally important. The K-12 mathematics curriculum should provide the appropriate experiences, physical models, and manipulatives to assist in the construction of these meanings. Appropriate technology should also be regularly used to help students develop their number sense. Through regular and frequent experiences emphasizing the measurement of real objects, the counting and grouping of sets of discrete objects, and the well designed use of calculators, elementary students develop **place value concepts**, a **sense of magnitude (size)**, and **approximation and estimation skills** for whole numbers, decimals, and fractions. Real-world situations should be incorporated into their experiences to help young students become aware of the existence of other numbers, such as negative numbers.

The development of personal meaning for numbers should be reinforced in the middle grades with an extension to other numbers and notations such as integers, percents, exponents, and roots. High school students should extend their meaning of number to the real number system and recognize that still other number systems exist. They should have the opportunity to develop intuitive proofs of the fundamental properties of closure, commutativity, associativity, and distributivity.

Students must develop a facility for working with the different types of numbers we use in day-to-day activities. Statements about one particular quantity might best be expressed with a fraction, a percent, a decimal, a ratio, an approximated whole number, or some other number form; **comparisons** often have to be made among numbers in different forms. Therefore, students will need to be able to transform numbers from one form to **equivalent numbers** in another form, and to intelligently select the right form of numbers to use in a particular situation. The correct choice depends on the context of the situation, and therefore students must possess an adequate understanding of each form and the interrelationships among them.

One way to achieve such an understanding in the classroom is through the identification and description of number patterns and the use of pattern-based thinking. For example, examining and modeling many pairs of fractions with equal numerators help students develop the understanding and generalization that the fractions with larger denominators represent smaller quantities. Activities promoting pattern-based thinking can assist students in making similar generalizations about other number forms and their relationships, as well as build initial notions of still other types of important number concepts such as odd and even, prime numbers, and factors and multiples.

Graduates of our schools must be able to use numbers intelligently and understand them wherever they are encountered in real life. They must develop an **awareness of numbers and their uses**. Numbers are used as counts, measures, labels, and locations, and each use has unique characteristics and restrictions on the appropriate forms and operations. The opportunity to develop the needed familiarity with all of these uses comes through the regular presentation of problem situations which utilize them. Some activities should focus on the explicit uses of the numbers themselves, however. A discussion of why it makes no sense to add the numbers on a town's roadway welcome sign (see diagram on the next page) which lists its population, altitude, and year of founding would serve a dual purpose: to provide an example of some the standard uses of numbers, and to challenge the thoughtless computational manipulation of numbers.



IN SUMMARY, the commitment to develop number sense requires a dramatic shift in the way students learn mathematics. Our students will only develop strong number sense to the extent that their teachers encourage the **understanding** of mathematics as opposed to the memorization of rules and mechanical application of algorithms. Every child has the capability to succeed as a user of mathematics, but the degree of success is directly related to the strength of their number sense. The way to assure that all students acquire a good sense of number is to have them consistently engage in activities which require them to think about numbers and number relationships and to make the connection with quantitative information encountered in their daily lives.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 6 — Number Sense — Grades K-2

Overview

Students can develop a clear sense of number from consistent ongoing experiences in classroom activities where a variety of manipulatives and technology are used. The key components of number sense, as identified in the K-12 Overview, include an **awareness of numbers and their uses** in the world around us, a good sense of **place value concepts, approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Kindergarten, first, and second graders are just beginning to develop their concepts of number. They have most likely come to school with some ability to count, but with differing notions of what that activity means. It is in these grades that they begin to attach meaning to the numbers that they hear about and see all around them. One useful activity that can be repeated many times throughout this age range is the keeping of a scrapbook reflecting all the **uses of numbers** that the children can identify. It would probably include telephone numbers, addresses, ages, page numbers, clothing sizes, room numbers, and many others. Discussions of the similarities and differences in all of these uses can provide some interesting insights.

In terms of **numeration**, students in these grades start by constructing meaning for one-digit numbers and build up to formal work with three-digit numbers. The regular and consistent use of concrete models for that development is essential. Kindergartners need a variety of things to count, from poker chips to marbles to beans. Both concrete and rote counting are critically important in developing a sense of number. Adequate attention to counting activities throughout these grades will help to assure both a good sense of **magnitude (size) of numbers** and a real readiness for all four basic operations. (See Standard 8.) Counting by ones should be followed by counting back; skip counting by twos, fives, and tens; counting from a given starting number to a given target number by ones and by other numbers; counting on by tens from non-multiples of ten like 43; and so on.

As students are able to handle larger numbers, place value and base-ten ideas are introduced through grouping activities. Many of the models with which they are comfortable for single units can translate nicely into beginning base-ten models; poker chips can be put in groups of ten into small paper cups; beans can be pasted in tens onto tongue depressors, and so on. These newly enhanced models, along with the single digit units, are then used to represent two-digit numbers. As the next step, of course, groups of ten tens can be made to create hundreds. These first models of base-ten number are the best ones to use with young children who are first encountering these notions because they can actually build larger units from smaller units. Such models are called *bundle-able*. Another property these have is *proportionality*, because the model for a ten is actually ten times as large as the model for a one. A widely used model which is both bundle-able and proportional involves popsicle sticks which are wrapped into tens and hundreds with rubber bands.

The next type of model to be used would be one which is still proportional, but no longer bundle-able. The best examples of this type are the standard base-ten blocks. They require the child to trade ten ones for a ten rather than directly constructing a ten from the ones, and, as a result, are slightly more sophisticated. The last level of sophistication in this sequence of models includes those that are neither proportional nor bundle-able.

Two models of this type which are regularly used are chip trading materials and play money. With chip trading materials, there is no inherent concrete ten-to-one relationship between the red chips and the green chips; the red chips are not ten times as large as the green ones. The relationship holds only because of an external rule that is made up and followed. Similarly, there is no inherent concrete ten-to-one relationship that exists between dimes and pennies. The relationship only exists because of a rule that is external to the coins themselves. As a result, these most sophisticated models should be used *after* the underlying concepts are developed with the earlier models.

Children at these grade levels also begin to learn about **equivalence**. When youngsters find as many "names" as they can for the number *7* (such as *2 + 5*, *9 − 2*, and *one more than 6*), they are creating equivalent forms of the same number. Slightly older students should be using similar activities to generate equivalent forms of multi-digit numbers, partly in preparation for operations involving them: *67 = 6 tens and 7 ones = 5 tens and 17 ones = 4 tens and 27 ones*.

Estimation should be a routine part, not only of daily mathematics lessons, but also of the entire school day. Children should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the day, like: *About how many kids do you think there are out here in the playground? About how many pieces of construction paper will we need for this project if everyone needs three different colors?* and *How many of your great graphs do you think will fit on the bulletin board without overlapping?* After several children have had chances to make estimates about numbers like these, they should defend their estimates by giving some rationale for thinking they are close to the actual number. These discussions can be invaluable in helping them to develop good number sense.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful teaching tools when programmed to count forward and backward by some constant. Children can do the programming easily themselves and try to anticipate the calculator display. Appropriate computer software provides environments in which students can first develop a sense of small whole numbers and then build an understanding of place-value and base-ten ideas.

The topics that should comprise the number sense focus of the kindergarten through second grade mathematics program are:

whole number meanings through three digits
place value and number base
counting and grouping

Standard 6 — Number Sense — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Use real-life experiences, physical materials, and technology to construct meanings for whole numbers, commonly used fractions, and decimals.**

   - Young students make and use a variety of models for "number" ranging from poker chips to dot patterns on a paper plate, to Cuisenaire Rods, to tally marks, to domino and dice combinations. A large component of their early work with number focuses on the various *parts* into which any given number can be broken.
   - Students play the *Broken Key* game on their calculators. Kindergartners try to get the calculator display to show *7* while pretending that the *7* key is broken and cannot be pressed. Second graders might try to get the display to show *45* without pressing the *4* or the *5* key.
   - Students use *5-frames* and *10-frames* to help develop initial ideas of small numbers. By filling up a 5-cell grid with counters first and then putting out 2 more while trying to show "7 in all," the child not only learns about "7" but also about its relationship to "5."
   - Students use numbers throughout the school day as they discuss the date, attendance, time, snacks, money, etc.
   - Students investigate fractions by listening to the story *Gator Pie* by Louise Mathews and by discussing how Alvin and Alice can share their pie with more and more alligators.
   - Second-graders record prices as decimals ($0.39) and use this notation to find totals over $1 on a calculator.
   - Students find half of a sheet of paper by folding horizontally, by folding vertically, and by folding diagonally. They compare the results and discuss how they are alike and how they are different.
   - Students use *Balancing Bear* software to find combinations of numbered weights that will balance a seesaw or that will be greater or less than a given weight.

2. **Develop an understanding of place value concepts and numeration in relationship to counting and grouping.**

   - Calendar activities at the beginning of the school day incorporate a *Daily Count* feature where each day another popsicle stick is added to a collection representing all of the days of school to date. Whenever 10 single sticks are available, they are bundled with a rubber band and are thereafter counted as a *ten*. On the hundredth day of school, the ten *tens* are wrapped together to make a *hundred*, and the class celebrates the event with a party.

- Students progress from a *proportional* and *bundle-able* base ten model like popsicle sticks to a *proportional* but not *bundle-able* model like base-ten blocks to a model that is neither *proportional* nor *bundle-able* like pennies and dimes. (See K-2 Overview)

- Pairs of students play *Race to One Hundred* with base ten blocks. Each, in turn, rolls one or two dice and takes that many unit cubes. Whenever there are ten unit cubes in a player's collection, the player *must* trade for a ten block. The first player able to trade ten ten blocks for a hundred block is the winner.

- Students have 3 dimes and 4 pennies to spend on a variety of items that are displayed in a classroom store. The items have tags ranging from 3¢ to 56¢ and the children are asked: *Which of these items can be bought for exactly the amount of money that you have (requiring no change)? Which items can you buy and have some money left over? Which of these items cannot be bought because you do not have enough money? What items are left?*

- Student understanding of place value for two-digit numbers is assessed by asking each student to represent a different number using popsicle sticks or base 10 blocks.

3. See patterns in number sequences, and use pattern-based thinking to understand extensions of the number system.

- Students find patterns in a hundred number chart. When asked to describe patterns that they see, some children see a counting by ones pattern horizontally, others see the tens digit increasing and the ones digit staying the same as they move down the chart vertically, and still others see in the last column the numbers that they use to count by tens.

- Students use the constant function feature of their calculators to program a *skip count*. They press *+ 2 ===* to watch the display count by twos, try to anticipate what number comes next and make predictions to each other. Any number can replace the "*2*" to add difficulty to the activity.

- Students play *Find the Number* on a hundred number chart located at the front of the room, with each of the numbers covered by a Post-it or a small tag. One child calls out a number, like *45*, and a volunteer tries to identify where it is on the chart. The indicated Post-it is then lifted to check the guess.

4. Develop a sense of the magnitudes of whole numbers, commonly used fractions, and decimals.

- Children are presented with four jars of jelly beans — one with 3 beans in it, one with 19 beans in it, one with 52 beans in it, and one with 156. The teacher then asks *Which of these jars do you think has about 50 beans in it?* The students discuss their reasons for believing as they do.

- Second graders are challenged to guess how many sheets of paper are in the ream of paper on the front table. After everyone has made a guess, one student counts out 25 sheets from the top of the pile and places them next to the rest of the pile. Everyone is offered a chance to change their estimates and to discuss the reason for their change. Then students agree on a way to verify their guesses before trying to guess how many such reams it would take to reach the ceiling!

- Students work through the *Will a Dinosaur Fit?* lesson that is described in the First Four Standards of this *Framework*. They discuss how many dinosaurs of different types might fit into the classroom.
- Students fold paper circles into halves, fourths, and eighths and are asked questions like: *Which would you rather have, a half of a cherry pie or a fourth of the pie? How about three-eighths of a pizza or one-fourth?*
- Students read or listen to a piece of children's literature that has fractions as its theme, such as *Eating Fractions* by Bruce McMillan.

5. Understand the various uses of numbers including counting, measuring, labeling, and indicating location.

- A kindergarten teacher announces to her class: *Boys and Girls! Great News! The principal told me that our class has just won FIVE!* A discussion then ensues regarding the need for that number to exist in some context, to have some unit or label before it makes sense.
- Second graders are given a stack of old magazines. They cut out any information which uses numbers and sort them according to how they are used: as page numbers, as prices, as dates, as addresses, and so on.
- The class takes a walk around the school or neighborhood pointing out to each other the numbers they see, and discuss how they are used.

6. Count and perform simple computations with money.

- Students use play money to show different combinations of coins that can be used to "buy" an object. For example, an 11¢ pencil can be bought with 11 pennies, a dime and a penny, one nickel and six pennies, or two nickels and a penny.
- Students earn 2¢ each day for attendance and 1¢ for good behavior. They keep their play money in a bank, count it regularly and use it to buy objects from a treasure chest.
- Students play *Spend a Dollar*. They each start with $1 (either as a bill or in change) and then roll one or two dice to find out how much they "spend" on that turn. They trade coins as needed. The student who spends all of her money first wins.
- Students play a shopping board game. They each begin with a given amount of money in coins. They roll two dice to determine how far they move each turn. As they land on a space, they must buy whatever is shown. Some spaces may provide refunds. The winner is the first person to go around the board and still have money left.
- Students' abilities to recognize coins and find the value of a group of coins are assessed by having each student select three objects to "buy," identify and name the coins needed to purchase each object, and find the total amount of money required to purchase all three.

7. **Use models to relate whole numbers, commonly used fractions, and decimals to each other, and to represent equivalent forms of the same number.**

   - When modeling 2-digit numbers with base-ten models such as popsicle sticks, base-ten blocks, or pennies and dimes, students are frequently asked to show all the ways they can make a given number. Children then begin to see that *3 tens and 7 ones*, *2 tens and 17 ones*, *1 ten and 27 ones,* and *37 ones* all represent the same number *37*.

   - Students each develop questions whose answers are all equivalent to some target number. For example, if the target is *8*, students may ask the following questions: *What is 4+4? What is 9-1? What is 8+0? How many hands do four children have? How many days is one more than a week?* or *How much is a nickel and three pennies?*

   - Students use geoboards, pattern blocks, Cuisenaire Rods, paper folding, and tangrams to explore simple common fractions like halves, thirds, and fourths. For instance, they may be challenged to model 1/2 with all of the different models.

8. **Compare and order whole numbers, commonly used fractions, and decimals.**

   - Young students use dot pattern cards or dominoes to practice *more, less,* and *same*. For example, given a card with 6 dots on it, students use counters to make a set that is more, another that is less, and one that is the same. They can then label the sets with cards that show the appropriate words. With dominoes, students work in pairs to compare the dots on the two halves and state which is more and by how much.

   - Students play the old favorite card game *war* with either dot cards or with a deck of regular playing cards minus the face cards. Every now and then, the rule changes so that the student with the card that is *less* wins the play.

   - Students play *Guess the Point*. A long number line with endpoints of 20 and 75, for example, is drawn on the board where all of the intermediary points are labeled above the line. The labels are then covered by a long piece of paper that can be lifted to reveal them. A student places a finger somewhere on the line and others must estimate the numerical label of the point chosen. The paper is then lifted to check the accuracy of their responses.

9. **Explore real-life settings which give rise to negative numbers.**

   - Primary classrooms are equipped with Celsius thermometers, in addition to Fahrenheit ones, so that “below zero” outdoor temperatures can be recorded. Temperature reports, possibly in both scales, become a part of the everyday calendar routine.

References

Baratta-Lorton, Mary. *Mathematics Their Way*. Menlo Park, CA: Addison-Wesley, 1995.

Mathews, Louise. *Gator Pie.* New York: Dodd, Mead, and Co., 1979.

McMillan, Bruce. *Eating Fractions*. New York: Scholastic, 1991.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

Software

Balancing Bear. Sunburst Communication.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense — Grades 3-4

Overview

In third and fourth grade, students continue to develop their number sense by using manipulatives and technology. The key components of number sense, as identified in the K-12 Overview, include an **awareness of numbers and their uses** in the world around us, a good sense of **place value concepts, approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Third and fourth graders are refining their understanding of whole numbers and are just beginning to develop understanding of numbers like decimals and fractions that require substantially different ways of thinking about numbers. An excellent activity that can be used to impress upon these students the omnipresence of numbers around them is the keeping of a journal reflecting all of the **uses of numbers** that they can find in magazines and books. In fourth grade, they focus particularly on uses of fractions and decimals. They may include decimal prices in advertisements, fraction-off sales, and decimal or fraction measurements. Discussions of the uses found and the meanings of the numbers involved can provide interesting insights.

Their **numeration** work in earlier grades, having focused on models of number, has enabled students to use relatively sophisticated models like play money or chip trading to represent whole numbers up to three digits. The regular and consistent use of concrete models is essential for the continuing development of their understanding of numeration. They should use base-ten models not only to extend their experiences with whole numbers to four places, and then symbolically beyond that, but also to create meaning for decimals.

In addition, models are essential for the initial explorations of the meaning of fractions. Fraction Circles or Fraction Bars help children establish rudimentary meaning for fractions, but have the drawback of using the same size unit for all of the pieces. Cuisenaire Rods or paper folding can be used to accomplish the same goals without this drawback.

Children at these grade levels also continue their learning about **equivalence**. They should be engaged in activities using concrete models to generate equivalent forms of many different kinds of numbers. For multi-digit numbers, equivalences such as: *367 = 3 hundreds, 6 tens and 7 ones = 3 hundreds, 5 tens and 17 ones = 2 hundreds, 14 tens and 27 ones* are useful in promoting a confident feeling about place value and will help in understanding multi-digit computation. Early explorations of equivalent fractions (1/2 = 2/4) and equivalent decimals (3 tenths = 30 hundredths) can accompany the exploration of the basic equivalences between fractions and decimals (1/2 = 0.5).

Estimation should be a routine part not only of mathematics lessons, but of the entire school day. Children should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the day like: *About how many kids do you think there are in the auditorium? About how many paper cranes will each student have to fold if the class needs to make 200 altogether?* and *How many floor tiles do you think are on the floor?* After several children have had chances to make estimates about numbers like these, they should defend their estimates by giving some rationale for thinking they are close to the actual number. These discussions can be invaluable in helping them develop number sense.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful exploration tools when examining new numbers. Students will themselves raise questions about decimals when someone divides 30 by 60 inadvertently instead of 60 by 30 and wonder what the 0.5 in the display means. Computers provide software that creates environments in which students manipulate base-ten models on-screen and explore initial fraction and decimal concepts.

The topics that should comprise the number sense focus of the third and fourth grade mathematics program are:

whole number meanings through many digits
place value and number base
initial meanings for fractions and decimals

Standard 6 — Number Sense — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. Use real-life experiences, physical materials, and technology to construct meanings for whole numbers, commonly used fractions, and decimals.

- Students are comfortable using a full array of base-ten models including money, base-ten blocks, and chip trading materials to represent both whole numbers and decimals.
- Students use computer software that provides easy pictorial representation of large whole numbers, decimals, and fractions like the *MECC Math Tools,* the *Silver Burdett Math Workshop,* and the *Wasatch Math Construction Tools*.
- Students use geoboards to model common fractions. For example, they search for multiple ways to show 1/4 on the geoboard.
- Students use Cuisenaire Rods to model fractions, frequently switching the rod or length used as the *whole* to avoid the misconception that, for instance, the yellow rod is always one-half.

2. Develop an understanding of place value concepts and numeration in relationship to counting and grouping.

- Pairs of students play *Race to Five Hundred* and its opposite, *Race to Zero,* with base ten blocks. In the first game, each student, in turn, rolls a red die and a green die and makes a two digit number from the faces showing (using the red die as the tens digit). He or she then takes that many tens and ones from the bank. Whenever there are ten tens or ten ones in a player's collection, the player must trade for a larger block. The first player to collect 5 hundreds is the winner. In *Race to Zero,* the players *start* with 5 hundreds and *give back* blocks according to their dice rolls.
- Students use a die to generate random digits from 1 to 6. After each roll, they decide where to place the digit in a 4-digit whole number. The goal is to produce as large a number as possible. If a ten-sided die or spinner with ten equal sectors is available, students should use it to generate random digits from 0 to 9 and repeat the activity.
- Students work in groups to decide what the next base-ten block after the thousands block would look like.
- Students read and listen to children's literature that is related to a numeration theme like *Millions of Cats* and *The 500 Hats of Bartholomew Cubbins*.

3. **See patterns in number sequences, and use pattern-based thinking to understand extensions of the number system.**

   - Students use the constant function feature of their calculators to program a *skip count*. They press + *12 ===* to watch the calculator display count by twelves, trying to anticipate what number will come next and making predictions to each other. Any number can replace the *12* to change the difficulty level of the activity.

   - Students also use their calculators to play *Guess My Rule* games. One student secretly programs the calculator by typing something like x *2 =*. Thereafter, every time a number is pressed followed by an equals sign, the original number will be multiplied by two. A second student must guess the rule that was programmed. Rules like + *3 =*, ÷ *4 =*, and **-** *2 =* also work.

   - Students create and solve *arrow puzzles* on a hundred number chart. By naming a number and then giving directions for movement on the chart, instructions are given to arrive at some other number. For example: *72, down, down, right, right, up* leaves the student at the number *84*. After examples of different patterns are demonstrated on the chart, students point out patterns and try to solve puzzles mentally.

   - Students make a table to reflect how many handshakes there would be if everyone shook hands in groups of different sizes. For example, for 2 people, 1 handshake; for 3 people, 3 handshakes; for 4 people, 6 handshakes. As they extend the table for larger groups, the students look for a pattern in the emerging numbers.

   - Students search for patterns in the addition of even and odd numbers by using unifix cubes to represent the numbers and trying to arrange the sum into two stacks of equal height. (This will work for even numbers, but not for odd ones.)

4. **Develop a sense of the magnitudes of whole numbers, commonly used fractions, and decimals.**

   - Students imagine collecting 10,000 of something. They discuss what objects would be reasonable to collect (such as bottle caps, pennies for charity, or pebbles from the beach), how much space this collection might take up, and how much it would weigh?

   - Students cut and paste sheets of base-ten graph paper to make models of the different powers of ten: *1, 10, 100, 1000, 10,000.*

   - Students locate numbers as points on a number line strung across the room, continuing to attach labels as they learn more about numbers. Paper clips or tape are used to fasten equivalent forms of a number to the same point.

   - Students estimate and investigate how long a million seconds is using calculators.

   - Students write a Logo or BASIC computer program which will count to 100, printing the numbers to the screen as it runs, and timing how long it takes. Then they predict how long it would take the program to count to one thousand, to one hundred thousand, and to one million. They make the required changes to the program and check their predictions.

5. **Understand the various uses of numbers including counting, measuring, labeling, and**

indicating location.

- Students keep a 24-hour diary recording all of the ways they use or see others use numbers. They pool all of these uses of numbers and classify them into categories that they design.
- As part of a geography unit, students make a map of a fantasy island, using a Cartesian coordinate system to help describe the location of various places on the island. They use other numbers to describe geographical properties of the sites: elevation, longitude and latitude, population, and the like.
- The students brainstorm ways to describe their math book in terms of numbers: its width, the number of pages, the publication date, a student-generated "quality rating" of the book, the area of the cover, and so on.

6. Count and perform simple computations with money.

- Students establish a school store and make transactions on a regular basis, with different students assigned as clerk each day.
- Students read *Dollars and Cents for Harriet* and then decide how they would spend five dollars.
- Students practice making change with coins by *counting up* to the amount given. For example, if the bill is $1.73, and $2.00 is the amount given, the students would count up to $2.00 by starting with two pennies and saying, "$1.74" and "$1.75"; then they add one quarter to bring the total to $2.00. They would then count this change to find its value of $0.27.
- Students play *Treasure Math Storm* on the computer or use IBM's *Exploring Measurement, Time and Money*.

7. Use models to relate whole numbers, commonly used fractions, and decimals to each other, and to represent equivalent forms of the same number.

- When modeling 3- and 4-digit numbers with a base-ten model like base-ten blocks or place value chips, the students are frequently asked questions like: *Show all the ways you can make 327.* Children thus begin to see that *3 hundreds, 2 tens, and 7 ones; 2 hundreds, 12 tens, and 7 ones; 2 hundreds, 11 tens, and 17 ones;* and *32 tens and 7 ones* all represent the same number. Students are assessed by asking them to show 327 in two different ways.
- Students use shadings on ten-by-ten grids to represent fractions and decimals that are equivalent. For example, the representation for 0.4 is the same as that for 4/10.
- Students develop their own questions, the answers to which are equivalent to some target number. For example, if the target number is *24*, students may ask the following questions: *What is 20 + 4? What is 2* x *12? What is 2* x *2* x *2* x *3? How much is 2 dozen? How many is 3 less than the number of children in our class?* or *How much would something cost if you paid a quarter and got back a penny in change?*
- Students use geoboards, pattern blocks, Cuisenaire Rods, paper folding, and tangrams to explore common fractions. They may be challenged to model 3/4, for instance, with all of the different models.

- Students use money to represent decimals. For example, 8 dimes = \$0.80 = .8. They also represent fractional parts of a dollar as a decimal (a quarter = 1/4 = 25¢ = .25).
- Students use graham crackers, candy bars, pizzas, and other food to illustrate fractions.
- Students work through the *Sharing Cookies* lesson that is described in the First Four Standards of this *Framework*. They realize that 8 is not readily divisible by 5 and try to find ways to solve that sharing problem using real cookies.
- Students play *Bowl a Fact* by rolling three dice and using the numbers shown to make number sentences whose answers equal numbers from *1* to *10*. For each different answer, they knock down the bowling pin labeled with that number. For example, if they roll 2, 5, and 3, they can make these number sentences: $2 + 5 + 3 = 10$, $5 + 3 - 2 = 6$, $5 \times 2 - 3 = 7$, $5 - 3 + 2 = 4$, and $3 \times 2 - 5 = 1$, and therefore knock down the *10, 6, 7, 4* and *1* pins. If they cannot knock down all ten pins on the first roll, they roll the dice again and try to get the remaining pins. The students are assessed by giving all of them the same outcomes of two rolls of the three dice to play the game.

8. Compare and order whole numbers, commonly used fractions, and decimals.

- Students use base-ten materials such as, blocks, sticks or money to make models of pairs of 3- or 4-digit numbers like *405* and *450* and compare them to see which is larger. Responses and reasons can be written in a journal.
- Students play *Guess the Point*. A long number line with endpoints of 130 and 470, for example, is drawn on the board with the intermediary points labeled as multiples of ten above the line. The labels are then covered by a long piece of butcher paper that can be lifted to reveal them. A student places a finger somewhere on the line and others must estimate the numerical label of the point chosen. The paper is then lifted to check the accuracy of their responses.
- Pairs of students play *Hi-Lo* with whole numbers and decimals. One student thinks of a number within a given range such as 1 to 1000. The other student tries to guess the number, receiving feedback after each guess as to whether the guess was too high or too low, and keeping a written record of the guesses and the feedback. The goal is to find the number using as few guesses as possible.
- When using Cuisenaire Rods, students choose a *base rod* to represent one whole, and then determine the values of all of the rest of the rods. They then use the rods to model the comparison of the relative sizes of two fractions with different denominators.

9. Explore real-life settings which give rise to negative numbers.

- Students record daily low Celsius temperatures throughout the winter and draw a line graph of those temperatures. Students discuss changes in the graph and the meaning of the line dipping below the zero degree line.
- Students examine a videotape of a section of a football game and record the results of a series of plays as a series of integers — gains as positive integers and losses as negative integers (for example: −3, +5, +9, first down; −5, −4, +6, punt). They use their record to determine the total yardage gained during the drive.

- Students use an almanac to find the altitudes of selected cities around the country, and discuss what it means for a city to be below sea level.

References

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Seuss, Dr. *The 500 Hats of Bartholomew Cubbins*. New York: Vanguard, 1938.

Software

Exploring Measurement, Time, and Money. IBM.

Math Construction Tools. Wasatch.

Math Tools. Minnesota Educational Computing Consortium (MECC).

Math Workshop. Silver Burdett.

Treasure Math Storm. The Learning Company.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense — Grades 5-6

Overview

Fifth and sixth graders should have a good sense of whole numbers and their orders of magnitude and should be focusing mostly on developing number sense with decimals, fractions, and rational numbers, which require substantially different ways of thinking about numbers. They also should be exploring two relatively new topics: ratios and integers. The key components of number sense, as explained in the K-12 Overview, are an **awareness of the uses of numbers** in the world around us, a good sense of **approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Students at this age are capable of categorizing all of the ways in which numbers are used in our society. An excellent activity is to have them collect ways in which they see numbers being used during a twenty-four hour period. Their **uses of numbers** would probably include telephone numbers, addresses, ages, page numbers, clothing sizes, library book numbers, room numbers, and many others. Discussions of the similarities and differences among these uses should resolve themselves into some of the standard categorizations: counts, measures, labels, and indicators of location. The students' data can then be graphed according to these categories.

Their **numeration** work in earlier grades, having focused on models of whole numbers, has taken these students to the point where they are able to use relatively sophisticated models like play money or chip trading to represent whole numbers up to three digits. The regular and consistent use of concrete models is essential for the continuing development of their understanding of numeration. The focus now shifts to a real sense of the meanings of decimals and fractions and to providing models which adequately serve that purpose. Money continues to provide a superb setting for the learning of decimal concepts (at least up to two decimal places) because of the students' increasing familiarity with it, because of the vast array of real-world applications that it makes available, and because of its inherent motivational quality. Base-ten blocks are also useful as a slightly more abstract model. They have the added advantage of being able to represent *any number with four digits* in the place value system. A block model of 3274 with which the students are familiar can become a model of the decimal 3.274 if, instead of thinking of the smallest block as one unit, they think of the largest block as one unit.

In addition, models are essential for the continued exploration of fraction meaning and fraction operations. Fraction Circles or Fraction Bars help children establish rudimentary meaning for fractions, but have the drawback of using the same size unit for all the pieces. This is a fairly serious drawback leading to the misconception, for instance, that 1/3 is *always* less than 1/2 without regard to the *units* in which those fractions are expressed; students need to be aware, for example, that 1/3 of a large pizza is frequently larger than 1/2 of a small one. Cuisenaire Rods or paper folding can also be used to accomplish many of the same goals without the same drawback. A sample unit on fractions for the sixth-grade level can be found in Chapter 17 of this *Framework.*

Work with ratios, percents, and integers in grades 5 and 6 should be limited to informal exploration, with no use of more formal, symbolic procedures. Students should use models as they explore these topics. They might use 2 red tiles and 1 yellow tile to illustrate mixing paint in the ratio 2:1 and extend this pattern in order to make larger (and smaller) quantities. By using base ten blocks and 10 x 10 grids, they can visualize percent more easily. Two-color counters or the number line might be used to model positive and negative numbers (integers).

Students in grades 5 and 6 should begin to understand the ways in which different types of numbers are related. For example, they should understand that every whole number is a rational number, since it can be written as a fraction. Similarly, every decimal is a rational number. By the end of sixth grade, they should have had sufficient experiences with integers to realize that the integers consist of the whole numbers and their opposites (additive inverses).

Students at these grade levels continue their learning about **equivalence**, but there is a significant shift in what that means. As third and fourth graders, they have explored simple fractions and decimals, and their work with equivalence has focused primarily on the multiple ways to represent whole numbers (*8 = 2 + 6 = 9 – 1, 23 = 2 tens + 3 ones = 1 ten + 13 ones,* and so on). Now, as fifth- and sixth-graders, they should begin to focus on the representation of the same quantity with different *types* of numbers. Their work with equivalent fractions (1/2 = 2/4) and equivalent decimals (3 tenths = 30 hundredths), for example, should lead to exploration of the basic equivalences of fractions and decimals (1/2 = 0.5). They should be engaged in activities using concrete models to generate equivalent forms of many different kinds of numbers. They also begin to explore the role of ratios and percents in this mix. Ten-by-ten grid paper helps enormously with these activities, since all forms of a quantity can frequently be represented on it.

Estimation should be a routine part not only of mathematics lessons, but of the entire school day. Children should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the day, like: *About what fraction (percentage) of the kids in the playground do you think are wearing gloves? About one-third of our students stay for the after-school program in the afternoon; if there are 500 students in the school about how many of them stay?* After children have had several chances to make estimates about numbers like these, they should defend their estimates by giving some rationale for thinking they are close to the actual number. These discussions can be invaluable in helping them develop good number sense.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful exploration tools when examining new relationships. Many insights about the relationships between fractions and decimals, for instance, can be achieved by simply dividing the numerators of fractions by their denominators. Generalizations about what kinds of fractions produce what kinds of decimals start to flow very freely in such open-ended explorations. Computer software also creates environments in which students manipulate decimal models on-screen and explore fraction and decimal relationships.

The topics that should comprise the number sense focus of the fifth and sixth grade mathematics program are:

fractions
decimals
equivalence
integers
ratio and percent

Standard 6 — Number Sense — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

10. Understand money notations, count and compute money, and recognize the decimal nature of United States currency.

- Students try to identify fake "bargains" or misleading information from store advertisements, and try to determine which of several options is the best. For example: *Which of the following offers the cheapest price?*

XYZ Mega-Deal Tuna	*ABC Tuna*	*Aunt Betty's Best Tuna*
10 oz can $2.80	*3 oz can $0.87*	*4 oz can $1.04*

- Students plan a fantasy driving trip to Walt Disney World for their family. They research and consider the variety of expenses to be incurred — lodging along the way and at the park, meals, souvenirs, gasoline, admission, and so on. A reasonable budget for the trip is the centerpiece of a report prepared by each student group. Their reports are assessed using a scoring rubric that includes mathematical correctness as well as creativity.
- Students use play money to model decimal numbers. They use decimal language and find fractional equivalents for each coin. For example: *a dime = 0.1 = 1/10 = 1 tenth.*
- Students first estimate and then use a calculator to find out *how long* it would take to spend one million dollars at a rate of one dollar per second.

11. Extend their understanding of the number system by constructing meanings for integers, rational numbers, percents, exponents, roots, absolute values, and numbers represented in scientific notation.

- Students each develop questions, the answers to all of which are equivalent to some target number. For example, if the target number is *24*, students may ask the following questions: *What is 8 x 3? What is* $(-25) - (-49)$*? What is* $5^2 - 1$*? What is 3 more than the sixth triangular number? What is 1 less than one-fourth of 100? or What is the smallest positive number with 8 factors?*
- Students continue to refine their concepts of fractions using all available models to answer questions like: *Is 1/4 always larger than 1/8? Is 1/4 of every pizza larger than 1/8 of every other pizza?* Issues that point out the importance of defining the unit are special topics for discussion.
- Students use two-color counters to construct models of the set/subset meaning of fraction. You might ask: *Given 3 red counters in a set of 12, what are the equivalent fractions that*

represent the reds as a part of the set?

- Students also use two-color counters to model and begin to make sense of positive and negative integers. In this system, a positive 1 is represented by one color and a negative 1 by the other. Students determine the value of a pile of counters by pairing up counters, one of each color, setting aside all pairs, and counting the remaining counters.
- Students read Shel Silverstein's poem *A Giraffe and a Half* and discuss how to describe an amount that is more than one whole but less than two.
- Students read *The Phantom Tollbooth* and discuss the relationships between decimals and fractions in the book. For example, Milo meets half a child (actually, .58 of a child since the average family has 2.58 children).
- Students construct a time line to scale to show the history of the earth. Significant periods and events are shown along the line with numbers reflecting the number of years since the earth's beginning.

12. Develop number sense necessary for estimation.

- Students imagine collecting a million of something. They discuss what objects would be reasonable to collect (such as toothpicks, punched holes from fan-folded computer paper or pages in telephone books to be recycled), how much space this collection would take up, and how much it would weigh.
- Students make estimates to answer the question: *How much drinking water do you think Columbus' ships carried with them on their trip across the ocean?* Then they gather the data they need to make more informed estimates. (*Addenda Series Grade 4 Book.)*
- Students determine the number of decimal places in a simple decimal multiplication product, not by mechanically adding the number of places in the factors, but by estimating a reasonable range for the product and placing the decimal point so that their computed product falls within that range.
- Students investigate the question: *What size room would be needed to hold one million ping-pong balls?*
- Students read *Counting on Frank* and estimate how many dogs would fill their classroom.
- Students use estimates to compare fractions. For example, 3/7 < 9/16 since 3/7 is less than half and 9/16 is more than half.

13. Expand the sense of magnitudes of different number types to include integers, rational numbers, and roots.

- Students challenge each other to find target numbers on a number line. First one student asks another to find 3.2. The second target number then must be between 0 and 3.2, say 1.74. The third must then be between 0 and 1.74 and so on.
- Students use their calculators to explore the types of decimal expansions for common fractions. They discover that some such decimals terminate, some repeat, and some appear to do neither. (Actually, if the calculators could exhibit more digits, each such decimal would either terminate or repeat.)

- Students use calculators to come as close as they can to an answer to: *What number multiplied by itself gives an answer of 2?*

14. Understand and apply ratios, proportions, and percents in a variety of situations.

- Students begin to see a ratio as both the comparison of two quantities and as a number in its own right. They are challenged to find ratios that they frequently use like *$0.65 per pound, 55 miles per hour,* and so on.
- In a social studies unit, students use population and area data for countries in South America to compute population densities, and then compare their results to those for other areas of the world.
- Students use two different sizes of grid paper to copy a simple drawing of a house from the smaller grid to the larger grid, investigating and discussing the change from one to the other and exploring ways to represent it numerically. They then copy the same drawing onto a third grid, smaller than the second but larger than the first.
- Students are challenged to use any combination of the digits 3, 4, 5, and 9 to make a ratio as close as possible to 90%. As follow-up, they invent other *closeness* problems for each other.
- Students search for as many uses of percent as they can find over the course of a week. The sources for the uses, however, are to be exclusively within the school setting. Likely entries in the resulting list are: *grades on tests, foul shot success of the basketball team, a measure of how close the PTA is to their fund-raising goal for the new playground equipment,* and so on. For each use found, the students explain what 100% would represent and whether percentages above 100% would make any sense in the given context.

15. Develop and use order relations for integers and rational numbers.

- Students use a deck of *fraction cards* for a variety of tasks. A deck consists of one card containing each of a number of fractions, for example, 1/4, 9/10, 2/19, 4/7, 7/9, 1/3, 12/15, 2/5, and 5/8. They are asked to: *Find the smallest fraction in the set. Sort into two groups more than 1/2 and less than 1/2. Determine which pairs have a value close to 1.*
- Students use a number line, including both positive and negative integers, to graph inequalities stated verbally. For example: *Show all of the numbers larger than −2.*
- Students gain understanding of the order relationships among fractions and integers by comparing them with similar ones for whole numbers. *How is the comparison of 4/7 to 5/7 like the comparison of 4 to 5? How is comparing 4/7 to 4/8 like comparing 7 to 8? How is comparing −4 to −7 like comparing 4 to 7?* They answer similar questions on their test.
- Students use 4 digits, say *2, 3, 4,* and *5* to construct as many true fraction sentences as they can. For example, *2/3 < 5/4*.

16. Recognize and describe patterns in both finite and infinite number sequences involving whole

numbers, rational numbers, and integers.

- Given the first four rows, students formulate a rule for generating succeeding rows of Pascal's triangle. They look for other patterns in the triangle.
- Students explore the well-known problem of taking a long walk by first doing half of it, then half of what remains, half again of what remains, and so on. They write the series as 1/2 + 1/4 + 1/8 + ... *What happens to the walker?*
- Students solve this classic problem: *Which would you choose as the method for getting your allowance next month: $1.00 every day; or 1 cent the first day, 2 cents the second, 4 cents the third, 8 cents the fourth, and so on?*

17. Develop and apply number theory concepts, such as, primes, factors, and multiples, in real-world and mathematical problem situations.

- Students build rectangular arrays with square tiles to determine which of the first fifty counting numbers are *rectangular* (composite) and which are *non-rectangular* (prime).
- Students use the Sieve of Eratosthenes to generate a list of all the primes in the first 100 counting numbers.
- Students use common multiples to solve problems like this: *Hot dog buns come in packages of 8. Hot dogs come in packages of 6. What is the smallest number of packages of each that can be bought so that there are no extra buns or hot dogs?*

18. Investigate the relationships among fractions, decimals, and percents, and use all of them appropriately.

- Students address the questions *How are 0.50 and 40/100 alike?* and *How are they different?* Answers can be written in their math journals.
- Students use shadings on a ten by ten grid to discuss all of the different equivalences. For example, the same shading can be named 3/10, 30/100, 0.3, 0.30, .3, .30, and 30%. Thinking of the grid as $1.00 leads to some interesting insights about two-place decimals.
- Students explore the density property of numbers by addressing problems like: *Find 4 decimals between 0.456 and 0.457. Find 3 fractions between 3/5 and 4/5.*

19. Identify, derive, and compare properties of numbers.

- Students use Venn diagrams to explore the multiple sets to which particular number belong. For example, a Venn diagram is created for these three sets of numbers less than 25: multiples of 3, factors of 24, primes; the Venn diagram is used to answer questions like: *How many numbers are in exactly two of these sets*? A similar question is used on their test.
- Students explore the property of *closure* for a variety of sets of numbers under various operations. For example: *Using subtraction, is there always an answer within the set of positive whole numbers for any member of the set minus any other?* (no); *Is there always an answer within the set of integers?* (yes); *Is there always an answer within the set of even integers?* (yes); *within the set of odd integers?* (no).

- Students explore the properties of odd and even numbers under various operations. For instance: *What can always be said about the sum of two even numbers? of two odd numbers? of an even and an odd number?*
- Students create a book about *zero* for second-graders.
- Students explore the concepts of place value and zero by learning about other number systems. For example, they might use the computer program *Maya Math* to learn about the Mayan number system.

References

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Software

Maya Math. Sunburst Communications.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense— Grade 7-8

Overview

Seventh and eighth graders should have a good sense of whole numbers and their orders of magnitude and should be focusing on further developing number sense with decimals and fractions. They should be extending their understanding of whole numbers to negative numbers, including comparison and ordering. They also should be working on incorporating ratio, proportion and percent, powers and roots, and scientific notation into their conception of the number system. The key components of number sense, as explained in the K-12 Overview, are an **awareness of the uses of numbers** in the world around us, a good sense of **approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Students at this age are capable of categorizing the ways in which numbers are used in our society. One interesting activity is to have them collect data on how numbers appear in a portion of a newspaper, categorize these uses, and then graph their results.

In their work with **numeration**, seventh- and eighth-graders should begin to see mathematics as a coherent body of knowledge. They should begin to see the integers and the rationals as logical and necessary extensions of the whole number system. Only with these extensions can expressions like *3 − 7* and *4 divided by 3* have answers.

In grades 7 and 8, students are focusing on ratios, proportions, and percents, topics which they were just beginning to consider in grades 5 and 6. Their work with these concepts and the relationship of these numbers to fractions, decimals, and whole numbers form the foundation for a very powerful problem solving skill: proportional reasoning. New topics at these grade levels are exponents, roots, and scientific notation. Students should also explore irrational numbers, such as π, square roots of numbers which are not perfect squares, and other decimals which neither end nor repeat.

Students at these grade levels need to continue learning about **equivalence**, but there is a vast array of kinds of equivalence to be considered here. Students focus on the representation of the same quantity using different *types* of numbers and the selection of the appropriate number type given a particular problem context. It is particularly important that students understand the difference between the exact value of a fraction, such as 2/3, and its approximation of .667, especially since they now use calculators routinely. Relationships among decimals, fractions, ratios, and percents comprise the largest emphasis, but work with exponents and roots and their relationship to scientific notation is also a focus in these grades. Number theory provides a rich context for interesting problems in this area. Questions about infinity, division by zero, and primes and composites, combine with discussions about finite and infinite sequences and series and searches for patterns to open up the full richness of a mathematical world.

Estimation should be a routine part of mathematics classes. Students should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the class with answers which demonstrate confident and well-conceived use of estimation strategies and sense of number.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful exploration tools when examining numerical relationships. Many insights about the relationships between fractions and decimals, for instance, can be attained by simply dividing the numerators of fractions by their denominators. Generalizations about what kinds of fractions produce what kinds of decimals start to flow very freely in such open-ended explorations. Computer software can also be very useful. Spreadsheets, for example, can show a great many ratios on the screen at the same time. For example, the five ingredients of a waffle recipe that makes 4 waffles can be listed across the top row of the spreadsheet, with following rows showing how the recipe changes to make 2, 8, and 12 waffles (*Curriculum and Evaluation Standards*, 1989, p. 89).

The topics that should comprise the number sense focus of the seventh and eighth grade mathematics program are:

- rational numbers (both positive and negative)
- equivalence
- integers
- ratio, proportion, and percent
- exponents, roots, and scientific notation

Standard 6 — Number Sense— Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

10. Understand money notations, count and compute money, and recognize the decimal nature of United States currency.

- Students can solve a variety of real-world money problems such as: *If you make $750.00 a month, would you rather have a 12% raise or an $85 a month raise?* or *Which sale is better on a $17.00 sweater, a 1/3 off sale, a $5.00 discount, or a 30% discount?*
- Students use time-cards and pay rates to compute weekly wages and deductions for various workers. Issues such as, time-and-a-half for overtime, double time for holidays, and percentages of total wages to be deducted for various taxes all come into play.
- Students investigate the details and make plans to dispose of the proceeds of the $27,000,000 lottery which they just won.

11. Extend their understanding of the number system by constructing meanings for integers, rational numbers, percents, exponents, roots, absolute values, and numbers represented in scientific notation.

- Students develop a *scientific notation Olympics* by creating events like the 9.144×10^3 *centimeter sprint* (100 yard dash) or the 7.272×10^6 *milligram hurl* (shot put).
- Students use a bookkeeping simulation to explore the effect of bills and credits coming into, or going out of, their business. These financial activities are recorded as various actions on positive and negative integers, all affecting the net worth of the business.
- Students view the *Powers of Ten* video, developed to show how one's view of the world is affected by changes in the order of magnitude of one's position.
- Students construct a hypothetical stock portfolio, using $10,000 to "buy" shares of stock, and track the performance of the portfolio and each individual stock day by day.
- Students explore absolute values as the distance between two points on a number line and compare this to subtraction.
- Students measure the circumference and diameter of 15 to 20 round objects, recording the results in a table. They make a scatterplot (with diameter on the horizontal axis) and use a piece of spaghetti to draw the line of best fit. They discuss why π is represented in the drawing as the slope of the line.
- Students construct line segments of varying irrational lengths on a geoboard or dot paper.

For example, the diagonal of a unit square has length $\sqrt{2}$, the diagonal of a 1 x 2 rectangle has length $\sqrt{5}$, and the circumference of a circle with unit diameter has length π. Through exploring these common irrational numbers that arise in problem situations, students learn that not all numbers can be represented as a ratio of two integers.

- Students construct their own graph for the square roots of the numbers from 1 to 25, using trial-and-error to approximate each root to the nearest tenth. They plot the numbers on the horizontal scale, and their square roots on the vertical scale.
- Students work on traditional "systems of equations" problems, involving two unknowns, by devising non-algebraic solution strategies for them. Some samples are: *Two numbers have a sum of 32; they have a product of 240. What are the numbers?* or *Sally is 22 years younger than her Dad. In 3 years, her Dad will be 3 times as old as she. How old is Sally?*

12. Develop number sense necessary for estimation.

- Students wrestle with this classic problem: *After spending most of the day looking for her missing pet cat, Whiskers, the eccentric billionaire, Ms. Money Bags, received a ransom demand. The caller said she was to bring a suitcase packed with $1,000,000 in one- and five-dollar bills to the bus station and leave it in Locker #26-C. Then her pet would be returned to her. How did she respond?*
- Students describe a scale model of the solar system built on the premise that the earth is represented by a ping-pong ball.
- Students make estimates of the number of times various events happen in an average lifetime, discuss their strategies for estimation, and then check their estimates against some reference. A good reference for this activity is *In an Average Lifetime* by Tom Heymann. Among other things, in an average lifetime, an American consumes 10,231 gallons of beverages, spends $1,331 on home-delivered food, and spends 911 hours brushing his or her teeth.

13. Expand the sense of magnitudes of different number types to include integers, rational numbers, and roots.

- Students play *Locate the Point*. A number line with end points -5 and 5 is suspended in the classroom, using a long string with tabs to indicate the positions of the integers between the two end numbers. Students are given cards with different types of numbers on them. (For example: $-12/3$, 1.1, 1.01, $\sqrt{2}$, π, -2^2, $(-2)^2$, $\sqrt{3}$, $\sqrt{8}$, $1.\overline{9}$, 2, $\sqrt[3]{8}$, etc.) They take turns and attach their card on the appropriate spot on the number line. Classmates decide whether the position is correct. If more than one expression is used for the same number, the cards with those numbers are attached by tape.
- Students use only the multiplication and division functions on their calculators to perform a series of successive approximations to find acceptable values for several roots: the square roots of 2, 3, 7, and 10, and the cube roots of 10 and 100.
- Pairs of students play *Hi-Lo* with decimals as a way to emphasize the density of the rational numbers. One student thinks of a number between 0 and 10 with up to 4 decimal places.

The other student tries to guess the number, receiving feedback after each guess as to whether the guess was too high or too low. Written records of the guesses and the feedback are kept. The goal is to find the number using as few guesses as possible.

14. Understand and apply ratios, proportions, and percents in a variety of situations.

- Students take consumer price data from 10 and 25 years ago and figure out the percentage increase or decrease in the prices of various products over those periods of time. They discuss questions such as: *What makes a price go up? What would make it go down?*
- Students predict, and then determine, which body part ratios are fairly constant from person to person. Some interesting ones are height/arm span, wrist circumference/hand span, and waist/neck circumference.
- Students make a three-dimensional model of the classroom with different groups taking responsibility for modeling different objects in the class. First the desired size of the model is discussed and a scale factor agreed upon. Then each of the groups measures and applies that scaling factor to their objects, determines appropriate materials and means of construction, and builds the models.
- Students examine whether it is better to take a discount of 20% and then add a 6% sales tax or add the sales tax and then take a 20% discount. (The answer may surprise the students!)
- Students examine different statements involving proportions and discuss which ones make sense and which do not. For example: *If one girl can mow the yard in 30 minutes, then two girls can mow the yard in 15 minutes. If one boy can walk to school in 20 minutes, then two boys can walk to school in 10 minutes.*
- Students compare magazine subscription prices for 6, 9, and 12 months in order to decide which is the better buy.
- Students estimate what percent of plain M&M's are red, green, yellow, blue, brown, and orange. They test their guesses by counting the number of each color in a small bag and finding the percentages. They also discuss whether they improve their estimates by combining their data.
- Students simulate running a business using the computer program *The Whatsit Corporation* or *Survival Math*.

15. Develop and use order relations for integers and rational numbers.

- Students use concrete and pictorial models to develop order relations among fractions and integers. Using Cuisenaire Rods and varying the *unit*, students demonstrate that one fraction is larger than another. Similar arguments and conclusions are made on a number line for integers.
- Students' abilities to order rational numbers (both positive and negative) are assessed by asking them to identify points on a number line between, say, −3 and −5.
- Students are each given a rational number on a large card (−1.2, 4, 3/4, −2 1/4, −1, 3.14, 22/7, and so on). They then order themselves from least to greatest along the front or side of the classroom. They also respond to instructions like: *Hold up your card if it is between −2.5 and +0.7.*

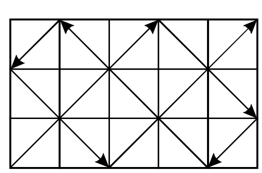
16. **Recognize and describe patterns in both finite and infinite number sequences involving whole numbers, rational numbers, and integers.**

- Students formulate a description of the *nth* row of Pascal's triangle.
- Students investigate the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, ...) to see how it is generated and then do library research to find theories about its startling occurrences in nature.
- Students explore the *golden ratio* discovered and used by the ancient Greeks. They find examples of golden rectangles (whose sides are in the golden ratio) in everyday objects (3 x 5 cards, bricks, cereal boxes), and in architecture (the Parthenon).
- Students discuss and predict the sum of this well-known series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

17. **Develop and apply number theory concepts, such as primes, factors, and multiples, in real-world and mathematical problem situations.**

- Students write a Logo or a BASIC computer program to find all the factors of any number that is provided as input. They can then use the same program to determine if any input number is prime.
- Students explore Goldbach's conjecture (a mathematical hunch which has never been proved nor disproved) which states that: *Any even number greater than two can be written as the sum of two prime numbers*. *For example: 14 = 11 + 3, 24 = 11 + 13, and 56 = 3 + 53. Can you find one that cannot be written this way?*
- Students develop rules of divisibility for all one-digit numbers and explain and apply these rules on a test.
- Students investigate the path of a ball on a billiard table with sides of whole number length when the ball starts in a corner and always travels at a 45 degree angle. For example, a ball on the 3 x 5 table in the diagram starts in the lower left corner and takes the path shown, hitting the perimeter eight times (including the first and last corners) and going through all 15 squares, before ending at the top right corners. They make a table which records the length and width of the billiard table, the number of hits against the perimeter, and the number of squares passed through, for billiard tables of various sizes, and look for relationships. The number of hits against the perimeter of the table, including the first and last corners, is the sum of the width and the length of the billiard table divided by their greatest common factor.



18. Investigate the relationships among fractions, decimals, and percents, and use all of them appropriately.

- Given a circle graph of some interesting data, students estimate the size of each section of the graph as a fraction, a percent, and as a decimal. Students also create their own circle graphs.
- Students use two different-color interlocking paper circles (each has a cut along a radius so they fit together), each marked off in wedges that are one hundredth of the circle, to show fractions that have denominators of 10 or 100, decimals to hundredths, and whole number percents to 100%.
- Students explore patterns in particular families of decimal expansions, such as those for the fractions, *1/7*, *2/7*, *3/7*, ... or *1/9*, *2/9*, *3/9*, ...

19. Identify, derive, and compare properties of numbers.

- Students work on this problem from the *Curriculum and Evaluation Standards for School Mathematics* (p. 93): *Find five examples of numbers that have exactly three factors. Repeat for four factors, and then again for five factors. What can you say about the numbers in each of your lists?*
- Students explore *perfect numbers,* those numbers that are equal to the sum of all of their factors including 1 but excluding themselves. Six is the first perfect number, where $6 = 1 + 2 + 3$. Interestingly, the next one has 2 digits, the third has 3 digits and the fourth has 4 digits. The pattern breaks down there, though, since the fifth perfect number has 8 digits. Students who have worked on a computer program to find all of the factors of numbers (see Indicator 17 on the previous page) may want to revise their program to see how many perfect numbers they can find.

References

Heyman, Tom. *On an Average Day*. New York: Fawcett Columbine, 1989.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

Software

Logo. Many versions of Logo are commercially available.

Survival Math. Sunburst Communications.

The Whatsit Corporation. Sunburst Communications.

Video

Powers of Ten. Philip Morrison, Phylis Morrison, and the office of Charles and Roy Eames. New York: Scientific American Library, 1982.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense — Grades 9-12

Overview

High school students build upon their knowledge of rational numbers as they increase their understanding of irrational numbers and generalize number relationships through their work with algebra. The key components of number sense, as identified in the K-12 Overview, are an **awareness of the uses of numbers** in the world around us, a good sense of **approximation, estimation, and magnitude**, the **concepts of numeration**, and an understanding of the **equivalence** of different representations and forms of numbers.

In their work with **numeration**, ninth- through twelfth-graders should view mathematics as a coherent body of knowledge. They should see the integers, the rational numbers, and the real numbers as logical and necessary extensions of the whole number system. Only with these extensions can expressions like *3 minus 7, 4 divided by 3,* and *the ratio of the circumference of a circle to its diameter* have values. High school students should understand how the integers, rational numbers, irrational numbers, and real numbers are related to each other and what properties are true for the numerical operations on these number systems.

Students at these grade levels continue their learning about **equivalence**, but with an increasing focus on approximation, particularly for irrational numbers (see Standard 14 — Building Blocks of Calculus). High school students should understand the difference between an exact value of an irrational number, such as $\sqrt{3}$, and its approximation (1.728). They should also be familiar with the use of scientific notation as an equivalent form of decimal number.

Estimation continues to be a regular part of mathematics classes, both estimation of quantities and estimation of the results of operations. Students should respond to questions that arise naturally during the course of the class with answers that illustrate confident and well-conceived use of estimation strategies and number sense.

Technology also plays an important role in number sense at these grade levels, particularly since calculators and computers use approximations for some fraction-to-decimal conversions and for irrational numbers. Calculators can be wonderful exploration tools when examining numerical relationships, and computer software which allows exploration of number relationships through conversion utilities and graphs opens up even more possibilities.

The topics that should comprise the number sense focus of the ninth through twelfth grade mathematics program are:

the real number system
exponents, roots, and scientific notation
properties of number systems

Standard 6 — Number Sense — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

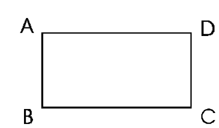
Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

20. Extend their understanding of the number system to include real numbers and an awareness of other number systems.

- Students explore alternative number bases and note their advantages and disadvantages. Binary and hexadecimal systems are of primary interest because of their use in computer programming.
- Students read about and understand the historical relationship between the square root of 2 and the Pythagoreans' attempt to express the length of the diagonal of a square in terms of its sides. They use geoboards to illustrate how the hypotenuse of a right triangle can represent a length of $\sqrt{2}$, $\sqrt{3}$, or $\sqrt{5}$, and so on.
- Students' understanding of real numbers is assessed by asking them to locate rational and irrational numbers on a number line.
- Students find two numbers between any two given real numbers, such as $.4\overline{2}$ and $.\overline{42}$.
- Students can give examples of irrational numbers such as 3.010010001 ... or .4323323332 ... .

21. Develop conjectures and informal proofs of properties of number systems and sets of numbers.

- Students discuss whether the transitive and reflexive properties hold for different relationships, such as "is a friend of", "is perpendicular to," or "is a factor of."
- Students make up a number system using the symbols ▲, ■, ★, ♥ and ●. They develop algorithms for adding and multiplying within their system and decide whether these operations are commutative and associative.
- Students explore the properties of clock arithmetic or a modular arithmetic system.
- Students examine properties involving addition of matrices, scalar multiplication, and matrix multiplication. They demonstrate that matrix multiplication is not commutative by providing a counter-example.
- Students investigate transformations of the rectangle ABCD: reflection about its the horizontal line of symmetry (H), reflection about its vertical line of symmetry (V), rotation by 180° (R), and rotation by 360° (the identity, I). They construct an operations table



(see below) which tells what happens if one of these transformations is followed by another. Thus, for example, if you reflect about the vertical line of symmetry (V) and then rotate by 180° (R), the result is the same as reflecting about the horizontal line of symmetry (H); this is indicated in the table by placing H as the entry in the row for V and column for R representing the conclusion that *V followed by R is H*. Students investigate the properties of this operation "followed by."

	I	**H**	**V**	**R**
I	**I**	**H**	**V**	**R**
H	**H**	**I**	**R**	**V**
V	**V**	**R**	**I**	**H**
R	**R**	**V**	**H**	**I**

22. Extend their intuitive grasp of number relationships, uses, and interpretations and develop an ability to work with rational and irrational numbers.

- Students create computer spreadsheets to help assess real-world and purely mathematical numerical situations and to ask *what if* questions regarding complex data.
- Students use calculators and the formula for compound interest to answer specific questions regarding the amount of money that will be in a particular bank account after 1, 10, and 100 years.
- Students informally solve maximum/minimum problems with the help of graphing calculators.
- Students use formulas for projectile motion to solve problems regarding distance traveled, time in flight, maximum height, and so on.
- Students compare different representations for π, including 3.14, 22/7, and the value given by their calculators. They discuss the accuracy of each approximation, suggesting appropriate circumstances for its use.
- Students work through the *Ice Cones* lesson that is described in the First Four Standards of this *Framework*. They use graphing calculations to determine the maximum volume of a cone created from a 10 inch circle which is cut along a radius.

23. Explore a variety of infinite sequences and informally evaluate their limits.

- Students explore the value of .99 ... (or $4.9\overline{9}$ or $3.2\overline{9}$) as an infinite series (9/10 + 9/100 + ...), and conclude that its value is exactly 1 (or 5 or 3.3).
- Students analyze and discuss the sums of infinite series such as the following:

$1 - 1 + 1 - 1 + 1 - 1 + \ldots$
$(1 - 1) + (1 - 1) + (1 - 1) + \ldots$
$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$

- Students informally find the limits of real-world series such as the total vertical distance traveled by a ball dropped from a height of 10 meters which always bounces back to 3/4 of its original height.
- Students investigate the sums of series such as:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \qquad or \qquad \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

- Students estimate the area of a circle with radius of 10 by informally judging the limiting value of the sequence produced by the areas of inscribed regular polygons as their sides increase by 1; that is, they calculate the area of an inscribed equilateral triangle, an inscribed square, an inscribed pentagon, and so on.
- Students discuss *pyramid schemes* and create a mathematical model to determine how many people would have to participate in the scheme for everyone at the fifth level to be paid, for everyone at the tenth level to be paid, and then for everyone who participated at any level to be paid. For example, if each person pays four others, then there are 4^n people at the nth level.

References

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

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